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A Borsuk-Ulak Theorem for Heisenberg Group Actions

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Abstract

Let $G = H_{2n+1}$ be a $(2n+1)$-dimensional Heisenberg Lie group acts on $M = C^m - \{0\}$ and $M^{'} = C^{m^{'}} - \{0\}$ exponentially. By using Cohomological Index we proved the following theorem.

If $f : M \rightarrow M^{'}$ is a G-equivariant map, then $m \leq m'$.

Key Words: Borsuk-Ulam Type Theorem, Cohomological Index, Group Action.

1. Introduction

By using cohomological index and relative index theories Fadell, Husseini and Rabinowitz proved Borsuk-Ulam type theorems for compact Lie groups. The ideal-valued index $Index_G(X)$ of a G-space X for a compact Lie group, is the kernel of the map $H^*_G(pt) \rightarrow H^*_G(X)$, where $H^*_G(pt)$ is the Borel cohomology of a point, which is isomorphic to $H^*(BG)$, the cohomology of the classifying space of G, [2,5]. If G is a non-compact Lie group, where BG may be acyclic, then the preceding method fails.

Fadell and Husseini introduced infinitesimal ideal-valued index theory to overcome difficulties of this type. Infinitesimal index is the kernel of the map from $B\mathcal{G}$, the basic subcomplex of G to $H^*_{\mathcal{G}}(X)$, the infinitesimal G-deRham cohomology of a G-space X.They proved a Borsuk-Ulam type theorem for the non-compact abelian Lie group $G = C$ [3,4].

In this work we would like to extend their results to the Heisenberg groups. The

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main theorem of this work is a Borsuk-Ulam type theorem about a Heisenberg Lie group action.

2. Preliminaries

First recall the definitions of the Lie derivative $\Theta(X)$, the substitution operation $i(X)$, and the differential operator δ.

The Lie derivative of a p-form α with respect to $X \in X(M)$ is the linear map $\Theta(X)$ homogeneous of degree zero, given by

$$\Theta(X)\alpha = X(\alpha(X_1, X_2, ...X_p)) - (\sum_{j=1}^{p} \alpha(X_1, ..., [X, X_j], ..., X_p)).$$

The substitution operator $i(X)$, induced by X define a $(p-1)$-form $i(X)\alpha$ by

$$i(X)\alpha(X_1, X_2, ..., X_{p-1}) = \alpha(X, X_1, X_2, ..., X_{p-1})$$

The map $i(X) : \Omega(M) \rightarrow \Omega(M)$ is a homogeneous operator of degree (-1).

The exterior derivative is the real linear map δ, homogeneous of degree 1, defined by

$$\delta\alpha(X_0, X_1, ..., X_p) = \sum_{j=0}^{p} (-1)^j X_j(\alpha(X_0, .., , \hat{X}_j, ..., X_p))$$

$$+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \alpha([X_i, X_j], ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p)$$

where α is a p-form.

Let G be a connected Lie group with its Lie algebra $\mathcal{G}$ and dual $\mathcal{G}^*$. The Weil algebra $W(\mathcal{G}) = \Lambda(\mathcal{G}^*) \otimes S(\mathcal{G}^*)$ where $\Lambda(\mathcal{G}^*)$ is the exterior algebra of the dual $\mathcal{G}^*$ and $S(\mathcal{G}^*)$ is the symmetric algebra generated by elements of degree 2. Let s_k be a basis of $S(\mathcal{G}^*)$, $h : \mathcal{G}^* \rightarrow S^2(\mathcal{G}^*)$ defined by $h(\alpha_k) = s_k$, where α_k is a basis for $\mathcal{G}^*$. Also we define $\Theta_S(X)s_k = h(\Theta_E(X)\alpha_k)$, on $S(\mathcal{G}^*)$, where $\Theta_E(X)$ is the usual Lie derivative defined on $\mathcal{G}^*$. The substitution operation $i(X)$ is as defined above on $\Lambda(\mathcal{G}^*)$ and 0 on $S(\mathcal{G}^*)$.

The differential operator δ on $W(\mathcal{G})$ defined as follows:

$$\delta = \delta_E + \delta_S + h$$

$$h = \sum_k i(X_k)\mu(h(\alpha_k))$$

$$\delta_E = (1/2)\sum_k \mu(\alpha_k)\Theta_E(X_k)$$

$$\delta_S = \sum_k \mu(\alpha_k)\Theta_S(X_k)$$

$X_k \in \mathcal{G}$, $\alpha_k \in \mathcal{G}^*$ and μ is a multiplication operator defined as $\mu(\alpha)\beta = \alpha \wedge \beta$, $\alpha, \beta \in \Lambda(\mathcal{G}^*)$.

Let L be a finite dimensional Lie algebra and i, δ and Θ are defined as above. Let $R = \sum_{p \geq 0} R^p$ be a graded commutative algebra with differential δ.
The horizontal subalgebra of R:

$$R_{i=0} = \bigcap_{X \in L} ker\ i(X).$$

The invariant subalgebra of R:

$$R_{\Theta=0} = \bigcap_{X \in L} ker\ \Theta(X).$$

The basic subalgebra of R:

$$R_{i=0,\Theta=0} = (R_{i=0}) \bigcap (R_{\Theta=0}).$$

The basic subalgebra of the Weil algebra of a Lie group G:

$$W(\mathcal{G})_{i=0,\Theta=0} = (\Lambda(\mathcal{G}^*) \otimes S(\mathcal{G}^*))_{i=0,\Theta=0} \cong S(\mathcal{G}^*)_{\Theta=0} = B\mathcal{G}.$$

The Basic Weil subalgebra serves as the algebraic analogue of the classifying space BG and we will denoted by $B\mathcal{G}$, [3].

Infinitesimal Index:

The infinitesimal deRham complex of a differentiable G-manifold M is $\Omega(M) \otimes W(\mathcal{G})$, with Θ, i and δ the differential operator, where $\Omega(M)$ denotes the differential forms on M. The basic subcomplex $\Omega_{\mathcal{G}}(M)$ of $\Omega(M) \otimes W(\mathcal{G})$ is defined as $\Omega_{\mathcal{G}}(M) = (\Omega(M) \otimes W(\mathcal{G}))_{i=0,\Theta=0}$ and the cohomology of $\Omega_{\mathcal{G}}(M)$ is called the infinitesimal deRham cohomology of M and denoted by $H^*_{\mathcal{G}}(M)$. The inclusion map

$$j_M : W(\mathcal{G}) \rightarrow \Omega(M) \otimes W(\mathcal{G})$$

$$x \rightarrow 1 \otimes x$$

j_M induces morphisms

$$\tilde{j_M} : W(\mathcal{G})_{i=0,\Theta=0} \rightarrow (\Omega(M) \otimes W(\mathcal{G}))_{i=0,\Theta=0}$$

$$\tilde{j_M} : B\mathcal{G} \rightarrow \Omega_{\mathcal{G}}(M)$$

a morphism $\tilde{j_M}$ of differential graded algebras is called the classifying map for the G-space M. The classifying map $\tilde{j_M}$ induces

$$j^*_M : B\mathcal{G} \rightarrow H^*_{\mathcal{G}}(M)$$

since $\delta = 0$ on $S(\mathcal{G}^*)$.

The infinitesimal $\mathcal{G}$-index of M, $Index_{\mathcal{G}} M$, is the kernel of the map

$$j^*_M : B\mathcal{G} \rightarrow H^*_{\mathcal{G}}(M)$$

where j^*_M is induced by $\tilde{j_M} : B\mathcal{G} \rightarrow \Omega_{\mathcal{G}}(M)$.

The infinitesimal $\mathcal{G}$-index possesses the following properties:

Continuity. *[3] If $B\mathcal{G}$ is Noetherian, there is an open G-set V_0 such that $X \subset V_0$ and for every open G-set U, $X \subset U \subset V_0$,*

$$Index_{\mathcal{G}} X = Index_{\mathcal{G}} U.$$

Monotonicity. *[3] Let $B\mathcal{G}$ be Noetherian, and $f : M \rightarrow N$ is a differentiable G-map, $X \subset M$ and $Y \subset N$ are G-subsets with $f(X) \subset Y$, then*

$$Index_{\mathcal{G}} Y \subset Index_{\mathcal{G}} X.$$

Additivity. *[3] Let $X \cup Y \subset M$, X and Y be G-sets and $B\mathcal{G}$ is Noetherian, then*

$$(Index_{\mathcal{G}}X)(Index_{\mathcal{G}}Y) \subset Index_{\mathcal{G}}(X \cup Y).$$

Recall that, by definition, the $(2n+1)$-dimensional Heisenberg Lie group H_{2n+1} is the Lie group of real matrices of the form:

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & \cdots & 0 & y_1 \\ 0 & 0 & 1 & \cdots & 0 & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & y_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where x_i, y_i, and $z \in R$.
H_{2n+1} is a two-step, nilpotent Lie group. Let $\mathcal{H}_{2n+1}$ denote the Lie algebra of H_{2n+1}. $\mathcal{H}_{2n+1}$ is generated by $\{X_1, ..., X_n, Y_1, ..., Y_n, Z\}$ with all its commutators equal to zero except $[X_i, Y_i] = Z, i = 1, 2...n$. The dual $\mathcal{H}^*_{2n+1}$ is generated by $\{\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n, \gamma\}$, where

$$X_i = \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}$$

and $\alpha_i = dx_i$, $\beta_i = dy_i$, $\gamma = dz - \sum_{i=1}^{n} x_i dy_i$, $i = 1, ..., n$.

Proposition 1. *Let G be a $(2n+1)$-dimensional Heisenberg Lie group and $A = G/[G,G]$ be its abelinization , then $B\mathcal{G} \cong B\mathcal{A}$. Where $B\mathcal{A}$ is the polynomial algebra in $s_1, \ldots, s_n, t_1, \ldots, t_n$, and $s_i = h(\alpha_i)$,and $t_i = h(\beta_i)$.*

Proof. The basic subcomplex $B\mathcal{G} \cong (S\mathcal{G}^*)_{\Theta=0}$, where $\Theta = 0$ means that the Lie derivatives is zero with respect to all $X \in \mathcal{G}$. Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$, and $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma\}$ denote the generators of $\mathcal{G}$ and $\mathcal{G}^*$ respectively, then $\mathcal{A}$ and $\mathcal{A}^*$ are generated by $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$, and $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ respectively. It is known that, if A is abelian, all the generators of $\mathcal{A}^*$ are invariant differential forms, then $\Theta(X)\alpha_i = \Theta(X)\beta_j = 0$ for all $X \in \mathcal{G}$ and $1 \leq i, j \leq n$, [3]. Therefore

$$B\mathcal{A} = (S\mathcal{A}^*)_{\Theta=0} = S\mathcal{A}^* \subset (S\mathcal{G}^*)_{\Theta=0} = B\mathcal{G}$$

Now assume that $B\mathcal{G}$ has some elements ω which is not an element of $B\mathcal{A}$. Then ω contains a polynomial of $r = h(\gamma)$. Let

$$\wp(r) = \sum_{l=0}^{m} a_l r^l \neq 0$$

where a_l's are linearly independent in $B\mathcal{G}$.

$$\omega = \sum_{j=1}^{k} c_j s_{j_1} \wedge \ldots \wedge s_{j_p} \wedge t_{j_1} \wedge \ldots \wedge t_{j_q} \wedge \wp(r)$$

and $\Theta(X)\omega = 0$ for all $X \in \mathcal{G}$. Since $\Theta(X)s_i = h(\Theta(X)\alpha_i) = 0$ and $\Theta(X)t_j = h(\Theta(X)\beta_j) = 0$,

$$\Theta(X)\omega = (\sum_{j=1}^{k} c_j s_{j_1} \wedge \ldots \wedge s_{j_p} \wedge t_{j_1} \wedge \ldots \wedge t_{j_q}) \wedge \Theta(X)\wp(r) = 0$$

$$\Theta(X)\wp(r) = 0$$

$$\Theta(X)\wp(r) = \Theta(X)(a_0 + a_1 r + a_2 r^2 + \cdots + a_m r^m) = 0$$

$$a_1(\Theta(X)r) + a_2(\Theta(X)r^2) + \cdots + a_m(\Theta(X)r^m) = 0$$

$$a_1(\Theta(X)r) + 2a_2 r(\Theta(X)r) + \cdots + m a_m r^{m-1}(\Theta(X)r) = 0$$

$$(a_1 + 2a_2 r + \cdots + m a_m r^{m-1})(\Theta(X)r) = 0$$

since $\Theta(X)r = h(\Theta(X)\gamma) \neq 0$, then

$$a_1 + 2a_2 r + \cdots + m a_m r^{m-1} = 0$$

by linearly independence, $a_i = 0$ for $i = 1, ..., m$, thus $\wp(r) = a_0$. Therefore, $B\mathcal{G} \cong B\mathcal{A}$. □

Exponential G Action:

Let C^n denote the complex n-space and $M = C^n$. Fadell and Husseini defined the right C-action

$$M \times G \rightarrow M$$

$$(z_1, \ldots, z_n)(x+iy) = e^x(z_1(e^{iy})^{\lambda_1}, \ldots, z_n(e^{iy})^{\lambda_n})$$

where $(\lambda_1, \ldots, \lambda_n)$ is an n-tuple of non-zero real numbers, such that λ_i/λ_j are irrational for $i \neq j$. This action takes $C^n - \{0\}$ onto itself. Fadell and Husseini called this action an exponential action with parameters $\lambda_1, \ldots, \lambda_n$, and then proved the following Borsuk-Ulam type theorem,[3]:

If $G = C$ acts on C^n and C^m with exponential actions with parameters $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, respectively, with $m < n$. Then every G-map $f : C^n \to C^m$ has a non-trivial zero. Alternatively, there does not exist a G-map $f : C^n - \{0\} \to C^m - \{0\}$.

Here we want to use the same action for $G = G_1 \oplus \cdots \oplus G_n$ and $M = C^m - \{0\} = (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\}$ where $G_k = R + iR$ and $M_k = C^{m_k} - \{0\}$.

The G_k action on M_k is defined as follows:

$$\varphi_k : (\vec{z}, \xi_k, \vec{\lambda_k}) \to e^{x_k}(z_1(e^{iy_k})^{\lambda_1}, \ldots, z_{m_k}(e^{iy_k})^{\lambda_{m_k}}).$$

where $\xi_k = x_k + iy_k \in G_k$, $\vec{z} = (z_1, \ldots, z_{m_k}) \in M_k$, and $\vec{\lambda_k} = (\lambda_1, \ldots, \lambda_{m_k})$ is an m_k- tuple of non-zero real numbers such that λ_i/λ_j are irrational for $i \neq j$.

Since $z_j(e^{iy_k})^{\lambda_j} = z_j$ for all j, if and only if $\lambda_j y_k = 2q\pi$, $q \in Z$, the discrete subgroup $\Gamma_j = \{(2q\pi/\lambda_j)\}$, $q \in Z$ appears as non-trivial isotropy. If we require that λ_i/λ_j be irrational for $i \neq j$, then these would be the only non-trivial isotropy subgroups. We also note that the representation $iy_k \to diag((e^{iy_k})^{\lambda_1}, \ldots, (e^{iy_k})^{\lambda_{m_k}})$ has compact image $R/D = S^1$, D discrete, in $U(m_k)$ if all ratios λ_i/λ_j are rational. Otherwise, this imbedding of R in $U(m_k)$ has closure which is a torus of dimension ≥ 2.

The G-action on M is defined as follows:

$$M \times G \to M$$

$$(\oplus_{k=1}^n M_k) \times (\oplus_{k=1}^n G_k) \to (\oplus_{k=1}^n M_k)$$

$$((\vec{z_1}, \ldots, \vec{z_n}), (\xi_1, \ldots, \xi_n), (\vec{\lambda}_1, \ldots, \vec{\lambda}_n)) \to e^{(\sum_{j=1}^n x_j)}(\varphi_1(\vec{z_1}, iy_1, \vec{\lambda}_1), \ldots, \varphi_n(\vec{z_n}, iy_n, \vec{\lambda}_n)).$$

Here we have

$$G = (G_1 \oplus iG_1) \oplus \cdots \oplus (G_n \oplus iG_n).$$

This can be written as,

$$G = (G_1 \oplus \cdots \oplus G_n) \oplus iG_1 \oplus \cdots \oplus iG_n$$

$$G = R\oplus iR_1\oplus\cdots\oplus iR_n.$$

Let $G^{'} = R$ and $G^{''} = R_1\oplus\cdots\oplus R_n$. Here $G^{'}$ acts freely on $C^m - \{0\}$ and $C^m - \{0\}\cong R^+\times S^{2m-1}$ with orbit space $(C^m - \{0\})/G'\cong S^{2m-1}$. If Γ is a discrete subgroup of G', note that $G'/\Gamma = S^1$ is a compact group and

$$M/\Gamma = (C^m - \{0\})/\Gamma\cong S^1\times S^{2m-1}.$$

By lemmas 4.2, 4.3, and 4.4 of Fadell and Husseini [3], there is a natural chain equivalence

$$\nu : \Omega(M)_{\Theta=0}\rightarrow\Omega(S^1\times S^{2m-1}).$$

Now, consider $M = C^m - \{0\}$ as a $G^{'}$-space by restricting the G action. Then the natural projection $M\rightarrow M/G^{'} = S^{2m-1}$ is a locally trivial principle G-bundle by Palais' theorems [7] .

Proposition 2. *[3] There is a chain equivalence* $\gamma : \Omega_{\mathcal{G}'}(M)\rightarrow\Omega(S^{2m-1})$.

Atiyah and Bott showed that, since torus T is compact, the Borel cohomology $H_T^*(S^{2m-1}; R)$ is naturally isomorphic to the infinitesimal cohomology $H_{\mathcal{T}}^*(S^{2m-1})$ [1]. Furthermore, the ideal-valued index, $Index_T(S^{2m-1})$ and the infinitesimal ideal-valued index, $Index_{\mathcal{T}}(S^{2m-1})$ coincide when $H^*(BT; R)$ and $B\mathcal{T}$ are naturally identified. The inclusion map $T\subset T^m\subset U(m)$ induces homomorphisms $\lambda_j : T\rightarrow S^1$, $j = 1, 2, ..., m.$, and if $\mathcal{S}$ is the Lie algebra of S^1, λ_j induces $\lambda_j^* : B\mathcal{S}\rightarrow B\mathcal{T}$. If σ is the generator of $B\mathcal{S}$ set $\lambda_j^{'} = \lambda_j^*(\sigma)$. Then, the natural inclusion

$$B\mathcal{T}\rightarrow\Omega(M)_{\Theta_{\mathcal{T}}=0}\otimes B\mathcal{T}$$

induces a surjection

$$B\mathcal{T}\rightarrow H_{\mathcal{T}}^*(S^{2m-1})$$

with kernel P_T the principal ideal generated by $\varepsilon = \lambda_1^{'}\lambda_2^{'}...\lambda_m^{'}$. Here each $\lambda_j : T\rightarrow S^1$ is nontrivial because $(S^{2m-1})^{G^{''}} = (S^{2m-1})^T = \emptyset$. This implies that if $g : B\mathcal{T}\rightarrow B\mathcal{G}^{''}$ is induced by inclusion $g(\lambda_j')\neq 0$ for $j = 1, 2, ..., m$.

Lemma. *[6] $g(\varepsilon)$, is a polynomial of $t_1,\ldots t_n$, of degree m.*

3. Results

Now consider the exponential $G = G_1 \oplus \cdots \oplus G_n$ action on

$$M = (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\} = C^m - \{0\}$$

where $m = m_1 + \cdots + m_n$, and $M = C^m - \{0\} = R^+ \times S^{2m-1}$.

Theorem 1. *Let $G = G_1 \oplus \cdots \oplus G_n$ acts on $M = (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\}$ with an exponential action with parameters $(\lambda_1, \ldots, \lambda_m)$. Then, the following inclusion map*

$$j_M : W(G) \rightarrow \Omega_{\mathcal{G}}(M)$$

induces a surjection

$$j_M^* : B\mathcal{G} \rightarrow H_{\mathcal{G}}^*(M)$$

The kernel of this map is an ideal generated by $s_1 + \cdots + s_n$ and $\lambda_1' \lambda_2' \ldots \lambda_m'$.

$$Index_{\mathcal{G}}(M) = < s_1 + \cdots + s_n, \lambda_1' \lambda_2' \ldots \lambda_m' > .$$

Proof. I. First compare the spectral sequences $\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*)$ and $\Omega(M)_{\Theta'=0} \otimes S(\mathcal{G}'^*)$ via the filtration preserving map

$$\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*) \rightarrow \Omega(M)_{\Theta'=0} \otimes S(\mathcal{G}'^*)$$

induced by $G' \subset G$. Induced map on fibers $\Omega(M)_{\Theta=0} \rightarrow \Omega(M)_{\Theta'=0}$ is a chain equivalence and on the base $S(\mathcal{G}^*) = R[s_1, ..., s_n, t_1, ..., t_n] \rightarrow R[s_1 + ... + s_n] = S(\mathcal{G}'^*)$. At E_2-level we have the following diagram:

$$\begin{array}{ccc} H^*(\Omega(M)_{\Theta=0}) \otimes S(\mathcal{G}^*) & \rightarrow & H^*(\Omega(M)_{\Theta'=0}) \otimes S(\mathcal{G}'^*) \\ \downarrow d_2 & & \downarrow d_2' \\ H^*(\Omega(M)_{\Theta=0}) \otimes S(\mathcal{G}^*) & \rightarrow & H^*(\Omega(M)_{\Theta'=0}) \otimes S(\mathcal{G}'^*) \end{array}$$

Let $u' \in H^1(\Omega(M)_{\Theta'=0}) = H^1(S^1 \times S^{2m-1})$ denotes a generator corresponding to the S^1-factor. Since $H_{\mathcal{G}'}^*(M) = H^*(S^{2m-1})$ then $d_2' u' \neq 0$. We may assume without loss that, if $u \in H^1(\Omega(M)_{\Theta=0}) = H^1(S^1 \times S^{2m-1})$ is the generator corresponding to $u^{'}$, then $d_2 u = s_1 + \cdots + s_n$.

II. Now compare the spectral sequences for $\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*)$ and $\Omega(S^{2m-1})_{\Theta''=0} \otimes S(\mathcal{G}^{''*})$ via natural map

$$\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*) \to \Omega(S^{2m-1})_{\Theta''=0} \otimes S(\mathcal{G}^{''*})$$

induced by $M = C^m - \{0\} \to S^{2m-1}$ and $G^{''} \subset G$.

If we take a generator $v^{''} \in H^*(\Omega(S^{2m-1})_{\Theta''=0}) \cong H^*(S^{2m-1})$ and apply Husseini's Lemma, we will see that $d^{''}_m v^{''} = C\lambda^{'}_1 \lambda^{'}_2 ... \lambda^{'}_m.$, $C \neq 0$. Since $v^{''}$ may be choosen as the image of v, where $v \in H^{2m-1}(\Omega(M)_{\Theta=0})$, which denotes a generator corresponding to S^{2m-1} factor, then we have $d_m v = C\lambda^{'}_1 \lambda^{'}_2 ... \lambda^{'}_m$. □

Now assume $G \cong R^{2n}$, and $M = C^m - \{0\} = (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\}$.

Let $\varphi : G \longrightarrow Gl_m(C)$, where $m = m_1 + \cdots + m_n$ be a homomorphism. Also assume that *im* $\varphi \subset$ diagonal matrices and closure$(\varphi(G)) = R_1 \oplus iR_1 \oplus \cdots \oplus R_n \oplus iR_n$.

Corollary 1. *Let $G \cong R^{2n}$ and $M = C^m - \{0\}$, with exponential G action given as above. Then $Index_{\mathcal{G}}(M) =< s_1 + \cdots + s_n, \lambda^{'}_1 \lambda^{'}_2 ... \lambda^{'}_m >$.*

Now consider $G = H_{2n+1}$, $(2n+1)$ dimensional nilpotent Heisenberg Lie group of real matrices and let

$$\psi : G \longrightarrow G/[G,G] \cong R^{2n}$$

and $M = C^m - \{0\} = (C^{m_1} \oplus \cdots \oplus C^{m_n}) - \{0\}$, and G acts on M via ψ.

$$M \times G \longrightarrow M$$

$$((\vec{z_1}, \ldots, \vec{z_n}), g) \to e^{(\sum_{j=1}^n x_j)}(\varphi_1(\vec{z_1}, iy_1, \vec{\lambda}_1), \ldots, \varphi_n(\vec{z_n}, iy_n, \vec{\lambda}_n))$$

Proposition 3. *Let G be a $(2n+1)$-dimensional nilpotent Heisenberg Lie group and $A = G/[G,G]$ be its abelinization and $M = C^m - \{0\}$, the complex m-space. If the G-action $M \times G \to M$ is defined as above, then*

$$H^*_{\mathcal{G}}(M) \cong H^*_{\mathcal{A}}(M).$$

Proof. Let $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,Z\}$, and $\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\gamma\}$ denote the generators of $\mathcal{G}$, and $\mathcal{G}^*$ respectively, then $\mathcal{A}$ and $\mathcal{A}^*$ are generated by $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ and $\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\}$ respectively. Here $S\mathcal{A}^*$ is generated by

$$\{h(\alpha_1),\ldots,h(\alpha_n),h(\beta_1),\ldots,h(\beta_n)\}=\{s_1,\ldots,s_n,t_1,\ldots,t_n\}$$

also $S\mathcal{G}^*$ is generated by $\{s_1,\ldots,s_n,t_1,\ldots,t_n,r\}$ where $r=h(\gamma)$.

We want to show that $\Omega_{\mathcal{G}}(M)\cong\Omega_{\mathcal{A}}(M)$ where

$$\Omega_{\mathcal{G}}(M)=(\Omega(M)\otimes S(\mathcal{G}^*))_{\Theta_{\mathcal{G}}=0}$$

and

$$\Omega_{\mathcal{A}}(M)=(\Omega(M)\otimes S(\mathcal{A}^*))_{\Theta_{\mathcal{A}}=0}.$$

We need to check that Θ_X, for $X\in$ {Center of $\mathcal{G}$}. Since the center of $\mathcal{G}$ is generated by Z, $\Theta_Z=0$ on $\Omega(M)$ and also $S(\mathcal{G}^*)_{\Theta_Z=0}=S(\mathcal{A}^*)$, then

$$(\Omega(M)\otimes S(\mathcal{G}^*))_{\Theta_Z=0}=(\Omega(M)\otimes S(\mathcal{A}^*)).$$

Since

$$(\Omega(M)\otimes S(\mathcal{G}^*))_{\Theta_{\mathcal{G}}=0}=(\Omega(M)\otimes S(\mathcal{G}^*))_{\Theta_Z=0,\Theta_{\mathcal{A}}=0}$$

then

$$(\Omega(M)\otimes S(\mathcal{G}^*))_{\Theta_{\mathcal{G}}=0}=(\Omega(M)\otimes S(\mathcal{A}^*))_{\Theta_{\mathcal{A}}=0}.$$

This gives us

$$\Omega_{\mathcal{G}}(M)\cong\Omega_{\mathcal{A}}(M)$$

and then

$$H^*_{\mathcal{G}}(M)\cong H^*_{\mathcal{A}}(M).$$

□

Proposition 4. *Let $G=H_{2n+1}$ acts on $M=(C^{m_1}\oplus\cdots\oplus C^{m_n})-\{0\}$ with an exponential action with parameters $\{\lambda_1,\ \lambda_2,\ldots,\ \lambda_m\}$. Then,*

$$Index_{\mathcal{G}}(M) =< s_1 + \cdots + s_n, \lambda_1^{'}\lambda_2^{'}...\lambda_m^{'} > .$$

Proof. By Proposition.1. and Proposition.3. $B\mathcal{G}{\cong}B\mathcal{A}$ and $H_{\mathcal{G}}(M){\cong}H_{\mathcal{A}}(M)$. Then,

$$Index_{\mathcal{A}}(M) = Index_{\mathcal{G}}(M).$$

The Proposition follows since $Index_{\mathcal{A}}(M) =< s_1 + \cdots + s_n, \lambda_1^{'}\lambda_2^{'}...\lambda_m^{'} >$ from Proposition.3. □

We may now give the following Borsuk-Ulam type theorem.

Theorem 2. *Let $G = H_{2n+1}$ acts on $M = C^p - \{0\} = (C^{p_1}\oplus\cdots\oplus C^{p_n}) - \{0\}$ and $N = C^q - \{0\} = (C^{q_1}\oplus\cdots\oplus C^{q_n}) - \{0\}$ with an exponential actions with parameters $\{\lambda_1,\ \lambda_2, \ldots,\ \lambda_p\}$ and $\{\mu_1,\ \mu_2,\ \ldots, \mu_q\}$ respectively. If $f : M{\rightarrow}N$ is a G-equivariant map, then $p{\leq}q$.*

Proof. Proposition.4. gives that

$$Index_{\mathcal{G}}(M) =< s_1 + \cdots + s_n, \lambda_1^{'}\lambda_2^{'}...\lambda_p^{'} >$$

and

$$Index_{\mathcal{G}}(N) =< s_1 + \cdots + s_n, \mu_1^{'}\mu_2^{'}...\mu_q^{'} > .$$

Where $\lambda_1^{'}\lambda_2^{'}...\lambda_p^{'}$ and $\mu_1^{'}\mu_2^{'}...\mu_q^{'}$ are polynomials of $t_1, \ldots t_n$ with degrees p and q respectively. By monotonicity of $Index_{\mathcal{G}}$, if $f : M{\rightarrow}N$ is a G-equivariant map, then

$$Index_{\mathcal{G}}(M){\supset}Index_{\mathcal{G}}(N).$$

This implies that the degree of $\lambda_1^{'}\lambda_2^{'}...\lambda_p^{'}$ is smaller than or equal to the degree of $\mu_1^{'}\mu_2^{'}...\mu_q^{'}$. Thus $p{\leq}q$. □

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