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A Remark on the Asymptotic Properties of Positive Homogeneous Maps on Homogeneous Lattices

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Abstract

An abstract version of Lyapunov exponents is defined for positive homogeneous maps on Homogeneous Lattices and a sufficient conditon is given for the asymptotic stability of the map.

Key Words: Lyapunov Exponents, asymptotic stability

1. Introduction

We will start with a typical example. Let X be a compact Hausdorff space, and let $a : X \to \Re$ and $\varphi : X \to X$ both be continuous functions. Define $T : C(X; \Re) \to C(X; \Re)$ by $(Tf)(x) = a(x)f(\varphi(x))$, if $a > 0$ on X, then the operator T is linear and positive. For any continuous, positive function $f : X \to \Re$, define for $n = 1, 2, ...$

$$\gamma(f, n, x) \equiv (T^n f)^{1/n}(x) = \left[f(\varphi^n(x)) \prod_{k=0}^{n-1} a(\varphi^k(x)) \right]^{1/n}$$

and

$$\Gamma(f, n) = \max_{x \in X} \gamma(f, n, x).$$

It is well known that $\lim_{n\to\infty} \Gamma(f, n)$ exists as a positive number and we will denote it by

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$e^{\mu(f)}$. Similarly let us define $\mu(f,x)$ through

$$e^{\mu(f,x)} = \limsup_{n\to\infty} \gamma(f,n,x).$$

The numbers $\mu(f)$ are known as *Lyapunov exponents of T at f*, and $\mu(f,x)$ were named *Local Lyapunov exponents* in Eden et. al. [2]. Also in the same paper it was proven that there exists $x_0 \in X$ such that $\Gamma(f,n) \geq \gamma(f,n,x_0) \geq e^{\mu}, \forall n$. Hence

$$\lim_{n\to\infty} \gamma(f,n,x_0)$$

exists and is equal to e^{μ}. It turns out that the vector space structure of $C(X;\Re)$ plays a minimal role in the proof of the main theorem in Eden et. al. [2] (see also [3]) and a more general theorem can be established based on the order properties of the underlying space and the positivity of the operator T. This is the aim of this note.

2. Main Result

In this section we establish the existence of the L-exponent of a positive map on a Homogeneous Lattice with unity, see Definition 2.8. L-exponents is a measure of the asymptotic behviour of the map, e.g. if the L-exponent is negative then the iterates of the map converge to zero exponentially fast.

In order to impose minimal conditions on the underlying structures, we give the definitions explicitly.

Definition 2.1. Let $(\Lambda, \vee, \wedge)$ be a lattice that is closed under a "scalar multiplication" $\cdot : \Re \times \Lambda \to \Lambda$. Let us further assume that the lattice structure is compatible with the scalar multiplication, i.e. for $f, g \in \Lambda$, $\alpha > 0, \alpha(f \vee g) = \alpha f \vee \alpha g$ and $\alpha(f \wedge g) = \alpha f \wedge \alpha g$. We will call the structure $(\Lambda, \vee, \wedge, \cdot)$ a **Homogeneous Lattice**.

We will assume that the lattice Λ has two distinguished elements : the zero $\mathbf{0}$ and the unit $\mathbf{e}$, where $\mathbf{e} \vee \mathbf{0} = \mathbf{e}$, $\mathbf{e} \wedge \mathbf{0} = \mathbf{0}$.

Definition 2.2. The **Positive Cone** of the lattice Λ consists of the elements $\mathbf{f}$ such that $\mathbf{f} \vee \mathbf{0} = \mathbf{0}$ and will be denoted by Λ_+.

Definition 2.3. An order can be defined by $\mathbf{f} \leq \mathbf{g} \Leftrightarrow \mathbf{f} \vee \mathbf{g} = \mathbf{g}$ and $\mathbf{f} \wedge \mathbf{g} = \mathbf{f}$. It follows that $\mathbf{e} \geq \mathbf{0}$ and the order structure is compatible with the scalar multiplication.

Clearly, $\mathbf{f} \in \Lambda_+ \Leftrightarrow \mathbf{f} \geq \mathbf{0}$

Example 2.1. Furnish $C(X;\Re)$ with the standard max and min operators, i.e.

$\mathbf{f} \vee \mathbf{g} = \mathbf{h} \Leftrightarrow h(x) = \max\{f(x), g(x)\} \qquad \mathbf{f} \wedge \mathbf{g} = \mathbf{h} \Leftrightarrow h(x) = \min\{f(x), g(x)\}$.

Then $(C(X;\Re), \vee, \wedge)$ is a Homogeneous Lattice, with zero and unit defined by $\mathbf{0}(x) = 0$ and $\mathbf{e}(x) = 1$ for all $x \in X$. The positive cone of $C(X;\Re)$ consists of non-negative continuous functions.

Definition 2.4. An **order preserving valuation** on a Homogeneous Lattice $\|\cdot\| : \Lambda \to \Re$ is a non-negative map with the following properties:

(i) $\|\alpha \mathbf{f}\| = |\alpha| \, \|\mathbf{f}\|$, $\mathbf{f} \in \Lambda, \alpha \in \Re$;

(ii) $\|\mathbf{f}\| = 0$ implies that $\mathbf{f} = \mathbf{0}$;

(iii) $\mathbf{0} \leq \mathbf{f} \leq \mathbf{g}$ implies that $\|\mathbf{f}\| \leq \|\mathbf{g}\|$;

(iv) $\|\mathbf{e}\| = 1$;

(v) For every $\mathbf{f} \in \Lambda, -\|\mathbf{f}\| \, \mathbf{e} \leq \mathbf{f} \leq \|\mathbf{f}\| \, \mathbf{e}$.

Definition 2.5. A map $T : \Lambda \to \Lambda$ is called **positive homogeneous** if it is order preserving and homogeneous, i.e.

(i) $\mathbf{f} \leq \mathbf{g}$ implies that $T\mathbf{f} \leq T\mathbf{g}$,

(ii) $T(\alpha \mathbf{f}) = \alpha T(\mathbf{f})$, for $\alpha \in \Re$, $\mathbf{f} \in \Lambda$.

A positive homogeneous map T also preserves the lattice operations, i.e. $T(\mathbf{f} \vee \mathbf{g}) = T\mathbf{f} \vee T\mathbf{g}$ and $T(\mathbf{f} \wedge \mathbf{g}) = T\mathbf{f} \wedge T\mathbf{g}$.

Definition 2.6. The **norm** of a positive homogeneous map T is defined by $|T| = \sup\limits_{\mathbf{0} \leq \mathbf{f} \leq \mathbf{e}} \|T\mathbf{f}\|$.

Lemma 2.1. Let T and S be a positive homogeneous maps on Λ, then

(i) $|T| = \|Te\|$;

(ii) $|\alpha T| = \alpha \, |T|$ for $\alpha > 0$;

(iii) $\|T\mathbf{f}\| \leq |T| \, \|\mathbf{f}\|$ for $\mathbf{f} \in \Lambda_+$;

(iv) $|ST| \leq |S| \, |T|$.

Proof. (i) follows from the positivity of T combined with $\|T\mathbf{e}\| \leq |T|$.

(ii) follows from (i) and the homogeneity of the valuation $\| \quad \|$.

By the homogeneity of $T, T\mathbf{0} = \mathbf{0}$, hence for (iii) we can assume that $\|\mathbf{f}\| \neq 0$. Then $\frac{1}{\|\mathbf{f}\|}\mathbf{f} \leq \mathbf{e}$ implies that $\left\| T\left(\frac{1}{\|\mathbf{f}\|}\mathbf{f}\right) \right\| \leq |T|$. Combining with $\left\| T\left(\frac{1}{\|\mathbf{f}\|}\mathbf{f}\right) \right\| = \left\| \frac{1}{\|\mathbf{f}\|} T(\mathbf{f}) \right\| =$

$\frac{1}{\|\mathbf{f}\|}\left\| T\left(\mathbf{f}\right)\right\|$, (iii) follows.

(iv) follows from (i) and (iii) with $\mathbf{f} = S\mathbf{e}$. □

Definition 2.7. A sequence of real numbers $\{a_n\}$ is said to be **subadditive** if

$$a_{n+m} \leq a_n + a_m.$$

We state the following well-known fact about sub-additive sequences af real numbers (a proof can be found e.g. in Walters [4] pg.87, Theorem 4.9)

Lemma 2.2. If $\{a_n\}$ is a subadditive sequence then $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists and

$$\lim_{n\to\infty} \frac{a_n}{n} + \inf_{n\geq 1} \frac{a_n}{n}.$$

Lemma 2.3. The sequence of real numbers a_n defined by $a_n = \log\left\| T^n\mathbf{e}\right\|$ is subadditive. Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \log\left\| T^n\mathbf{e}\right\| = \inf_{n\geq 1} \frac{1}{n} \log\left\| T^n\mathbf{e}\right\|.$$

Proof. $a_{n+m} = \log\left\| T^{n+m}\mathbf{e}\right\| = \log\left| T^{n+m}\right| \leq \log\left| T^n\right| \left| T^m\right| = \log\left| T^n\right| + \log\left| T^m\right| = a_n + a_m$ by Lemma 2.1 (iv) and the result follows from Lemma 2.2. □

Definition 2.8. The number μ defined by $\lim\limits_{n\to\infty} \frac{1}{n} \log\left\| T^n\mathbf{e}\right\|$ is called the **L-exponent of T** (after Lyapunov).

The main theorem of this short note is the following.

Theorem 2.1. If there exists a number r such that $T\mathbf{e} \wedge T^2\mathbf{e} \wedge ... \wedge T^r\mathbf{e} < \theta\mathbf{e}$, for some $\theta < 1$ then $\mu < 0$, i.e. $\left\| T^n\mathbf{e}\right\|$ converges to zero exponentially.

Proof. We break the argument into smaller steps.

Step 1. Define $g_r = T\mathbf{e} \wedge T^2\mathbf{e} \wedge ... \wedge T^r\mathbf{e}$, then $g_r \leq \theta\mathbf{e} < \mathbf{e}$ combined with $g_r \leq T^i\mathbf{e}$ for $i = 1, 2, ..., r$ implies that $Tg_r \leq T^j\mathbf{e}$ for $j = 1, 2, ..., r+1$, consequently $Tg_r \leq g_r$.

Step 2. By induction, on k, $T^k g_r \leq g_r \leq \theta\mathbf{e}$, for $k = 1, 2, ..., r$. Hence $T^r g_r \leq \theta T^i e$, for $i = 1, 2, ..., r$, from which it follows that $T^r g_r \leq \theta g_r$.

Step 3. Let $a = \max\{1, \|Te\|\}$. From $Te \leq \|Te\|\, e \leq a\mathbf{e}$,and $a > 1$, we have $T^r e \leq a^{r-i} T^i e \leq a^r T^i e$, for $i = 0, 1, ..., r$, hence $T^r e \leq a^r g_r$.

Step 4. Choose n large enough such that $a^r \theta^n < 1$, and set $m = (n+1)r$ then $T^m\mathbf{e} = T^{nr} T^r\mathbf{e} \leq a^r T^{nr} g_r \leq a^r \theta^n \mathbf{e} \leq \mathbf{e}$. Consequently, $\left\| T^m\mathbf{e}\right\| < 1$.

Step 5. From the last step we infer that $\inf_{n\geq 1} \|T^n \mathbf{e}\|^{\frac{1}{n}} < 1$, hence $\mu = \lim_{n\to\infty} \frac{1}{n} \log \|T^n \mathbf{e}\| < 0$. □

Remark 2.1. The proof given above is the abstract version of a similar one given in Choquet and Foias, more specifically, there are three different proofs of the above mentioned result in the specific case of $C(X;\Re)$ only one of them is suitable for this generalization, i.e. the first one. [1]

Remark 2.2. The standard example is the one given in the introduction.

References

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