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Some Radius Problem for Certain Families of Analytic Functions

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Abstract

The aim of this paper is to give bounds of the radius of α-convexity for certain families of analytic functions in the unit disc. The radius of α-convexity is generalization of the radius of convexity and the radius of starlikeness, and introduced by S.S.Miller; P.T.Mocanu and M.O.Reade [3,4]

Key Words: Subordination principle, Carathedory functions, Janowski Starlike functions, Starlike functions of order β, The radius of Starlikeness, The radius of convexity, The radius of α-convexity.

1. Introduction

Most radius problems lead to functions p(z) with positive real part, or some more restrictive condition Re $p(z)$. Therefore, for our study we shall need the following definitions and the subordination principle.

Subordination Principle: Let $g(z)$ and $f(z)$ be regular and analytic in $D = \{z \Big| \mid z \mid < 1\}$, and let $f(z)$ be univalent there. Let further D_1 and D_2 denote the domains onto which the unit circle is mapped by $w = g(z)$ and $w = f(z)$ respectively. If $f(0) = g(0)$ and D_1 is contained in D_2 then

$$g(z) = f(w(z)), \tag{1.1}$$

where $w(z)$ is regular in D and

$$|w(z)| \leq |z|. \tag{1.2}$$

The sign of equality in (1.2) is possible only if the domains D_1 and D_2 coincide. If the functions $f(z)$ and $g(z)$ are related by (1.1) we say that $g(z)$ is subordinate to $f(z)$ and we write

$$g(z) \prec f(z). \tag{1.3}$$

The Class of Caratheodory Functions

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular and analytic in D and satisfies the condition $p(0) = 1$, Re $p(z) > 0$, then this function is Caratheodory functions. The class of these functions is denoted by P. If we use the subordination principle we have

$$p(z) \in p \quad \textbf{if and only if} \quad p(z) \prec \frac{1+z}{1-z}. \tag{1.4}$$

The Class of Janowski Functions

Let $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be regular and analytic in D and satisfies the condition

$$p(0) = 1, \text{Re} \quad p(z) > 0, \quad p(z) \prec \frac{1+Az}{1-Bz}, \quad -1 < A < 1, \quad -1 \leq B < A \tag{1.5}$$

then this functions is called a Janowski function. The class of this functions is denoted by $P(A, B)$

Geometrically, $p(z)$ is in $P(A, B)$ if and only if $p(0) = 1$ and $p(D)$ inside the open disc centred on the real axis with diameter end points

$$p(-1) = \frac{1-A}{1-B} \quad \textbf{and} \quad p(1) = \frac{1+A}{1+B}.$$

Special selections of A and B lead to familiar sets defined by inequalities under the condition, $p(0) = 1$, $M > \frac{1}{2}$, $0 \leq \beta < 1$, we have

1) $p(-1, 1) = p$ is the set defined by Re $p(z) > 0$ (Caratheodory's Class)

2) $p(1 - 2\beta, -1) = p(\beta)$ is the set defined by Re $p(z) > \beta$

3) $p(1, 0) = p(1)$ is the set defined by $|p(z) - 1| < 1$

4) $p(\beta,0)=p_*(\beta)$ is the set defined by $|p(z)-1|<\beta$

5) $p(1,\frac{1}{M}-1)=p(M)$ is the set defined by $|p(z)-M|<M$

6) $p(\beta,-\beta)=p_{**}(\beta)$ is the set defined by $\left|\frac{p(z)-1}{p(z)+1}\right|<\beta$

The Class of Janowski's Starlike Functions

Let $S^*(A,B)$ be the class of functions $f(z)$, $f(0)=0$, $f'(0)=1$ regular in D and satisfying the condition.

$$f(z)\in S^*(A,B) \quad \textbf{if and only if} \quad z.\frac{f'(z)}{f(z)}\in p(A,B). \tag{1.6}$$

Special selections of A and B lead to familiar sets defined by the inequalities under the condition $M>\frac{1}{2}$, $0\leq\beta<1$. We have

1) $S^*(1,-1)=S^*$ is the class of starlike functions with respect to the origin

2) $S^*(1-2\beta,-1)=S^*(\beta)$ is the class of starlike functions of order β

3) $S^*(1,-0)=S^*(1)$ is the class defined by $\left|z\frac{f'(z)}{f(z)}-1\right|<1,$

4) $S^*(\beta,-0)=S^*_*(\beta)$ is the class defined by $\left|z\frac{f'(z)}{f(z)}-1\right|<\beta$, $0\leq\beta<1$

5) $S^*(1,\frac{1}{M}-1)=S^*(M)$ is the class defined by $\left|z\frac{f'(z)}{f(z)}-M\right|<M$, $M>\frac{1}{2}$

6) $S^*(\beta,-\beta)=S^*_{**}(\beta)$ is the class defined by $\left|\frac{z\frac{f'(z)}{f(z)}-1}{z\frac{f'(z)}{f(z)}+1}\right|<\beta.$

2. The Radius of α-Convexity for the Class $\mathbf{S^*(A,B)}$

In this section we shall give the radius of α-convexity for the class $S^*(A,B)$.

Lemma 2.1. *Let $p_1(z) \in p(A,B)$, then*

$$p_1(z) = \frac{(1+A) \quad p(z) + (1-A)}{(1+B) \quad p(z) + (1-B)} \tag{2.1}$$

for some $p(z) \in p$, and conversly. This lemma was proved by Janowski [1].

Lemma 2.2. *Let $p_1(z) \in p(A,B)$, then*

$$Re\ p_1(z) \geq \frac{1-Ar}{1-Br} \tag{2.2}$$

This lemma was proved by Janowski [1].

Lemma 2.3. *Let $p(z) \in p$, then*

$$\operatorname{Re}\left(\frac{z\,p^{'}(z)}{p(z)+\frac{1-A}{1+A}}\right) \geq \frac{-(1+A)r}{(1-r)(1+Ar)} \tag{2.3}$$

Proof. Let $p(z) \in p$, then

$$\operatorname{Re}\left(\frac{zp^{'}(z)}{p(z)+\mu}\right) \geq \frac{-2r}{(1-r)[(1+\gamma)+(1-\gamma)r]}, \tag{2.4}$$

where Re $\mu = \gamma > 0$. The inequality (2.4) was proved by S.D.Bernardi [5]. On the other hand

$$\left\{\begin{array}{l} \left.-1 < A \leq +1 \Longrightarrow 1-A > 0\,,\ 1+A > 0\right\} \Longrightarrow \dfrac{1-A}{1+A} > 0 \Longrightarrow \mu = \dfrac{1-A}{1+A} > 0 \Longrightarrow \\ \operatorname{Re}\mu = \operatorname{Re}\left(\dfrac{1-A}{1+A}\right) = \dfrac{1-A}{1+A} > 0 \end{array}\right. \tag{2.5}$$

From the relation (2.4) and (2.5) we have the inequality (2.3). This shows that the lemma is true. □

Theorem 2.1. *The radius of α-convexity of the class of $S^*(A,B)$ is the unique root of polynomial*

$$R(A,B,\alpha,r) = (1-r)(1-Ar)(1+Ar)(1+Br) - \alpha r(1-Br)\Big[(1+A)(1+Br)+(1+B)(1+Ar)\Big]$$

in the interval (0, 1].

Proof. Let $f(z) \in S^*(A,B)$. From the definitions the classes $S^*(A,B)$, $p(A,B)$, and Lemma 2.1. we write

$$z.\frac{f'(z)}{f(z)} = p_1(z) = \frac{(1+A) \quad p(z) + (1-A)}{(1+B) \quad p(z) + (1-B)}, \tag{2.6}$$

where $p_1(z) \in p(A,B)$, $\quad p(z) \in p$. □

If we take the logarithmic derivative from the equality (2.6) we obtain

$$1 + z.\frac{f''(z)}{f'(z)} - z.\frac{f'(z)}{f(z)} = \frac{zp'(z)}{p(z) + \frac{1-A}{1+A}} + \frac{zp'(z)}{p(z) + \frac{1-B}{1+B}}.$$

Therefore, we have

$$\text{Re}\left[\left(1 + z.\frac{f''(z)}{f'(z)}\right) - z.\frac{f'(z)}{f(z)}\right] = \text{Re}\left[\frac{zp'(z)}{p(z) + \frac{1-A}{1+A}}\right] + \text{Re}\left[\frac{zp'(z)}{p(z) + \frac{1-B}{1+B}}\right]. \tag{2.7}$$

If we consider the result of lemma 2.3. and the relation (2.7) we obtain

$$\text{Re}\left[\left(1 + z.\frac{f''(z)}{f'(z)}\right) - z.\frac{f'(z)}{f(z)}\right] \geq \frac{-r[(1+A)(1+Br)+(1+B)(1+Ar)]}{(1-r)(1+Br)(1+Ar)} \tag{2.8}$$

On the other hand from the Lemma 2.2. and the equality (2.6) we have

$$\text{Re} \quad \frac{f'(z)}{f(z)} \geq \frac{1-Ar}{1-Br} \tag{2.9}$$

If we multiply both sides inequality (2.8) by $\alpha \quad (0 \leq \alpha \leq 1)$ we get

$$\mathrm{Re}\left[\alpha\left(1+z.\frac{f''(z)}{f'(z)}\right)-\alpha\left(z.\frac{f'(z)}{f(z)}\right)\right] \geq \frac{-\alpha r[(1+A)(1+Br)+(1+B)(1+Ar)]}{(1-r)(1+Br)(1+Ar)} \tag{2.10}$$

Summing the inequalities (2.9) and (2.10) we obtain that.

$$\begin{cases} \mathrm{Re}[J(A,B,\alpha,f(z))] = \mathrm{Re}\left[(1-\alpha)z\frac{f'(z)}{f(z)}+\alpha\left(1+z\frac{f''(z)}{f'(z)}\right)\right] \\ \geq \frac{(1-r)(1-Ar)(1+Ar)(1+Br)-\alpha r(1-Br)[(1+A)(1+Br)+(1+B)(1+Ar)]}{(1-r)(1+Ar)(1+Br)(1-Br)} \end{cases} \tag{2.11}$$

The inequality (2.11) shows that the theorem is true.

If we give special values to A and B we obtain the radius of α-convexity, the radius of convexity and the radius of starlikeness for the classes. $S^*(1,-1), S^*(1-2\beta,-1), S^*(1,0), S^*(\beta,0), S^*(\beta,-1)$.

(i) **For** $A=1, B=-1$ we obtain

$$\mathrm{Re}\ [J(1,-1,\alpha,f(z))] \geq \frac{(1-r)^2-2\alpha r}{1-r^2}$$

Then

$$r=(1+\alpha)-\sqrt{(1+\alpha)^2-1}$$

This is the radius of α-convexity for the class of starlike functions. This radius was obtained by S.S.Miller;P.T.Mocanu and M.O.Reade [3].

In this case

For $\alpha=1$ then we obtain $r=2-\sqrt{3}$ is the radius of convexity for the class of starlike functions. This result is well known.

(ii) **For** $A=1, B=0$ we obtain

$$\mathrm{Re}\ [J(1,0,\alpha,f(z))] \geq \frac{r^3-(\alpha+1)r^2-(3\alpha+1)r+1}{1-r^2}$$

Now we consider the polynomial.

$$g(r) = r^3 - (\alpha+1)r^2 - (3\alpha+1)r + 1$$

$g(0) = 1 > 0$, $g(1) = -4\alpha < 0$. Thus the smallest positive root r_0 of the equation $g(r) = 0$ lies between 0 and 1. Thus the inequality

$$\text{Re}\left[J(1,0,\alpha,f(z))\right] > 0$$

is valid for $|z| = r < r_0$. Hence the radius of α-convexity for $S^*(1,0)$ is not less than r_0.

On the other hand if we take $\alpha = 0$ in this case we obtain

$$g(r) = r^3-(\alpha+1)r^2-(3\alpha+1)r+1 \Longrightarrow g_1(r) = r^3-r^2-r+1 = (r-1)^2(r+1) \Longrightarrow r = 1$$

This shows that the radius of starlikeness for the class $S^*(1,0)$ is $r = 1$.

Similarly in this case for $\alpha = 1$. The polynomial $g(r)$ reduces to $g_2(r) = r^3 - 2r^2 - 4r + 1$. The polynomial $g_2(r)$ satisfies the condition $g_2(0) = 1 > 0$ and $g_2(1) = -4 < 0$. Therefore the equation $g_2(r) = 0$ has a positive real root r_{01} in the interval (0, 1], this root is smallest of the roots. Thus the inequality

$$\text{Re}\left[J(1,0,1,f(z))\right] > 0$$

is valid for $|z| = r < r_{01}$. Hence the radius of convexity for $S^*(1,0)$ is not less than r_{01} is obtained as follows

$$0 = r^3 - 2r^2 - 4r + 1 \equiv r^3 + br^2 + cr + d \Longrightarrow b = -2, c = -4, d = 1$$

$$p = c - \frac{b^2}{3} = -\frac{16}{3}, q = d - \frac{1}{3}bc + \frac{2}{27}b^3 = -\frac{61}{27}, \Delta = -4p^3 - 27q^2 = 12668 > 0$$

. Therefore all root of this equation is real and are distinct. On the other hand

$$\eta = \sqrt{\frac{-3}{4p}} = \frac{3}{8}, Cos3\theta = \frac{61}{28}, \quad \theta = \frac{1}{3}ArcCos\frac{61}{28}$$

$$r_1 = \frac{8}{3}Cos\left(\frac{1}{3}ArcCos\frac{61}{128}\right)$$

$$r_2 = \frac{8}{3}Cos\left(\frac{1}{3}ArcCos\frac{61}{128} + \frac{2\pi}{3}\right)$$

$$r_3 = \frac{8}{3}Cos\left(\frac{1}{3}ArcCos\frac{61}{128} + \frac{4\pi}{3}\right)$$

(iii) **For** $A = 1 - 2\beta, \quad B = 1$

$$\text{Re }[J(1-2\alpha, -1, \alpha, f(z))] \geq \frac{(1-2\beta)r^3 - \left[(1-2\beta)^2 + 2\alpha - 2\alpha\beta\right]r^2 - (2\alpha + 2\alpha\beta + 1)r + 1}{(1-r)(1+(1-2\beta)r)(1+r)}$$

Therefore the polynomial

$$g_3(r) = (1-2\beta)r^3 - \left[(1-2\beta)^2 + 2\alpha - 2\alpha\beta\right]r^2 - (2\alpha + 2\alpha\beta + 1)r + 1$$

satisfies the condition $g_3(0) = 1 > 0, \quad g_3(1) = -2\left(2\beta^2 - \alpha\beta + 2\alpha\right) < 0$. Thus the smallest positive real root r_{02} of the equation $g_3(r) = 0$ lies between 0 and 1. Thus the inequality

$$\text{Re }[J(1-2\alpha, -1, \alpha, f(z))] < 0$$

is valid for $|z| = r < r_{02}$. Hence the radius of α-convexity for $S^*(1-2\beta, -1)$ is not less than r_{02}

In this case.

For $\alpha = \mathbf{0}$;

$$\text{Re }[J(1-2\beta, -1, 0, f(z))] \geq \frac{(1-2\beta)r^3 - (1-2\beta)^2 r^2 - (1+2\beta)r + 1}{(1-r)(1+(1-2\beta)r)(1+r)}$$

Thus the polynomial $g_4(r) = (1-2\beta)r^3 - (1-2\beta)r^2 - (1+2\beta)r + 1$ satisfies the condition $g_4(0) = 1 > 0, g_4(1) = -4\beta^2 < 0$. Thus the smallest positive real root r_{03} of the equation $g_4(r) = 0$ lies between 0 and 1. Thus the inequality

$$\text{Re }[J(1-2\beta, -1, 0, f(z))] > 0$$

is valid for $|z| = r < r_{03}$. Hence the radius of starlikeness for $S^*(1-2\beta, 1)$ is not less than r_{03}

For $\alpha = \mathbf{1}$;

$$\text{Re }[J(1-2\beta,-1,1,f(z))] \geq \frac{(1-2\beta)r^3 - ((1-2\beta)^2 - 2\beta - 2)r^2 - (3+2\beta)r + 1}{(1-r)(1+(1-2\beta)r)(1+r)}$$

Thus the polynomial $g_5(r) = (1-2\beta)r^3 - \left((1-2\beta)^2 - 2\beta - 2\right)r^2 - (3+2\beta)r + 1$ satisfies the condition $g_5(0) = 1 > 0,\ g_5(1) = -4\left(\beta^2 + 1\right) < 0$. Thus the positive smallest r_{04} of the equation $g_5(r) = 0$ lies between 0 and 1. Thus the inequality

$$\text{Re }[J(1-2\beta,-1,1,f(z))] > 0$$

is valid for $|z| = r < r_{04}$ .Hence the radius of convexity for $S^*(1-2\beta,-1)$ is not less than r_{04}

For $\beta = \frac{1}{2}$;

$$\text{Re }[J(\frac{1}{2},-1,1,f(z))] \geq \frac{\alpha r^2 - 2(\alpha+1)+1}{(1-r^2)} \Longrightarrow r = (1+\alpha) - \sqrt{(1+\alpha)^2 - \alpha}$$

is the radius of α-convexity for the class starlike function of order $\frac{1}{2}$.

(iv) **For** $A = \beta, B = -\beta$;

$$\text{Re }[J(\beta,-\beta,\alpha,f(z))] \geq \frac{(\beta - \alpha\beta)r^2 - (2\alpha+\beta+1)r + 1}{1-\beta r}$$

Thus the radius of α-convexity for the class $S^*(\beta,-\beta)$ is

$$r = \frac{(2\alpha+\beta+1) - \sqrt{(2\alpha+\beta+1)^2 - 4\beta(1-2\alpha)}}{\beta(1-2\alpha)}$$

In this case.

For $\alpha = 0; r = \frac{2}{\beta}$ is the radius of starlikeness for $S^*(\beta,-\beta)$

For $\alpha = 1; r = \frac{\sqrt{\beta^2+10\beta+9-(3+\beta)}}{\beta}$ is the radius of convexity for the class $S^*(\beta,-\beta)$.

(v) **For** $A = \beta, B = 0$**;**

$$\text{Re }[J(\beta,0,\alpha,f(z))] \geq \frac{\beta^2 r^3(\beta^2 + \alpha\beta)r^2 - (\beta\alpha + 2\alpha + 1)r + 1}{(1-r)(1+\beta r)}$$

Now we consider the polynomial

$g_6(r) = \beta^2 r^3 - \Big(\beta^2 - \alpha\Big) r^2 - (1+\alpha+\alpha\beta) r + 1, \quad g_6(0) = 1 > 0, \quad g_6(1) = -\alpha < 0$. Thus the smallest positive root r_{05} of the equation $g_6(r) = 0$ lies between 0 and 1.

Thus the inequality

$$\text{Re}\,[J(\beta, 0, \alpha, f(z))] > 0$$

is valid for $|z| = r < r_{05}$. Hence the radius of α-convexity for $S^*(\beta, 0)$ is not less than r_{05}

In this case.

For $\alpha = 0$;

$$\text{Re}\,[J(\beta, 0, 0, f(z))] \geq \frac{\beta^2 r^3 - \beta^2 r^2 - r + 1}{(1-r)(1+\beta r)}$$

Thus $\beta^2 r^3 - \beta^2 r^2 - r + 1 = \Big(\beta^2 r^2 - 1\Big)(r-1) \Longrightarrow r = \frac{1}{\beta}$ is the radius of starlikeness. For the class $S^*(\beta, 0)$

For $\alpha = 1$;

$$\text{Re}\,[J(\beta, 0, 1, f(z))] \geq \frac{\beta^2 r^3 - (\beta^2 + \beta) r^2 - (\beta + 3) r + 1}{(1-r)(1+\beta r)}$$

On the other hand the polynomial $g_7(r) = \beta^2 r^3 - (\beta^2 - \beta) r^2 - (2+\beta) r + 1$ satisfies the condition $g_7(0) = 1 > 0$, $g_7(1) = -2(\beta + 1) < 0$. Thus the positive smallest root r_{06} of the equation $g_7(r) = 0$ lies between 0 and 1.

Thus the inequality

$$\text{Re}\,[J(\beta, 0, 1, f(z))] > 0$$

is valid for $|z| = r < r_{06}$. Thus the radius of convexity for $S^*(\beta, 0)$ is not less than r_{06}.

(vi) **For** $A = 1, B = 1 - \frac{1}{M}$ $\Big(M = 1 - \frac{1}{M}\Big)$;

$$\text{Re}\left[J\Big(1, (1 - \frac{1}{M}), \alpha, f(z)\Big)\right] \geq$$

$$\frac{(1-r)^2(1+r)(1+Mr) - \alpha r(1-Mr)[2(1+Mr) + (1+M)(1+r)]}{(1-r^2)(1-M^2r^2)}$$

Now we consider the polynomial

$$g_8(r) = (1-r)^2(1+r)(1+Mr) - \alpha r(1-Mr)[2(1+Mr) + (1+M)(1+r)]$$

$g_8(0) = 1,\ g_8(1) = -4\alpha(1-M)(1+M) < 0$. Thus the positive smallest root r_{07} of the equation $g_8(r) = 0$ lies between 0 and 1. Thus the inequality.

$$\text{Re}\ [J(1, M, \alpha, f(z))] > 0$$

is valid for $|z| = r < r_{07}$. Thus the radius of α -convexity for $S^*\left(1, (1 - \frac{1}{M})\right)$ is not less than r_{07}.

In this case.

For $\alpha = 0$ **;**

$$\text{Re}\ [J(1, M, 0, f(z))] \geq \frac{1-r}{1+Mr}$$

This shows that the radius of starlikeness for the class $S^*(1, M)$ is $r = 1$. This radius was obtained by Janowski [7].

For $\alpha = 1$**;**

$$Re[J(1, M, 1, f(z))] \geq \frac{(1-r)^2(1+r)(1+Mr) - r(1-Mr)[2(1+Mr) + (1+M)(1+r)]}{(1-r^2)(1-M^2r^2)}$$

Thus the polynomial

$$g_9(r) = (1-r)^2(1+r)(1+Mr) - r(1-Mr)[2(1+Mr) + (1+M)(1+r)]$$

$g_9(0) = 1 > 0,\ g_9(1) = -4(1-M)(1+M) < 0$.Thus the positive smallest root r_{08} of the equation $g_9(r) = 0$ lies between 0 and 1. Thus the inequality.

$$Re[J(1, M, 1, f(z))] > 0$$

is valid for $|z| = r < r_{08}$. Thus the radius of convexity for $S^*(1, M)$ is not less than r_{08}.

References

[1] Hassan. S. Al. Amiri, On the radius of β- Convexity of starlike functions of order α. *Proo. Amer. Math. Soc.* **39** (1973), 101-109.

[2] S. K. BaJpai: On regions of α-Convexity for starlike functions.

[3] S. S. Miller, P. T. Mocanu and M. O. Reade Bozilevic functions and generalized *Convexity. Rev. Roum de Math. Pures at Appl.* **16,** (1971), 1541-1544.

[4] S. S. Miller, P. T. Mocanu and M. O. Reade. On generalized Convexity in Conformal mapping II. Rev. Roum. Math. Pures. Et Appl. Tome XXI. No. 2 219-225 (1976).

[5] S. D. Bernardi. New distorsion theorems for functions of positive real parts and appli- cation to the partial sum of univalent convex functions. *Proc. of Amer. Math. Soc.* **34,** (1974), 113-118.

[6] W. Janowski. Extremal problems for a family of functions with positive real part and some related families. *Ann. Polon. Math.* **23**, (1970) 159-177.

[7] W. Janowski. Some extremal problems for certain families of analytic functions. *Ann. Polon. Math.* **28**, (1973). 297-326.

[8] Y. Polatoğlu. The radius of α-convexity for the class of λ- spirallike functions. *Revue. Roum. Math. Pures et Appl.* **34.** (1989) 263-267.

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