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Symplectic maps to projective spaces and symplectic invariants

Denis Auroux

1. Introduction

Let (X^{2n},ω) be a compact symplectic manifold. We will throughout this text assume that the cohomology class $\frac{1}{2\pi}[\omega]\in H^2(X,\mathbb{R})$ is integral. This assumption makes it possible to define a complex line bundle L over X such that $c_1(L)=\frac{1}{2\pi}[\omega]$. We also endow X with a compatible almost-complex structure J, and endow L with a Hermitian metric and a Hermitian connection of curvature $-i\omega$.

The line bundle L should be thought of as a symplectic version of an ample line bundle over a complex manifold. Indeed, although the lack of integrability of J prevents the existence of holomorphic sections, it was observed by Donaldson in [8] that, for large k, the line bundles $L^{\otimes k}$ admit many approximately holomorphic sections.

Observe that all results actually apply as well to the case where $\frac{1}{2\pi}[\omega]$ is not integral, with the only difference that the choice of the line bundle L is less natural : the idea is to perturb ω into a symplectic form ω' whose cohomology class is rational, and then work with a suitable multiple of ω'. One chooses an almost-complex structure J' which simultaneously is compatible with ω' and satisfies the positivity property $\omega(v,J'v)>0$ for all tangent vectors. All the objects that we construct are then approximately J'-holomorphic, and therefore symplectic with respect to not only ω' but also ω.

Donaldson was the first to show in [8] that, among the many approximately holomorphic sections of $L^{\otimes k}$ for $k\gg 0$, there is enough flexibility in order to obtain nice transversality properties ; this makes it possible to imitate various classical topological constructions from complex algebraic geometry in the symplectic category. Let us mention in particular the construction of smooth symplectic submanifolds ([8], see also [2] and [15]), symplectic Lefschetz pencils ([10], see also [9]), branched covering maps to $\mathbb{CP}^2$ ([3],[5]), Grassmannian embeddings and determinantal submanifolds ([15]).

Intuitively, the main reason why the approximately holomorphic framework is suitable to imitate results from algebraic geometry is that, for large values of k, the increasing curvature of $L^{\otimes k}$ provides access to the geometry of X at very small scale ; as one zooms into X, the geometry becomes closer and closer to a standard complex model, and the lack of integrability of J becomes negligible.

The introduction of approximately holomorphic sections was motivated in the first place by the observation that, if suitable transversality properties are satisfied, then every geometric object that can be defined from these sections automatically becomes symplectic. Therefore, in order to perform a given construction using such sections, the strategy is always more or less the same : starting with a sequence of approximately holomorphic sections of $L^{\otimes k}$ for all $k \gg 0$, the goal is to perturb them in order to ensure uniform transversality properties that will guarantee the desired topological features.

For example, the required step in order to construct symplectic submanifolds is to obtain bounds of the type $|\nabla s_k|_{g_k} > \eta$ along the zero set of s_k for a fixed constant $\eta > 0$ independent of k, while approximate holomorphicity implies a bound of the type $|\bar\partial s_k|_{g_k} = O(k^{-1/2})$ everywhere (see §3.1). Here $g_k = kg$ is a rescaled metric which dilates everything by a factor of $k^{1/2}$ in order to adapt to the decreasing "characteristic scale" imposed by the increasing curvature $-ik\omega$ of the line bundles $L^{\otimes k}$. The desired topological picture, similar to the complex algebraic case, emerges for large k as an inequality of the form $|\bar\partial s_k| \ll |\partial s_k|$ becomes satisfied at every point of the zero set : this can easily be shown to imply that the zero set of s_k is smooth, approximately pseudo-holomorphic, and symplectic. Indeed, the surjectivity of ∇s_k implies the smoothness of the zero set, while the fact that $|\bar\partial s_k| \ll |\partial s_k|$ implies that the tangent space to the zero set, given by the kernel of $\nabla s_k = \partial s_k + \bar\partial s_k$, is very close to the complex subspace $\mathrm{Ker}(\partial s_k)$, hence its symplecticity (see also [8]).

The starting points for the construction, in all cases, are the existence of very localized approximately holomorphic sections of $L^{\otimes k}$ concentrated near any given point $x \in X$, and an effective transversality result for approximately holomorphic functions defined over a ball in $\mathbb{C}^n$ with values in $\mathbb{C}^r$ due to Donaldson (see [8] for the case $r = 1$ and [10] for the general case). These two ingredients imply that a small localized perturbation can be used to ensure uniform transversality over a small ball. Combining this local result with a globalization argument ([8], see also [3] and [15]), one obtains transversality everywhere.

The interpretation of the construction of submanifolds as an effective transversality result for sections extends verbatim to the more sophisticated constructions (Lefschetz pencils, branched coverings) : in these cases the transversality properties also concern the covariant derivatives of the sections, and this can be thought of as an effective analogue in the approximately holomorphic category of the standard generalized transversality theorem for jets.

This is especially clear when looking at the arguments in [15], [10] or [3] : the perturbative argument is now used to obtain uniform transversality of the holomorphic parts of the 1-jets or 2-jets of the sections with respect to certain closed submanifolds in the space of holomorphic jets. Successive perturbations are used to obtain transversality to the various strata describing the possible singular models ; one uses that each stratum is smooth away from lower dimensional strata, and that transversality to these lower dimensional strata is enough to imply transversality to the higher dimensional stratum near its singularities.

An extra step is necessary in the constructions : recall that desired topological properties only hold when the antiholomorphic parts of the derivatives are much smaller than the holomorphic parts. In spite of approximate holomorphicity, this can be a problem when the holomorphic part of the jet becomes singular. Therefore, a small perturbation is needed to kill the antiholomorphic part of the jet near the singularities ; this perturbation is in practice easy to construct. The reader is referred to [10] and [3] for details.

Although no general statement has yet been formulated and proved, it is completely clear that a very general result of uniform transversality for jets holds in the approximately holomorphic category. Therefore, the observed phenomenon for Lefschetz pencils and maps to $\mathbb{CP}^2$, namely the fact that near every point $x \in X$ the constructed maps are given in approximately holomorphic coordinates by one of the standard local models for generic holomorphic maps, should hold in all generality, independently of the dimensions of the source and target spaces. This approach will be developed in a forthcoming paper [4].

In the remainder of this paper we focus on the topological monodromy invariants that can be derived from the various available constructions. In Section 2 we study symplectic Lefschetz pencils and their monodromy, following the results of Donaldson [10] and Seidel [16]. In Section 3 we describe symplectic branched covers of $\mathbb{CP}^2$ and their monodromy invariants, following [3] and [5] ; we also discuss the connection with 4-dimensional Lefschetz pencils. In Section 4 we extend this framework to the higher dimensional case, and investigate a new type of monodromy invariants arising from symplectic maps to $\mathbb{CP}^2$. We finally show in Section 5 that a dimensional induction process makes it possible to describe a compact symplectic manifold of any dimension by a series of words in braid groups and a word in a symmetric group.

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2. Symplectic Lefschetz pencils

Let (X^{2n}, ω) be a compact symplectic manifold as above, and let s_0, s_1 be suitably chosen approximately holomorphic sections of $L^{\otimes k}$. Then X is endowed with a structure of *symplectic Lefschetz pencil*, which can be described as follows.

For any $\alpha \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, define $\Sigma_\alpha = \{x \in X,\ s_0 + \alpha s_1 = 0\}$. Then the submanifolds Σ_α are symplectic hypersurfaces, smooth except for finitely many values of the parameter α ; for these parameter values Σ_α contains a singular point (a normal crossing when $\dim X = 4$). Moreover, the submanifolds Σ_α fill all of X, and they intersect transversely along a codimension 4 symplectic submanifold $Z = \{x \in X,\ s_0 = s_1 = 0\}$, called the set of *base points* of the pencil.

Define the projective map $f = (s_0 : s_1) : X - Z \to \mathbb{CP}^1$, whose level sets are precisely the hypersurfaces Σ_α. Then f is required to be a *complex Morse function*, i.e. its critical points are isolated and non-degenerate, with local model $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$ in approximately holomorphic coordinates.

The following result due to Donaldson holds :

Theorem 2.1 (Donaldson [10]). *For $k \gg 0$, two suitably chosen approximately holomorphic sections of $L^{\otimes k}$ endow X with a structure of symplectic Lefschetz pencil, canonical up to isotopy.*

This result is proved by obtaining uniform transversality with respect to the strata $s_0 = s_1 = 0$ (of complex codimension 2) and $\partial f = 0$ (of complex codimension n) in the space of holomorphic 1-jets of sections of $\mathbb{C}^2 \otimes L^{\otimes k}$, by means of the techniques described in the introduction. A small additional perturbation ensures the compatibility requirement that $\bar\partial f$ vanishes at the points where $\partial f = 0$. These properties are sufficient to ensure that the structure is that of a symplectic Lefschetz pencil. For details, the reader is referred to [10].

The statement that the constructed pencils are canonical up to isotopy for $k \gg 0$ is to be interpreted as follows. Consider two sequences $(s_k^0)_{k\gg 0}$ and $(s_k^1)_{k\gg 0}$ of approximately holomorphic sections of $\mathbb{C}^2 \otimes L^{\otimes k}$ for increasing values of k. Assume that they satisfy the three above-described transversality and compatibility properties and hence define symplectic Lefschetz pencils. Then, for large enough k (how large exactly depends on the estimates on the given sections), there exists an interpolating family $(s_k^t)_{t\in[0,1]}$ of approximately holomorphic sections, depending continuously on the parameter t, such that for all values of t the sections s_k^t satisfy the transversality and compatibility properties. In particular, for large enough k the symplectic Lefschetz pencils defined by s_k^0 and s_k^1 are isotopic to each other. Moreover, the same result remains true if the almost-complex structures J_0 and J_1 with respect to which s_k^0 and s_k^1 are approximately holomorphic differ, so the topology of the constructed pencils depends only on the topology of the symplectic manifold X (and on k of course). However, because isotopy holds only for large values of k, this is only a weak (asymptotic) uniqueness result.

A convenient way to study the topology of a Lefschetz pencil is to blow up X along the submanifold Z. The resulting symplectic manifold $\hat{X}$ is the total space of a *symplectic Lefschetz fibration* $\hat{f} : \hat{X} \to \mathbb{CP}^1$. Although in the following description we work on the blown up manifold $\hat{X}$, it is actually preferrable to work directly on X ; verifying that the discussion applies to X itself is a simple task left to the reader.

The fibers of $\hat{f}$ can be identified with the submanifolds Σ_α, made mutually disjoint by the blow-up process. It is then possible to study the *monodromy* of the fibration $\hat{f}$ around its singular fibers.

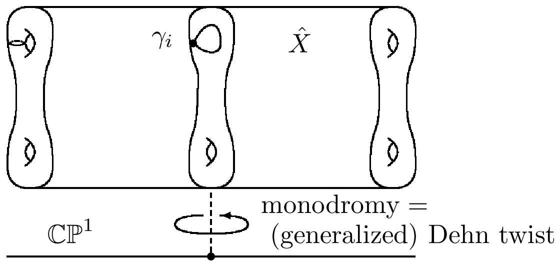
One easily checks that this monodromy consists of symplectic automorphisms of the fiber Σ_α. Moreover, the exceptional divisor obtained by blowing up the set of base points Z is a subfibration of $\hat{f}$, with fiber Z, which is unaffected by the monodromy ; after restricting to an affine slice, the normal bundle to the exceptional divisor can be trivialized, so that it becomes natural to consider that the monodromy of $\hat{f}$ takes values in the symplectic mapping class group $\mathrm{Map}^\omega(\Sigma, Z) = \pi_0(\{\phi \in \mathrm{Symp}(\Sigma,\omega), \phi_{|U(Z)} = \mathrm{Id}\})$, i.e. the set of isotopy classes of symplectomorphisms of the generic fiber Σ which coincide with the identity near Z.

In the four-dimensional case, Z consists of a finite number n of points, and Σ is a compact surface with a certain genus g (note that Σ is always connected because it satisfies a Lefschetz hyperplane type property) ; $\mathrm{Map}^\omega(\Sigma, Z)$ is then the classical mapping class group $\mathrm{Map}_{g,n}$ of a genus g surface with n boundary components.

In fact, the image of the monodromy map is contained in the subgroup of *exact* symplectomorphisms in $\mathrm{Map}^\omega(\Sigma, Z)$: the connection on $L^{\otimes k}$ induces over $\Sigma - Z$ a 1-form α such that $d\alpha = \omega$. This endows $\Sigma - Z$ with a structure of exact symplectic manifold. Monodromy transformations are then exact symplectomorphisms in the sense that they preserve not only ω but also the 1-form α : every monodromy transformation f satisfies $f^*\alpha - \alpha = dh$ for some function h vanishing near Z (see [17] for details).

It is well-known (see e.g. [16], [17]) that the singular fibers of a Lefschetz fibration are obtained from the generic fiber by collapsing a *vanishing cycle* to a point. The vanishing cycle is an embedded closed loop in Σ in the four-dimensional case ; more generally, it is an embedded Lagrangian sphere $S^{n-1} \subset \Sigma$. Then, the monodromy of $\hat{f}$ around one of its singular fibers consists in a *generalized Dehn twist* in the positive direction along the vanishing cycle.

The picture is the following :



Because the normal bundle to the exceptional divisor is not trivial, the monodromy map cannot be defined over all of $\mathbb{CP}^1$, and we need to restrict ourselves to the preimage of an affine subset $\mathbb{C}$ (the fiber at infinity can be assumed regular). The monodromy around the fiber at infinity of $\hat{f}$ is given by a mapping class group element δ_Z corresponding to a *twist around* Z. In the four-dimensional case Z consists of n points, and δ_Z is the product of positive Dehn twists along n loops each encircling one of the base points ; in the higher-dimensional case δ_Z is a positive Dehn twist along the unit sphere bundle in the normal bundle of Z in Σ (i.e. it restricts to each fiber of the normal bundle as a Dehn twist around the origin).

It follows from the above observations that the monodromy of the Lefschetz fibration $\hat{f}$ with critical levels $p_1, \dots, p_d$ is given by a group homomorphism

$$\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_d\}) \to \mathrm{Map}^\omega(\Sigma^{2n-2}, Z) \tag{1}$$

which maps the *geometric generators* of $\pi_1(\mathbb{C} - \{p_1, \dots, p_d\})$, i.e. loops going around one of the points p_i, to Dehn twists.

Alternately, choosing a system of generating loops in $\mathbb{C} - \{p_1, \dots, p_d\}$, we can express the monodromy by a *factorization* of δ_Z in the mapping class group :

$$\delta_Z = \prod_{i=1}^{d} \tau_{\gamma_i}, \tag{2}$$

where γ_i is the image in a chosen reference fiber of the vanishing cycle of the singular fiber above p_i and τ_{γ_i} is the corresponding positive Dehn twist. The identity (2) in $\mathrm{Map}^\omega(\Sigma, Z)$ expresses the fact that the monodromy of the fibration around the point at infinity in $\mathbb{CP}^1$ decomposes as the product of the elementary monodromies around each of the singular fibers.

The monodromy morphism (1), or equivalently the mapping class group factorization (2), completely characterizes the topology of the Lefschetz fibration $\hat{X}$. However, they are not entirely canonical, because two choices have been implicitly made in order to define them.

First, a base point in $\mathbb{C} - \{p_1, \dots, p_d\}$ and an identification symplectomorphism between Σ and the chosen reference fiber of $\hat{f}$ are needed in order to view the monodromy transformations as elements in the mapping class group of Σ. The choice of a different identification affects the monodromy morphism ψ by conjugation by a certain element $g \in \mathrm{Map}^\omega(\Sigma, Z)$. The corresponding operation on the mapping class group factorization (2) is a *simultaneous conjugation* of all factors : each factor τ_{γ_i} is replaced by $\tau_{g(\gamma_i)} = g^{-1}\tau_{\gamma_i} g$.

Secondly, a system of generating loops has to be chosen in order to define a factorization of δ_Z. Different choices of generating systems differ by a sequence of *Hurwitz operations*, i.e. moves in which two consecutive generating loops are exchanged, one of them being conjugated by the other in order to preserve the counterclockwise ordering. On the level of the factorization, this amounts to replacing two consecutive factors τ_1 and τ_2 by respectively τ_2 and $\tau_2^{-1}\tau_1\tau_2$ (or, by the reverse operation, $\tau_1\tau_2\tau_1^{-1}$ and τ_1).

It is quite easy to see that any two factorizations of δ_Z describing the Lefschetz fibration $\hat{f}$ differ by a sequence of these two operations (simultaneous conjugation and Hurwitz moves). Therefore, Donaldson's uniqueness statement implies that, for large enough values of k, the mapping class group factorizations associated to the symplectic Lefschetz pencil structures obtained in Theorem 2.1 are, up to simultaneous conjugation and Hurwitz moves, *symplectic invariants* of the manifold (X, ω).

Conversely, given any factorization of δ_Z in $\mathrm{Map}^\omega(\Sigma, Z)$ as a product of positive Dehn twists, it is possible to construct a symplectic Lefschetz fibration with the given monodromy. It follows from a result of Gompf that the total space of such a fibration is always a symplectic manifold. In fact, because the monodromy preserves the symplectic submanifold $Z \subset \Sigma$, it is also possible to reconstruct the blown down manifold X. More precisely, the following result holds :

Theorem 2.2 (Gompf). *Let (Σ, ω_Σ) be a compact symplectic manifold, and $Z \subset \Sigma$ a codimension 2 symplectic submanifold such that $[Z] = PD([\omega_\Sigma])$. Consider a factorization* *of* δ_Z *as a product of positive Dehn twists in* $\mathrm{Map}^\omega(\Sigma, Z)$. *In the case* $\dim(\Sigma) = 2$, *assume moreover that all the Dehn twists in the factorization are along loops that are not homologically trivial in* $\Sigma - Z$.

Then the total space X *of the corresponding Lefschetz pencil carries a symplectic form* ω_X *such that, given a generic fiber* Σ_0 *of the pencil,* $[\omega_X]$ *is Poincaré dual to* $[\Sigma_0]$*, and* $(\Sigma_0, \omega_{X|\Sigma_0})$ *is symplectomorphic to* (Σ, ω_Σ). *This symplectic structure on* X *is canonical up to symplectic isotopy.*

The strategy of proof is to first construct a symplectic structure in the correct cohomology class on a neighborhood of any fiber of the pencil, which is easily done as Σ already carries a symplectic structure and the monodromy lies in the exact symplectomorphism group. More precisely, the symplectic structure on $\Sigma - Z$ is exact, and Dehn twists along exact Lagrangian spheres are exact symplectomorphisms [17]. When $\dim \Sigma \geq 4$, the exactness condition is always trivially satisfied, while in the case $\dim \Sigma = 2$ it can be ensured by suitably choosing the vanishing loop in its homotopy class provided that it does not separate Σ into connected components without base points. With this understood, it is possible to define local symplectic structures over neighborhoods of the singular fibers, coinciding with a fixed standard symplectic form near Z, and to combine them into a globally defined symplectic form, singular near the base locus Z. Since the total monodromy is δ_Z, the structure of X near Z is completely standard, and so a non-singular symplectic form on X can be recovered (this process can also be viewed as a symplectic blow-down along the exceptional hypersurface $\mathbb{CP}^1 \times Z$ in the total space of the corresponding Lefschetz fibration). This operation changes the cohomology class of the symplectic form on X, but one easily checks that the resulting class is a nonzero multiple of the Poincaré dual to a fiber ; scaling the symplectic form by a suitable factor then yields ω_X. The proof that this process is canonical up to symplectic isotopy is a direct application of Moser's stability theorem. The reader is referred to [11] and references therein for details.

In conclusion, the study of the monodromy of symplectic Lefschetz pencils makes it possible to define invariants of compact symplectic manifolds, which in principle provide a complete description of the topology. However, the complexity of mapping class groups and the difficulties in computing the invariants in concrete situations greatly decrease their usefulness in practice. This motivates the introduction of other similar topological constructions which may lead to more usable invariants.

3. Branched covers of $\mathbb{CP}^2$ and invariants of symplectic 4-manifolds

Throughout §3, we assume that (X, ω) is a compact symplectic 4-manifold. In that case, three generic approximately holomorphic sections s_0, s_1 and s_2 of $L^{\otimes k}$ never vanish simultaneously, and so they define a projective map $f = (s_0 : s_1 : s_2) : X \to \mathbb{CP}^2$. It was shown in [3] that, if the sections are suitably chosen, this map is a *branched covering*, whose branch curve $R \subset X$ is a smooth connected symplectic submanifold in X.

There are two possible local models in approximately holomorphic coordinates for the map f near the branch curve. The first one, corresponding to a generic point of R, is the map $(x,y) \mapsto (x^2,y)$; locally, both the branch curve R and its image by f are smooth. The other local model corresponds to the isolated points where f does not restrict to R as an immersion. The model map is then $(x,y) \mapsto (x^3 - xy, y)$, and the image of the smooth branch curve $R : 3x^2 - y = 0$ has equation $f(R) : 27z_1^2 = 4z_2^3$ and presents a cusp singularity. These two local models are the same as in the complex algebraic setting.

It is easy to see by considering the two model maps that R is a smooth approximately holomorphic (and therefore symplectic) curve in X, and that $f(R)$ is an approximately holomorphic symplectic curve in $\mathbb{CP}^2$, immersed away from its cusps. After a generic perturbation, we can moreover require that the branch curve $D = f(R)$ satisfies a self-transversality property, i.e. that its only singular points besides the cusps are transverse double points ("nodes"). Even though D is approximately holomorphic, it is not immediately possible to require that all of its double points correspond to a positive intersection number with respect to the standard orientation of $\mathbb{CP}^2$; the presence of (necessarily badly transverse) negative double points is a priori possible.

It was also shown in [3] that the branched coverings obtained from sections of $L^{\otimes k}$ are, for large values of k, canonical up to isotopy (this weak uniqueness statement holds in the same sense as that of Theorem 2.1). Therefore, the topology of the branch curve $D = f(R)$ can be used to define symplectic invariants, provided that one takes into account the possibility of cancellations or creations of pairs of nodes with opposite orientations in isotopies of branched coverings.

Most of the results cited below were obtained in a joint work with L. Katzarkov [5].

3.1. Quasiholomorphic maps to $\mathbb{CP}^2$

In order to study the topology of the singular plane curve D, it is natural to try to adapt the braid group techniques previously used by Moishezon and Teicher in the algebraic case (see e.g. [13], [14], [18]). However, in order to apply this method it is necessary to ensure that the branch curve satisfies suitable transversality properties with respect to a generic projection map from $\mathbb{CP}^2$ to $\mathbb{CP}^1$. This leads naturally to the notion of *quasiholomorphic covering* introduced in [5], which we now describe carefully.

We slightly rephrase the conditions listed in [5] in such a way that they extend naturally to the higher dimensional case ; the same definitions will be used again in §4. It is important to be aware that these concepts only apply to sequences of objects obtained for increasing values of the degree k ; the general strategy is always to work simultaneously with a whole family of sections indexed by the parameter k, in order to ultimately ensure the desired properties for large values of k. We start with the following terminology :

Definition 3.1. A sequence of sections s_k of complex vector bundles E_k over X (endowed with Hermitian metrics and connections) is *asymptotically holomorphic* if there exist constants C_j independent of k such that $|\nabla^j s_k|_{g_k} \le C_j$ and $|\nabla^{j-1}\bar\partial s_k|_{g_k} \le C_j k^{-1/2}$ for all j, all norms being evaluated with respect to the rescaled metric $g_k = kg$ on X.

The sections s_k are *uniformly transverse to* 0 if there exists a constant $\gamma > 0$ such that, at every point $x \in X$ where $|s_k(x)| \le \gamma$, the covariant derivative $\nabla s_k(x)$ is surjective and has a right inverse of norm less than γ^{-1} w.r.t. g_k (we then say that s_k is γ-transverse to 0).

In the case where the rank of the bundle E_k is greater than the dimension of X, the surjectivity condition imposed by transversality is never satisfied ; γ-transversality to 0 then means that the norm of the section is greater than γ at every point of X.

As mentioned in the introduction, it is easy to check that, if sections are asymptotically holomorphic and uniformly transverse to 0, then for large k their zero sets are smooth approximately holomorphic symplectic submanifolds. This principle, which plays a key role in Donaldson's construction of symplectic submanifolds [8], can also be applied to the Jacobian of the maps defined below and now implies the symplecticity of their branch curves.

Definition 3.2. A sequence of projective maps $f_k : X \to \mathbb{CP}^2$ determined by asymptotically holomorphic sections $s_k = (s_k^0, s_k^1, s_k^2)$ of $\mathbb{C}^3 \otimes L^{\otimes k}$ for $k \gg 0$ is *quasiholomorphic* if there exist constants C_j, γ, δ independent of k, almost-complex structures $\tilde{J}_k$ on X, and finite sets $\mathcal{C}_k, \mathcal{T}_k, \mathcal{I}_k \subset X$ such that the following properties hold (using $\tilde{J}_k$ to define the $\bar{\partial}$ operator) :

(0) $|\nabla^j(\tilde{J}_k - J)|_{g_k} \le C_j k^{-1/2}$ for every $j \ge 0$; $\tilde{J}_k = J$ outside of the 2δ-neighborhood of $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$; $\tilde{J}_k$ is integrable in the δ-neighborhood of $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$;

(1) the section s_k of $\mathbb{C}^3 \otimes L^{\otimes k}$ is γ-transverse to 0 ;

(2) $|\nabla f_k(x)|_{g_k} \ge \gamma$ at every point $x \in X$;

(3) the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \bigwedge^2 \partial f_k$ is γ-transverse to 0 ; in particular it vanishes transversely along a smooth symplectic curve $R_k \subset X$ (the branch curve).

(3′) the restriction of $\bar{\partial} f_k$ to $\mathrm{Ker}\, \partial f_k$ vanishes at every point of R_k ;

(4) the quantity $\partial(f_{k|R_k})$, which can be seen as a section of a line bundle over R_k, is γ-transverse to 0 and vanishes at the finite set $\mathcal{C}_k$ (the cusp points of f_k) ; in particular $f_k(R_k) = D_k$ is an immersed symplectic curve away from the image of $\mathcal{C}_k$;

(4′) f_k is $\tilde{J}_k$-holomorphic over the δ-neighborhood of $\mathcal{C}_k$;

(5) the section (s_k^0, s_k^1) of $\mathbb{C}^2 \otimes L^{\otimes k}$ is γ-transverse to 0 ; as a consequence D_k remains away from the point $(0:0:1)$;

(6) let $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ be the map defined by $\pi(x:y:z) = (x:y)$, and let $\phi_k = \pi \circ f_k$. Then the quantity $\partial(\phi_{k|R_k})$ is γ-transverse to 0 over R_k, and it vanishes over the union of $\mathcal{C}_k$ with the finite set $\mathcal{T}_k$ (the tangency points of the branch curve D_k with respect to the projection π) ;

(6′) f_k is $\tilde{J}_k$-holomorphic over the δ-neighborhood of $\mathcal{T}_k$;

(7) the projection $f_k : R_k \to D_k$ is injective outside of the singular points of D_k, and the self-intersections of D_k are transverse double points. Moreover, all special points of D_k (cusps, nodes, tangencies) lie in different fibers of the projection π, and none of them lies in $\pi^{-1}(0:1)$;

(8) the section s_k^0 of $L^{\otimes k}$ is γ-transverse to 0 ;

(8′) R_k intersects the zero set of s_k^0 at the points of $\mathcal{I}_k$; f_k is $\tilde{J}_k$-holomorphic over the δ-neighborhood of $\mathcal{I}_k$.

Remark 3.1. Definition 3.2 is slightly stronger than the definition given in [5]. Most notably, property (8), which ensures that the fiber of $\pi \circ f_k$ above $(0:1)$ enjoys suitable genericity properties, has been added for our purposes. Similarly, condition (6′) is significantly stronger than in [5], where it was only required that $\bar\partial f_k$ vanish at the points of $\mathcal{T}_k$. These extra conditions only require minor modifications of the arguments, while allowing the inductive construction described in §5 to be largely simplified.

Observe that, because of property (0), the notions of asymptotic holomorphicity with respect to J or $\tilde J_k$ coincide. Moreover, even though $\tilde J_k$ is used implicitly thoughout the definition, the choice of J or $\tilde J_k$ is irrelevant as far as transversality properties are concerned since they differ by $O(k^{-1/2})$.

Property (1) means that s_k is everywhere bounded from below by γ ; this implies that the projective map f_k is well-defined, and that $|\nabla^j f_k|_{g_k} = O(1)$ and $|\nabla^{j-1}\bar\partial f_k|_{g_k} = O(k^{-1/2})$ for all j. The second property can be interpreted in terms of transversality to the codimension 4 submanifold in the space of 1-jets given by the equation $\partial f = 0$. Properties (3) and (3′) yield the correct structure near generic points of the branch curve : the transverse vanishing of $\mathrm{Jac}(f_k)$ implies that the branching order is 2, and the compatibility property (3′) ensures that $\bar\partial f_k$ remains much smaller than ∂f_k in all directions, which is needed to obtain the correct local model.

Properties (4) and (4′) determine the structure of the covering near the cusp points. More precisely, observe that along R_k the tangent plane field TR_k and the plane field $\mathrm{Ker}\,\partial f_k$ coincide exactly at the cusp points ; condition (4) expresses that these two plane fields are transverse to each other (in [3] and [5] this condition was formulated in terms of a more complicated quantity; the two formulations are easily seen to be equivalent). This implies that cusp points are isolated and non-degenerate. The compatibility condition (4′) then ensures that the expected local model indeed holds.

The remaining conditions are used to ensure the compatibility of the branch curve $D_k = f_k(R_k)$ with the projection π to $\mathbb{CP}^1$. In particular, the transversality condition (6) and the corresponding compatibility condition (6′) imply that the points where the branch curve D_k fails to be transverse to the fibers of π are isolated non-degenerate tangency points. Moreover, property (7) states that the curve D_k is transverse to itself. This implies that D_k is a *braided curve* in the following sense :

Definition 3.3. A real 2-dimensional singular submanifold $D \subset \mathbb{CP}^2$ is a *braided curve* if it satisfies the following properties : (1) the only singular points of D are cusps (with positive orientation) and transverse double points (with either orientation) ; (2) the point $(0:0:1)$ does not belong to D ; (3) the fibers of the projection $\pi : (x:y:z) \mapsto (x:y)$ are everywhere transverse to D, except at a finite set of nondegenerate tangency points where a local model for D in orientation-preserving coordinates is $z_2^2 = z_1$; (4) the cusps, nodes and tangency points are all distinct and lie in different fibers of π.

We will see in §3.2 that these properties are precisely those needed in order to apply the braid monodromy techniques of Moishezon-Teicher to the branch curve D_k.

The main result of [5] can be formulated as follows :

Theorem 3.1 ([3],[5]). *For $k \gg 0$, it is possible to find asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ such that the corresponding projective maps $f_k : X \to \mathbb{CP}^2$ are quasiholomorphic branched coverings. Moreover, for large k these coverings are canonical up to isotopy and up to cancellations of pairs of nodes in the branch curves D_k.*

The uniqueness statement is to be understood in the same weak sense as for Theorem 2.1 : given two sequences of quasiholomorphic branched coverings (possibly for different choices of almost-complex structures on X), for large k it is possible to find an interpolating one-parameter family of quasiholomorphic coverings, the only possible non-trivial phenomenon being the cancellation or creation of pairs of nodes in the branch curve for certain parameter values.

The proof of Theorem 3.1 follows a standard pattern : in order to construct quasiholomorphic coverings, one starts with any sequence of asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ and proceeds by successive perturbations in order to obtain all the required properties, starting with uniform transversality. Since transversality is an open condition, it is preserved by the subsequent perturbations.

So the first part of the proof consists in obtaining, by successive perturbation arguments, the transversality properties (1), (2), (3) and (4) of Definition 3.2 as in [3], (5) and (6) as in [5], and also (8) by a direct application of the result of [8]. The argument is notably more technical in the case of (4) and (6) because the transversality conditions involve derivatives along the branch curve, but these can actually all be thought of as immediate applications of the general transversality principle mentioned in the Introduction.

The second part of the proof, which is comparatively easier, deals with the compatibility conditions. The idea is to ensure these properties by perturbing the sections s_k by quantities bounded by $O(k^{-1/2})$, which clearly affects neither holomorphicity nor transversality properties. One first chooses suitable almost-complex structures $\tilde{J}_k$ differing from J by $O(k^{-1/2})$ and integrable near the finite set $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$. It is then possible to perturb f_k near these points in order to obtain conditions (4′), (6′) and (8′), by the same argument as in §4.1 of [3]. Next, a generic small perturbation yields the self-transversality of D (property (7)). Finally, a suitable perturbation yields property (3′) along the branch curve without modifying R_k and D_k and without affecting the other compatibility properties.

The uniqueness statement is obtained by showing that, provided that k is large enough, all the arguments extend verbatim to one-parameter families of sections. Therefore, given two sequences of quasiholomorphic coverings, one starts with a one-parameter family of sections interpolating between them in a trivial way and perturbs it in such a way that the required properties hold for all parameter values (with the exception of (7) when a node cancellation occurs). Since this construction can be performed in such a way that the two end points of the one-parameter family are not affected by the perturbation, the isotopy result follows immediately.

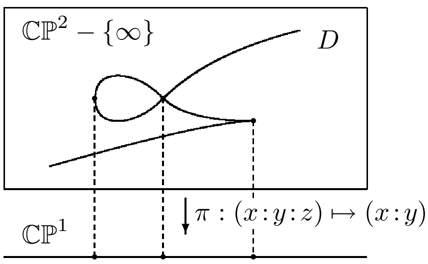
The reader is referred to [3] and [5] for more details (incorporating requirement (8) in the arguments is a trivial task).

3.2. Braid monodromy invariants

We now describe the monodromy invariants that naturally arise from the quasiholomorphic coverings described in the previous section. This is a relatively direct extension to the symplectic framework of the braid group techniques studied by Moishezon and Teicher in the algebraic case (see [13], [14], [18]).

Recall that the braid group on d strings is the fundamental group $B_d = \pi_1(\mathcal{X}_d)$ of the space $\mathcal{X}_d$ of unordered configurations of d distinct points in the plane $\mathbb{R}^2$. A braid can therefore be thought as a motion of d points in the plane. An alternate description involves compactly supported orientation-preserving diffeomorphisms of $\mathbb{R}^2$ which globally preserve a set of d given points : $B_d = \pi_0(\mathrm{Diff}^+_c(\mathbb{R}^2, \{q_1, \dots, q_d\}))$. The group B_d is generated by *half-twists*, i.e. braids in which two of the d points rotate around each other by 180 degrees while the other points are preserved. For more details see [6].

Consider a braided curve $D \subset \mathbb{CP}^2$ (see Definition 3.3) of fixed degree d, for example the branch curve of a quasiholomorphic covering as given by Theorem 3.1. Projecting to $\mathbb{CP}^1$ via the map π makes D a singular branched covering of $\mathbb{CP}^1$. The picture is the following :



Let $p_1, \dots, p_r$ be the images by π of the special points of D (nodes, cusps and tangencies). Observing that the fibers of π are complex lines (or equivalently real planes) which generically intersect D in d points, we easily get that the monodromy of the map $\pi_{|D}$ around the fibers above $p_1, \dots, p_r$ takes values in the braid group B_d.

The monodromy around one of the points $p_1, \dots, p_r$ is as follows. In the case of a tangency point, a local model for the curve D is $y^2 = x$ (with projection to the x factor), so one easily checks that the monodromy is a half-twist exchanging two sheets of $\pi_{|D}$. Since all half-twists in B_d are conjugate, it is possible to write this monodromy in the form $Q^{-1}X_1Q$, where $Q \in B_d$ is any braid and X_1 is a fixed half-twist (aligning the points $q_1, \dots, q_d$ in that order along the real axis, X_1 is the half-twist exchanging the points q_1 and q_2 along a straight line segment). In the case of a transverse double point with positive intersection, the local model $y^2 = x^2$ implies that the monodromy is the square of a half-twist, which can be written in the form $Q^{-1}X_1^2Q$. The monodromy around a double point with negative intersection is the mirror image of the previous case, and can therefore be written as $Q^{-1}X_1^{-2}Q$. Finally, the monodromy around a cusp (local model $y^2 = x^3$) is the cube of a half-twist and can be expressed as $Q^{-1}X_1^3Q$.

However, in order to describe the monodromy automorphisms as braids, one needs to identify up to compactly supported diffeomorphisms the fibers of π with a reference plane $\mathbb{R}^2$. This implicitly requires a trivialization of the fibration π, which is not available over all of $\mathbb{CP}^1$. Therefore, as in the case of Lefschetz pencils, it is necessary to restrict oneself to the preimage of an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, by removing the fiber above the point at infinity (which may easily be assumed to be regular). So the monodromy map is only defined as a group homomorphism

$$\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d. \tag{3}$$

Since the fibration π defines a line bundle of degree 1 over $\mathbb{CP}^1$, the monodromy around the fiber at infinity is given by the *full twist* Δ^2, i.e. the braid which corresponds to a rotation of all points by 360 degrees (Δ^2 generates the center of B_d).

Therefore, choosing as in §2 a system of generating loops in $\mathbb{C} - \{p_1, \dots, p_r\}$, we can express the monodromy by a factorization of Δ^2 in the braid group :

$$\Delta^2 = \prod_{j=1}^{r} Q_j^{-1} X_1^{r_j} Q_j, \tag{4}$$

where the elements $Q_j \in B_d$ are arbitrary braids and the degrees $r_j \in \{1, \pm 2, 3\}$ depend on the types of the special points lying above p_j.

As in the case of Lefschetz pencils, this braid factorization, which completely characterizes the braided curve D up to isotopy, is only well-defined up to two algebraic operations: simultaneous conjugation of all factors by a given braid in B_d, and Hurwitz moves. As previously, simultaneous conjugation reflects the different possible choices of an identification diffeomorphism between the fiber of π above the base point and the standard plane $(\mathbb{C}, \{q_1, \dots, q_d\})$, while Hurwitz moves arise from changes in the choice of a generating system of loops in $\mathbb{C} - \{p_1, \dots, p_r\}$.

Starting with any braid factorization of the form (4), it is possible to reconstruct a braided curve D in a canonical way up to isotopy (see [5]; similar statements were also obtained by Moishezon, Teicher and Catanese). Moreover, one easily checks that factorizations which differ only by global conjugations and Hurwitz moves lead to isotopic braided curves (each such operation amounts to a diffeomorphism isotopic to the identity, obtained in the case of a Hurwitz move by lifting by π a diffeomorphism of $\mathbb{CP}^1$, and in the case of a global conjugation by a diffeomorphism in each of the fibers of π).

Moreover, it is important to observe that every braided curve D can be made symplectic by a suitable isotopy. In fact, it is sufficient to perform a radial contraction in all the fibers of π, which brings the given curve into an arbitrarily small neighborhood of the zero section of π (the complex line $\{z = 0\}$ in $\mathbb{CP}^2$). The tangent space to D is then very close to that of the complex line (and therefore symplectic) everywhere except near the tangency points; verifying that the property also holds near tangencies by means of the local model, one obtains that D is symplectic.

We now briefly describe the structure of the fundamental group $\pi_1(\mathbb{CP}^2 - D)$. Consider a generic fiber of π, intersecting D in d points $q_1, \dots, q_d$. Then the inclusion map $i : \mathbb{C} - \{q_1, \dots, q_d\} \to \mathbb{CP}^2 - D$ induces a surjective homomorphism on fundamental groups. Therefore, a generating system of loops $\gamma_1, \dots, \gamma_d$ in $\mathbb{C} - \{q_1, \dots, q_d\}$ provides a set of generators for $\pi_1(\mathbb{CP}^2 - D)$ (*geometric generators*). Because the fiber of π can be compactified by adding the pole of the projection, an obvious relation is $\gamma_1 \dots \gamma_d = 1$. Moreover, each special point of the curve D, or equivalently every term in the braid factorization, determines a relation in $\pi_1(\mathbb{CP}^2 - D)$ in a very explicit way.

Namely, recall that there exists a natural right action of B_d on the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$, that we shall denote by $*$, and consider a factor $Q_j^{-1} X_1^{r_j} Q_j$ in (4). Then, if $r_j = 1$, the tangency point above p_j yields the relation $\gamma_1 * Q_j = \gamma_2 * Q_j$ (the two elements $\gamma_1 * Q_j$ and $\gamma_2 * Q_j$ correspond to small loops going around the two sheets of $\pi_{|D}$ that merge at the tangency point). Similarly, in the case of a node ($r_j = \pm 2$), the relation is $[\gamma_1 * Q_j, \gamma_2 * Q_j] = 1$. Finally, in the case of a cusp ($r_j = 3$), the relation becomes $(\gamma_1\gamma_2\gamma_1) * Q_j = (\gamma_2\gamma_1\gamma_2) * Q_j$. It is a classical result that $\pi_1(\mathbb{CP}^2 - D)$ is exactly the quotient of $F_d = \langle \gamma_1, \dots, \gamma_d \rangle$ by the above-listed relations.

Given a branched covering map $f : X \to \mathbb{CP}^2$ with branch curve D, it is easy to see that the topology of X is determined by a group homomorphism from $\pi_1(\mathbb{CP}^2 - D)$ to the symmetric group S_n of order $n = \deg f$. Considering a generic fiber of π which intersects D in d points $q_1, \dots, q_d$, the restriction of f to its preimage Σ is a n-sheeted branched covering map from Σ to $\mathbb{C}$ with branch points $q_1, \dots, q_d$. This covering is naturally described by a monodromy representation

$$\theta : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to S_n. \tag{5}$$

Because the branching index is 2 at a generic point of the branch curve of f, the group homomorphism θ maps geometric generators to transpositions. Also, θ necessarily factors through the surjective homomorphism $i_* : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to \pi_1(\mathbb{CP}^2 - D)$, because the covering f is defined everywhere, and the resulting map from $\pi_1(\mathbb{CP}^2 - D)$ to S_n is exactly what is needed to recover the 4-manifold X from the branch curve D. The properties of θ are summarized in the following definition due to Moishezon :

Definition 3.4. A geometric monodromy representation associated to a braided curve $D \subset \mathbb{CP}^2$ is a surjective group homomorphism θ from the free group $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) = F_d$ to the symmetric group S_n of order n, mapping the geometric generators γ_i (and thus also the $\gamma_i * Q_j$) to transpositions, and such that

$\theta(\gamma_1 \dots \gamma_d) = 1$,
$\theta(\gamma_1 * Q_j) = \theta(\gamma_2 * Q_j)$ if $r_j = 1$,
$\theta(\gamma_1 * Q_j)$ and $\theta(\gamma_2 * Q_j)$ are distinct and commute if $r_j = \pm 2$,
$\theta(\gamma_1 * Q_j)$ and $\theta(\gamma_2 * Q_j)$ do not commute if $r_j = 3$.

Observe that, when the braid factorization defining D is affected by a Hurwitz move, θ remains unchanged and the compatibility conditions are preserved. On the contrary, when the braid factorization is modified by simultaneously conjugating all factors by a certain braid $Q \in B_d$, the system of geometric generators $\gamma_1, \dots, \gamma_d$ changes accordingly, and so the geometric monodromy representation θ should be replaced by $\theta \circ Q_*$, where Q_* is the automorphism of F_d induced by the braid Q.

One easily checks that, given a braided curve $D \subset \mathbb{CP}^2$ and a compatible monodromy representation $\theta : F_d \to S_n$, it is possible to recover a compact 4-manifold X and a branched covering map $f : X \to \mathbb{CP}^2$ in a canonical way. Moreover, as observed above we can assume that the curve D is symplectic; in that case, the branched covering map makes it possible to endow X with a symplectic structure, canonically up to symplectic isotopy (see [3],[5] ; a similar result has also been obtained by Catanese).

The above discussion leads naturally to the definition of symplectic invariants arising from the quasiholomorphic coverings constructed in Theorem 3.1. However, things are complicated by the fact that the branch curves of these coverings are only canonical up to cancellations of double points.

On the level of the braid factorization, a pair cancellation amounts to removing two consecutive factors which are the inverse of each other (necessarily one must have degree 2 and the other degree -2); the geometric monodromy representation is not affected. The opposite operation is the creation of a pair of nodes, in which two factors $(Q^{-1}X_1^{-2}Q).(Q^{-1}X_1^2Q)$ are added anywhere in the factorization ; it is allowed only if the new factorization remains compatible with the monodromy representation θ, i.e. if $\theta(\gamma_1 * Q)$ and $\theta(\gamma_2 * Q)$ are commuting disjoint transpositions.

Definition 3.5. Two braid factorizations (along with the corresponding geometric monodromy representations) are m-equivalent if there exists a sequence of operations which turns one into the other, each operation being either a global conjugation, a Hurwitz move, or a pair cancellation or creation.

In conclusion, we get the following result :

Theorem 3.2 ([5]). *The braid factorizations and geometric monodromy representations associated to the quasiholomorphic coverings obtained in Theorem 3.1 are, for $k \gg 0$, canonical up to m-equivalence, and define symplectic invariants of (X^4, ω).*

Conversely, the data consisting of a braid factorization and a geometric monodromy representation, or a m-equivalence class of such data, determines a symplectic 4-manifold in a canonical way up to symplectomorphism.

3.3. The braid group and the mapping class group

Let $f : X \to \mathbb{CP}^2$ be a branched covering map, and let $D \subset \mathbb{CP}^2$ be its branch curve. It is a simple observation that, if D is braided, then the map $\pi \circ f$ with values in $\mathbb{CP}^1$ obtained by forgetting one of the components of f topologically defines a Lefschetz pencil. This pencil is obtained by lifting via the covering f the pencil of lines on $\mathbb{CP}^2$ defined by π, and its base points are the preimages by f of the pole of the projection π.

Moreover, if one starts with the quasiholomorphic coverings given by Theorem 3.1, then the corresponding Lefschetz pencils coincide for $k \gg 0$ with those obtained by Donaldson in [10] and described in §2.

As a consequence, in the case of a 4-manifold, the invariants described in §3.2 (braid factorization and geometric monodromy representation) completely determine those described in §2 (factorizations in mapping class groups). It is therefore natural to look for a more explicit description of the relation between branched coverings and Lefschetz pencils. This description involves the group of *liftable braids*, which has been studied in a special case by Birman and Wajnryb in [7]. We recall the following construction from §5 of [5].

Let $\mathcal{C}_n(q_1, \dots, q_d)$ be the (finite) set of all surjective group homomorphisms $F_d \to S_n$ which map each of the geometric generators $\gamma_1, \dots, \gamma_d$ of F_d to a transposition and map their product $\gamma_1 \cdots \gamma_d$ to the identity element in S_n. Each element of $\mathcal{C}_n(q_1, \dots, q_d)$ determines a simple n-fold covering of $\mathbb{CP}^1$ branched at $q_1, \dots, q_d$.

Let $\mathcal{X}_d$ be the space of configurations of d distinct points in the plane. The set of all simple n-fold coverings of $\mathbb{CP}^1$ with d branch points and such that no branching occurs above the point at infinity can be thought of as a covering $\tilde{\mathcal{X}}_{d,n}$ above $\mathcal{X}_d$, in which the fiber above the configuration $\{q_1, \dots, q_d\}$ identifies with $\mathcal{C}_n(q_1, \dots, q_d)$. Therefore, the braid group $B_d = \pi_1(\mathcal{X}_d)$ acts on the fiber $\mathcal{C}_n(q_1, \dots, q_d)$ by deck transformations of the covering $\tilde{\mathcal{X}}_{d,n}$. In fact, the action of a braid $Q \in B_d$ on $\mathcal{C}_n(q_1, \dots, q_d)$ is given by $\theta \mapsto \theta \circ Q_*$, where $Q_* \in \mathrm{Aut}(F_d)$ is the automorphism induced by Q on the fundamental group of $\mathbb{C} - \{q_1, \dots, q_d\}$.

Fix a base point $\{q_1, \dots, q_d\}$ in $\mathcal{X}_d$, and consider an element θ of $\mathcal{C}_n(q_1, \dots, q_d)$ (i.e., a monodromy representation $\theta : F_d \to S_n$). Let p_θ be the corresponding point in $\tilde{\mathcal{X}}_{d,n}$.

Definition 3.6. The subgroup $B_d^0(\theta)$ of liftable braids is the set of all the loops in $\mathcal{X}_d$ whose lift at the point p_θ is a closed loop in $\tilde{\mathcal{X}}_{d,n}$. Equivalently, $B_d^0(\theta)$ is the set of all braids which act on $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ in a manner compatible with the covering structure defined by θ.

In other words, $B_d^0(\theta)$ is the set of all braids Q such that $\theta \circ Q_* = \theta$, i.e. the stabilizer of θ with respect to the action of B_d on $\mathcal{C}_n(q_1, \dots, q_d)$.

There exists a natural bundle $\mathcal{Y}_{d,n}$ over $\tilde{\mathcal{X}}_{d,n}$ (the *universal curve*) whose fiber is a Riemann surface of genus $g = 1 - n + (d/2)$ with n marked points. Each of these Riemann surfaces naturally carries a structure of branched covering of $\mathbb{CP}^1$, and the marked points are the preimages of the point at infinity.

Given an element Q of $B_d^0(\theta) \subset B_d$, it can be lifted to $\tilde{\mathcal{X}}_{d,n}$ as a loop based at the point p_θ, and the monodromy of the fibration $\mathcal{Y}_{d,n}$ along this loop defines an element of $\mathrm{Map}_{g,n}$ (the mapping class group of a Riemann surface of genus g with n boundary components), which we call $\theta_*(Q)$. This defines a group homomorphism $\theta_* : B_d^0(\theta) \to \mathrm{Map}_{g,n}$.

More geometrically, viewing Q as a compactly supported diffeomorphism of the plane preserving $\{q_1, \dots, q_d\}$, the fact that Q belongs to $B_d^0(\theta)$ means that it can be lifted via the covering map $\Sigma_g \to \mathbb{CP}^1$ to a diffeomorphism of Σ_g ; the corresponding element in the mapping class group is $\theta_*(Q)$.

It is easy to check that, when the given monodromy representation θ is compatible with a braided curve $D \subset \mathbb{CP}^2$, the image of the braid monodromy homomorphism $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$ describing D is entirely contained in $B_d^0(\theta)$: this is because the geometric monodromy representation θ factors through $\pi_1(\mathbb{CP}^2 - D)$, on which the braids in $\operatorname{Im}\rho$ act trivially. Therefore, we can take the image of the braid factorization describing D by θ_* and obtain a factorization in the mapping class group $\mathrm{Map}_{g,n}$. One easily checks that $\theta_*(\Delta^2)$ is, as expected, the twist δ_Z around the n marked points.

As observed in [5], all the factors of degree ± 2 or 3 in the braid factorization lie in the kernel of θ_* ; therefore, the only terms whose contribution to the mapping class group factorization is non-trivial are those arising from the tangency points of the branch curve D, and each of these is a Dehn twist. More precisely, the image in $\mathrm{Map}_{g,n}$ of a half-twist $Q \in B_d^0(\theta)$ can be constructed as follows. Call γ the path joining two of the branch points naturally associated to the half-twist Q (i.e. the path along which the twisting occurs). Among the n lifts of γ to Σ_g, only two hit the branch points of the covering ; these two lifts have common end points, and together they define a loop δ in Σ_g. Then the element $\theta_*(Q)$ in $\mathrm{Map}_{g,n}$ is the positive Dehn twist along the loop δ (see Proposition 4 of [5]).

In conclusion, the following result holds :

Proposition 3.3. *Let $f : X \to \mathbb{CP}^2$ be a branched covering, and assume that its branch curve D is braided. Let $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d^0(\theta)$ and $\theta : F_d \to S_n$ be the corresponding braid monodromy and geometric monodromy representation. Then the monodromy map $\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to \mathrm{Map}_{g,n}$ of the Lefschetz pencil $\pi \circ f$ is given by the identity $\psi = \theta_* \circ \rho$.*

In particular, for $k \gg 0$ the symplectic invariants obtained from Theorem 2.1 are obtained in this manner from those given by Theorem 3.2.

Remark 3.2. It is a basic fact that for $n \geq 3$ the group homomorphism $\theta_* : B_d^0(\theta) \to \mathrm{Map}_{g,n}$ is surjective, and that for $n \geq 4$ every Dehn twist is the image by θ_* of a half-twist. This makes it natural to ask whether every factorization of δ_Z in $\mathrm{Map}_{g,n}$ as a product of Dehn twists is the image by θ_* of a factorization of Δ^2 in $B_d^0(\theta)$ compatible with θ. This can be reformulated in more geometric terms as the classical problem of determining whether every Lefschetz pencil is topologically a covering of $\mathbb{CP}^2$ branched along a curve with node and cusp singularities (a similar question replacing pencils by Lefschetz fibrations and $\mathbb{CP}^2$ by ruled surfaces also holds ; presently the answer is only known in the hyperelliptic case, thanks to the results of Fuller, Siebert and Tian).

A natural approach to these problems is to understand the kernel of θ_*. For example, if one can show that this kernel is generated by squares and cubes of half-twists (factors of degree 2 and 3 compatible with θ), then the solution naturally follows : given a decomposition of δ_Z as a product of Dehn twists in $\mathrm{Map}_{g,n}$, any lift of this word to $B_d^0(\theta)$ as a product of half-twists differs from Δ^2 by a product of factors of degree 2 and 3 and their inverses. Adding these factors as needed, one obtains a decomposition of Δ^2 into factors of degrees 1, ± 2 and ± 3 ; the branch curve constructed in this way may have nodes and cusps with reversed orientation, but it can still be made symplectic.

Even if the kernel of θ_* is not generated by factors of degree 2 and 3, it remains likely that the result still holds and can be obtained by starting from a suitable lift to $B^0_d(\theta)$ of the word in $\mathrm{Map}_{g,n}$. A better understanding of the structure of $\mathrm{Ker}\,\theta_*$ would be extremely useful for this purpose.

4. The higher dimensional case

In this section we extend the results of §3 to the case of higher dimensional symplectic manifolds. In §4.1 we prove the existence of quasiholomorphic maps $X \to \mathbb{CP}^2$ given by triples of sections of $L^{\otimes k}$ for $k \gg 0$. The topological invariants arising from these maps are studied in §4.2 and §4.3, and the relation with Lefschetz pencils is described in §4.4.

4.1. Quasiholomorphic maps to $\mathbb{CP}^2$

Let (X^{2n}, ω) be a compact symplectic manifold, endowed with a compatible almost-complex structure J. Let L be the same line bundle as previously (if $\frac{1}{2\pi}[\omega]$ is not integral one works with a perturbed symplectic form as explained in the introduction). Consider three approximately holomorphic sections of $L^{\otimes k}$, or equivalently a section of $\mathbb{C}^3 \otimes L^{\otimes k}$. Then the following result states that exactly the same transversality and compatibility properties can be expected as in the four-dimensional case :

Theorem 4.1. *For $k \gg 0$, it is possible to find asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ such that the corresponding $\mathbb{CP}^2$ valued projective maps f_k are quasiholomorphic (cf. Definition 3.2). Moreover, for large k these projective maps are canonical up to isotopy and up to cancellations of pairs of nodes in the critical curves D_k.*

Before sketching a proof of Theorem 4.1, we briefly describe the behavior of quasiholomorphic maps, which will clarify some of the requirements of Definition 3.2.

Condition (1) in Definition 3.2 implies that the set Z_k of points where the three sections s^0_k, s^1_k, s^2_k vanish simultaneously is a smooth codimension 6 symplectic (approximately holomorphic) submanifold. The projective map $f_k = (s^0_k : s^1_k : s^2_k)$ with values in $\mathbb{CP}^2$ is only defined over the complement of Z_k. The behavior near the set of base points is similar to what happens for Lefschetz pencils : in suitable local approximately holomorphic coordinates, Z_k is given by the equation $z_1 = z_2 = z_3 = 0$, and f_k behaves like the model map $(z_1, \dots, z_n) \mapsto (z_1 : z_2 : z_3)$. In fact, a map defined everywhere can be obtained by blowing up X along the submanifold Z_k. The behavior near Z_k being completely specified by condition (1), it is implicit that all the other conditions on f_k are only to be imposed outside of a small neighborhood of Z_k.

The correct statement of condition (3) of Definition 3.2 in the case of a manifold of dimension greater than 4 is a bit tricky. Indeed, $\mathrm{Jac}(f_k) = \bigwedge^2 \partial f_k$ is a priori a section of the vector bundle $\Lambda^{2,0}T^*X \otimes f_k^*(\Lambda^{2,0}T\mathbb{CP}^2)$ of rank $n(n-1)/2$. However, transversality to 0 in this sense is impossible to obtain, as the expected complex codimension of R_k is

$n-1$ instead of $n(n-1)/2$. Indeed, the section $\mathrm{Jac}(f_k)$ takes values in the non-linear subbundle $\mathrm{Im}(\bigwedge^2)$, whose fibers are of dimension $n-1$ at their smooth points (away from the origin). However, transversality to 0 does not have any natural definition in this subbundle, because it is singular along the zero section. The problem is very similar to what happens in the construction of determinantal submanifolds performed in [15].

In our case, a precise meaning can be given to condition (3) by the following observation. Near any point $x \in X$, property (2) implies that it is possible to find local approximately holomorphic coordinates on X and local complex coordinates on $\mathbb{CP}^2$ in which the differential at x of the first component of f_k can be written $\partial f_k^1(x) = \lambda\, dz_1$, with $|\lambda| > \gamma/2$. This implies that, near x, the projection of $\bigwedge^2 \partial f_k$ to its components along $dz_1 \wedge dz_2, \dots, dz_1 \wedge dz_n$ is a quasi-isometric isomorphism. In other words, the transversality to 0 of $\mathrm{Jac}(f_k)$ is to be understood as the transversality to 0 of its orthogonal projection to the linear subbundle of rank $n-1$ generated by $dz_1 \wedge dz_2, \dots, dz_1 \wedge dz_n$.

Another equivalent approach is to consider the (non-linear) bundle $\mathcal{J}^1(X, \mathbb{CP}^2)$ of holomorphic 1-jets of maps from X to $\mathbb{CP}^2$. Inside this bundle, the 1-jets whose differential is not surjective define a subbundle Σ of codimension $n-1$, smooth away from the stratum $\{\partial f = 0\}$. Since this last stratum is avoided by the 1-jet of f_k (because of condition (2)), the transversality to 0 of $\mathrm{Jac}(f_k)$ can be naturally rephrased in terms of estimated transversality to Σ in the bundle of jets (this approach will be developed in [4]).

With this understood, conditions (3) and (3′) imply, as in the four-dimensional case, that the set R_k of points where the differential of f_k fails to be surjective is a smooth symplectic curve $R_k \subset X$, disjoint from Z_k, and that the differential of f_k has rank 2 at every point of R_k. Also, as before, conditions (4) and (4′) imply that $f_k(R_k) = D_k$ is a symplectic curve in $\mathbb{CP}^2$, immersed outside of the cusp points.

We now describe the proof of Theorem 4.1 ; most of the argument is identical to the 4-dimensional case, and the reader is referred to [3] and [5] for notations and details.

Proof of Theorem 4.1. The strategy of proof is the same as in the 4-dimensional case. One starts with an arbitrary sequence of asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ over X, and perturbs it first to obtain the transversality properties. Provided that k is large enough, each transversality property can be obtained over a ball by a small localized perturbation, using the local transversality result of Donaldson (Theorem 12 in [10]). A globalization argument then makes it possible to combine these local perturbations into a global perturbation that ensures transversality everywhere (Proposition 3 of [3]). Since transversality properties are open, successive perturbations can be used to obtain all the required properties : once a transversality property is obtained, subsequent perturbations only affect it by at most decreasing the transversality estimate.

Step 1. One first obtains the transversality statements in parts (1), (5) and (8) of Definition 3.2 ; as in the 4-dimensional case, these properties are obtained e.g. simply by applying the main result of [2]. Observe that all required properties now hold near the base locus Z_k of s_k, so we can assume in the rest of the argument that the points of X being considered lie away from Z_k, and therefore that f_k is locally well-defined.

One next ensures condition (2), for which the argument is an immediate adaptation of that in §2.2 of [3], the only difference being the larger number of coordinate functions.

Step 2. The next property we want to get is condition (3). Here a significant generalization of the argument in §3.1 of [3] is needed. The problem reduces, as usual, to showing that the uniform transversality to 0 of $\mathrm{Jac}(f_k)$ can be ensured over a small ball centered at a given point $x \in X$ by a suitable localized perturbation. As in [3] one can assume that $s_k(x)$ is of the form $(s_k^0(x), 0, 0)$ and therefore locally trivialize $\mathbb{CP}^2$ via the quasi-isometric map $(x:y:z) \mapsto (y/x, z/x)$; this reduces the problem to the study of a $\mathbb{C}^2$-valued map h_k. Because $|\partial f_k|$ is bounded from below, we can assume (after a suitable rotation) that $|\partial h_k^1(x)|$ is greater than some fixed constant. Also, fixing suitable approximately holomorphic Darboux coordinates $z_k^1, \dots, z_k^n$ (using Lemma 3 of [3], which trivially extends to dimensions larger than 4), we can after a rotation assume that $\partial h_k^1(x)$ is of the form $\lambda\, dz_k^1$, where the complex number λ is bounded from below.

By Lemma 2 of [3], there exist asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of $L^{\otimes k}$ with exponential decay away from x. Define the asymptotically holomorphic 2-forms $\mu_k^j = \partial h_k^1 \wedge \partial(z_k^j s_{k,x}^{\mathrm{ref}}/s_k^0)$ for $2 \le j \le n$. At x, the 2-form μ_k^j is proportional to $dz_k^1 \wedge dz_k^j$; therefore, over a small neighborhood of x, the transversality to 0 of $\mathrm{Jac}(f_k)$ in the sense explained above is equivalent to the transversality to 0 of the projection of $\mathrm{Jac}(h_k)$ onto the subspace generated by $\mu_k^2, \dots, \mu_k^n$. In terms of 1-jets, the 2-forms μ_k^j define a local frame in the normal bundle to the stratum of non-regular maps at $\mathcal{J}^1(f_k)$. Now, express $\mathrm{Jac}(h_k)$ in the form $u_k^2\mu_k^2 + \dots + u_k^n\mu_k^n + \alpha_k$ over a neighborhood of x, where $u_k^2, \dots, u_k^n$ are complex-valued functions and α_k has no component along dz_k^1. Then, the transversality to 0 of $\mathrm{Jac}(f_k)$ is equivalent to that of the $\mathbb{C}^{n-1}$-valued function $u_k = (u_k^2, \dots, u_k^n)$.

Since the functions u_k are asymptotically holomorphic, using suitable Darboux coordinates at x we can use Theorem 12 of [10] to obtain, for large enough k, the existence of constants $w_k^2, \dots, w_k^n$ smaller than any given bound $\delta > 0$ and such that $(u_k^2 - w_k^2, \dots, u_k^n - w_k^n)$ is η-transverse to 0 over a small ball centered at x, where $\eta = \delta(\log \delta^{-1})^{-p}$ (p is a fixed constant). Letting $\tilde{s}_k = (s_k^0, s_k^1, s_k^2 - \sum w_k^j z_k^j s_{k,x}^{\mathrm{ref}})$ and calling $\tilde{f}_k$ and $\tilde{h}_k$ the projective map defined by $\tilde{s}_k$ and the corresponding local $\mathbb{C}^2$-valued map, we get that $\mathrm{Jac}(\tilde{h}_k) = \mathrm{Jac}(h_k) - \sum w_k^j \mu_k^j$, and therefore that $\mathrm{Jac}(\tilde{f}_k)$ is transverse to 0 near x. Since the perturbation of s_k has exponential decay away from x, we can apply the standard globalization argument to obtain property (3) everywhere.

Step 3. The next properties that we want to get are (4) and (6). It is possible to extend the arguments of [3] and [5] to the higher dimensional case ; however this yields a very technical and lengthy argument, so we outline here a more efficient strategy following the ideas of [4]. Thanks to the previously obtained transversality properties (1) and (5), both f_k and ϕ_k are well-defined over a neighborhood of R_k, so the statements of (4) and (6) are well-defined. Moreover, observe that property (6) implies property (4), because at any point where $\partial(f_{k|R_k})$ vanishes, $\partial(\phi_{k|R_k})$ necessarily vanishes as well, and if it does so transversely then the same is true for $\partial(f_{k|R_k})$ as well. So we only focus on (6).

This property can be rephrased in terms of transversality to the codimension n stratum $S : \{\partial(\phi_{|R}) = 0\}$ in the bundle $\mathcal{J}^2(X, \mathbb{CP}^2)$ of holomorphic 2-jets of maps from X to $\mathbb{CP}^2$. However this stratum is singular, even away from the substratum S_{nt} corresponding to the non-transverse vanishing of $\mathrm{Jac}(f)$; in fact it is reducible and comes as a union $S_1 \cup S_2$, where $S_1 : \{\mathrm{Jac}(f) = 0,\ \partial(f_{|R}) = 0\}$ is the stratum corresponding to non-immersed points of the branch curve, and $S_2 : \{\partial\phi = 0\}$ is the stratum corresponding to tangency points of the branch curve. Therefore, one first needs to ensure transversality with respect to $S_0 = S_1 \cap S_2 : \{\partial\phi = 0,\ \partial(f_{|R}) = 0\}$, which is a smooth codimension $n+1$ stratum ("vertical cusp points of the branch curve") away from S_{nt}.

Step 3a. We first show that a small perturbation can be used to make sure that the quantity $(\partial\phi_k, \partial(f_{k|R_k}))$ remains bounded from below, i.e. that given any point $x \in X$, either $\partial\phi_k(x)$ is larger than a fixed constant, or x lies at more than a fixed distance from R_k, or x lies close to a point of R_k where $\partial(f_{k|R_k}))$ is larger than a fixed constant. Since this transversality property is local and open, we can obtain it by successive small localized perturbations, as for the previous properties.

Fix a point $x \in X$, and assume that $\partial\phi_k(x)$ is small (otherwise no perturbation is needed). By property (5), we know that necessarily (s_k^0, s_k^1) is bounded away from zero at x ; a rotation in the first two coordinates makes it possible to assume that $s_k^1(x) = 0$ and s_k^0 is bounded from below near x. As above, we replace f_k by the $\mathbb{C}^2$-valued map $h_k = (h_k^1, h_k^2)$, where $h_k^i = s_k^i / s_k^0$. By assumption, we get that $\partial h_k^1(x)$ is small. This implies in particular that $\mathrm{Jac}(f_k)$ is small at x, and therefore property (3) gives a lower bound on its covariant derivative. Moreover, by property (2) we also have a lower bound on $\partial h_k^2(x)$, which after a suitable rotation can be assumed equal to $\lambda\, dz_k^1$ for some $\lambda \neq 0$. So, as above we can express $\bigwedge^2 \partial f_k$ by looking at its components along $dz_k^1 \wedge dz_k^j$ for $2 \le j \le n$; we again define the 2-forms $\mu_k^j = \partial h_k^2 \wedge \partial(z_k^j s_{k,x}^{\mathrm{ref}} / s_k^0)$, and the functions $u_2, \dots, u_n$ are defined as previously. Define a $(n,0)$-form θ over a neighborhood of x by $\theta = \partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2$: at points of R_k, the vanishing of θ is equivalent to that of $\partial h^2_{k|R_k}$, or equivalently to that of $\partial f_{k|R_k}$. So our aim is to show that the quantity $(\partial h_k^1, \theta)$, which is a section of a rank $n+1$ bundle $\mathcal{E}_0$ near x, can be made bounded from below by a small perturbation.

For this purpose, we first show the existence of complex-valued polynomials (P_j^1, P_j^2) and local sections ϵ_j of $\mathcal{E}_0$, $1 \le j \le n+1$, such that :

(a) for any coefficients $w_j \in \mathbb{C}$, replacing the given sections of $L^{\otimes k}$ by $(s_k^0, s_k^1 + \sum w_j P_j^1 s_{k,x}^{\mathrm{ref}}, s_k^2 + \sum w_j P_j^2 s_{k,x}^{\mathrm{ref}})$ affects $(\partial h_k^1, \theta)$ by the addition of $\sum w_j \epsilon_j + O(w_j^2)$;

(b) the sections ϵ_j define a local frame in $\mathcal{E}_0$, and $\epsilon_1 \wedge \dots \wedge \epsilon_{n+1}$ is bounded from below by a universal constant.

First observe that, by property (3), $\partial u_2 \wedge \dots \wedge \partial u_n$ is bounded from below near x, whereas we may assume that $\theta = \partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2$ is small (otherwise no perturbation is needed). Therefore, ∂h_k^2 (which at x is colinear to dz_k^1) lies close to the span of the ∂u_j. In particular, after a suitable rotation in the $n-1$ last coordinates on X, we can assume that $\partial u_2 \wedge \partial h_k^2$ is small at x. On the other hand, we know that there exists

$j_0 \neq 1$ such that $dz_k^{j_0}$ lies far from the span of the $\partial u_j(x)$. We then define $P^1_{n+1} = z_k^2 z_k^{j_0}$ and $P^2_{n+1} = 0$. Adding to s_k^1 a quantity of the form $w\, z_k^2 z_k^{j_0}\, s_{k,x}^{\mathrm{ref}}$ does not affect $\partial h_k(x)$, but affects $\partial u_2(x)$ by the addition of a non-trivial multiple of $dz_k^{j_0}$, and similarly affects $\partial u_{j_0}(x)$ by the addition of a non-trivial multiple of dz_k^2. The other $\partial u_j(x)$ are not affected. Therefore, $\theta(x)$ changes by an amount of

$$cw\, dz_k^{j_0} \wedge \partial u_3 \wedge \cdots \wedge \partial u_n \wedge \partial h_k^2 + c' w\, \partial u_2 \wedge \cdots \wedge dz_k^2 \wedge \cdots \wedge \partial u_n \wedge \partial h_k^2 + O(w^2),$$

where the constants c and c' are bounded from above and below. The first term is bounded from below by construction, while the second term is only present if $j_0 \neq 2$ (this requires $n \geq 3$), and in that case it is small because $\partial u_2 \wedge \partial h_k^2$ is small. Therefore, the local section ϵ_{n+1} of $\mathcal{E}_0$ naturally corresponding to such a perturbation is of the form $(0, \epsilon'_{n+1})$ at x, where ϵ'_{n+1} is bounded from below.

Next, for $1 \leq j \leq n$ we define $P_j^1 = z_k^j$ and $P_j^2 = 0$, and observe that adding $w\, z_k^j\, s_{k,x}^{\mathrm{ref}}$ to s_k^1 affects $\partial h_k^1(x)$ by adding a nontrivial multiple of dz_k^j. Therefore, the local section of $\mathcal{E}_0$ corresponding to this perturbation is at x of the form $\epsilon_j(x) = (c'' dz_k^j, \epsilon'_j)$, where c'' is a constant bounded from below.

It follows from this argument that the chosen perturbations P_j^1 and P_j^2 for $1 \leq j \leq n+1$, and the corresponding local sections ϵ_j of $\mathcal{E}_0$, satisfy the conditions (a) and (b) expressed above. Observe that, because ϵ_j define a local frame at x and $\epsilon_1 \wedge \cdots \wedge \epsilon_{n+1}$ is bounded from below at x, the same properties remain true over a ball of fixed radius around x.

Now that a local approximately holomorphic frame in $\mathcal{E}_0$ is given, we can write $(\partial h_k^1, \theta)$ in the form $\sum \zeta_j \epsilon_j$ for some complex-valued functions ζ_j ; it is easy to check that these functions are asymptotically holomorphic. Therefore, we can again use Theorem 12 of [10] to obtain, if k is large enough, the existence of constants $w_1, \ldots, w_{n+1}$ smaller than any given bound $\delta > 0$ and such that $(\zeta_1 - w_1, \ldots, \zeta_{n+1} - w_{n+1})$ is bounded from below by $\eta = \delta(\log \delta^{-1})^{-p}$ (p is a fixed constant) over a small ball centered at x. Letting $\tilde{s}_k = (s_k^0, s_k^1 - \sum w_j P_j^1 s_{k,x}^{\mathrm{ref}}, s_k^2 - \sum w_j P_j^2 s_{k,x}^{\mathrm{ref}})$ and calling $\tilde{f}_k$, $\tilde{h}_k$ and $\tilde{\theta}$ the projective map defined by $\tilde{s}_k$ and the corresponding local maps, we get that $(\partial \tilde{h}_k^1, \tilde{\theta})$ is by construction bounded from below by $c_0 \eta$, for a fixed constant c_0 ; indeed, observe that the non-linear term $O(w^2)$ in the perturbation formula does not play any significant role, as it is at most of the order of $\delta^2 \ll \eta$. Since the perturbation of s_k has exponential decay away from x, we can apply the standard globalization argument to obtain uniform transversality to the stratum $S_0 \subset \mathcal{J}^2(X, \mathbb{CP}^2)$ everywhere.

Step 3b. We now obtain uniform transversality to the stratum $S : \{\mathrm{Jac}(f) = 0,\ \partial(\phi_{|R}) = 0\}$. The strategy and notations are the same as above. We again fix a point $x \in X$, and assume that x lies close to a point of R_k where $\partial(\phi_{k|R_k})$ is small (otherwise, no perturbation is needed). As above, we can assume that $s_k^0(x)$ is bounded from below and define a $\mathbb{C}^2$-valued map h_k. Two cases can occur : either $\partial h_k^1(x)$ is bounded away from zero, or it is small and in that case by Step 3a we know that $\partial(h^2_{k|R_k})$ is bounded from below near x.

We start with the case where ∂h_k^1 is bounded from below; in other words, we are not dealing with tangency points but only with cusps. In that case, we can use an argument similar to Step 3a, except that the roles of the two components of h_k are reversed. Namely, after a rotation we assume that $\partial h_k^1(x) = \lambda dz_k^1$ for some nonzero constant λ, and we define components $u_2, \dots, u_n$ of $\mathrm{Jac}(f_k)$ as previously (using ∂h_k^1 rather than ∂h_k^2 to define the μ_k^j). Let $\theta = \partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^1$: along R_k, the ratio between θ and $\partial(h^1_{k|R_k})$, or equivalently $\partial(\phi_{k|R_k})$, is bounded between two fixed constants, so the transverse vanishing of θ is what we are trying to obtain. More precisely, our aim is to show that the quantity $(u_2, \dots, u_n, \theta)$, which is a section of a rank n bundle $\mathcal{E}$ near x, can be made uniformly transverse to 0 by a small perturbation.

For this purpose, we first show the existence of complex-valued polynomials (P_j^1, P_j^2) and local sections ϵ_j of $\mathcal{E}$, $2 \le j \le n+1$, such that :

(a) for any coefficients $w_j \in \mathbb{C}$, replacing the given sections of $L^{\otimes k}$ by $(s_k^0, s_k^1 + \sum w_j P_j^1 s^{\mathrm{ref}}_{k,x}, s_k^2 + \sum w_j P_j^2 s^{\mathrm{ref}}_{k,x})$ affects $(u_2, \dots, u_n, \theta)$ by the addition of $\sum w_j \epsilon_j + O(w_j^2)$;

(b) the sections ϵ_j define a local frame in $\mathcal{E}$, and $\epsilon_2 \wedge \dots \wedge \epsilon_{n+1}$ is bounded from below by a universal constant.

By the same argument as in Step 3a, we find after a suitable rotation an index $j_0 \neq 1$ such that, letting $P_{n+1}^1 = 0$ and $P_{n+1}^2 = z_k^2 z_k^{j_0}$, the corresponding local section ϵ_{n+1} of $\mathcal{E}$ is, at x, of the form $(0, \dots, 0, \epsilon'_{n+1})$, with ϵ'_{n+1} bounded from below by a fixed constant.

Moreover, adding $w\, z_k^j s^{\mathrm{ref}}_{k,x}$ to s_k^2 amounts to adding w to u_j and does not affect the other u_i's, by the argument in Step 2. So, letting $P_j^1 = 0$ and $P_j^2 = z_k^j$, we get that the corresponding local sections of $\mathcal{E}$ are of the form $\epsilon_j = (0, \dots, 1, \dots, 0, \epsilon'_j)$, where the coefficient 1 is in j-th position.

So it is easy to check that both conditions (a) and (b) are satisfied by these perturbations. The rest of the argument is as in Step 3a : expressing $(u_2, \dots, u_n, \theta)$ as a linear combination of $\epsilon_2, \dots, \epsilon_{n+1}$, one uses Theorem 12 of [10] to obtain transversality to 0 over a small ball centered at x.

We now consider the second possibility, namely the case where $\partial h_k^1(x)$ is small, which corresponds to tangency points. By property (2) we know that $\partial h_k^2(x)$ is bounded from below, and we can assume that it is colinear to dz_k^1. We then define components $u_2, \dots, u_n$ of $\mathrm{Jac}(f_k)$ as usual (as in Step 3a and unlike the previous case, the μ_k^j are defined using ∂h_k^2 rather than ∂h_k^1). Letting $\theta = \partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^1$, we want as before to obtain the transversality to 0 of the quantity $(u_2, \dots, u_n, \theta)$, which is a local section of a rank n bundle $\mathcal{E}$ near x. For this purpose, as usual we look for polynomials P_j^1, P_j^2 and local sections ϵ_j satisfying the same properties (a) and (b) as above.

In order to construct P_{n+1}^i, observe that, by the result of Step 3a, the quantity $\partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2$ is bounded from below at x. So, adding to s_k^1 a small multiple of s_k^2 does not affect the u_j's, but it affects θ non-trivially. However, this perturbation is not localized, so it is not suitable for our purposes (we can't apply the globalization argument). Instead, let P_{n+1}^1 be a polynomial of degree 2 in the coordinates z_k^j and their complex conjugates, such that $P_{n+1}^1 s^{\mathrm{ref}}_{k,x}$ coincides with s_k^2 up to order two at x. Note that the coefficients of

P^1_{n+1} are bounded by uniform constants, and that its antiholomorphic part is at most of the order $O(k^{-1/2})$ (because s^2_k and $s^{\text{ref}}_{k,x}$ are asymptotically holomorphic); therefore, $P^1_{n+1}s^{\text{ref}}_{k,x}$ is an admissible localized asymptotically holomorphic perturbation. Also, define $P^2_{n+1} = 0$. Then one easily checks that the local section ϵ_{n+1} of $\mathcal{E}$ corresponding to P^1_{n+1} and P^2_{n+1} is, at x, of the form $(0, \dots, 0, \epsilon'_{n+1})$, where ϵ'_{n+1} is bounded from below.

Moreover, let $P^1_j = z^j_k$ and $P^2_j = 0$: as above, this perturbation affects u_j and not the other u_i's, and we get that the corresponding local sections of $\mathcal{E}$ are of the form $\epsilon_j = (0, \dots, 1, \dots, 0, \epsilon'_j)$, where the coefficient 1 is in j-th position.

Once again, these perturbations satisfy both conditions (a) and (b). Therefore, expressing $(u_2, \dots, u_n, \theta)$ as a linear combination of $\epsilon_2, \dots, \epsilon_{n+1}$, Theorem 12 of [10] yields transversality to 0 over a small ball centered at x by the usual argument. Now that both possible cases have been handled, we can apply the standard globalization argument to obtain uniform transversality to the stratum $S \subset \mathcal{J}^2(X, \mathbb{CP}^2)$. This gives properties (4) and (6) of Definition 3.2.

Step 4. Now that all required transversality properties have been obtained, we perform further perturbations in order to achieve the other conditions in Definition 3.2. These new perturbations are bounded by a fixed multiple of $k^{-1/2}$, so the transversality properties are not affected. The argument is almost the same as in the case of 4-manifolds (see §4 of [3] and §3.1 of [5]); the adaptation to the higher-dimensional case is very easy.

One first defines a suitable almost-complex structure $\tilde{J}_k$, by the same argument as in §4.1 of [3] (except that one also considers the points of $\mathcal{T}_k$ and $\mathcal{I}_k$ besides the cusps). As explained in §4.1 of [3], a suitable perturbation makes it possible to obtain the local holomorphicity of f_k near these points, which yields conditions (4′), (6′) and (8′) ; the argument is the same in all three cases. Next, a generically chosen small perturbation yields the self-transversality of D (property (7)). Finally, as described in §4.2 of [3], a suitable perturbation yields property (3′) along the branch curve without modifying R_k and D_k and without affecting the other compatibility properties. This completes the proof of the existence statement in Theorem 4.1.

Uniqueness. The uniqueness statement is obtained by showing that, provided that k is large enough, the whole argument extends to the case of families of sections depending continuously on a parameter $t \in [0, 1]$. Then, given two sequences of quasiholomorphic maps, one can start with a one-parameter family of sections interpolating between them in a trivial way and perturb it in such a way that the required properties hold for all parameter values (with the exception of (7) when a node cancellation occurs). If one moreover checks that the construction can be performed in such a way that the two end points of the one-parameter family are not affected by the perturbation, the isotopy result becomes an immediate corollary. Observe that, in the one-parameter construction, the almost-complex structure is allowed to depend on t.

Most of the above argument extends to 1-parameter families in a straightforward manner, exactly as in the four-dimensional case ; the key observation is that all the standard building blocks (existence of approximately holomorphic Darboux coordinates z^j_k and of localized approximately holomorphic sections $s^{\mathrm{ref}}_{k,x}$, local transversality result, globalization principle, ...) remain valid in the parametric case, even when the almost-complex structure depends on t. The only places where the argument differs from the case of 4-manifolds are properties (3), (4) and (6), obtained in Steps 2 and 3 above.

For property (3), one easily checks that it is still possible in the parametric case to assume, after composing with suitable rotations depending continuously on the parameter t, that $s^1_k(x) = s^2_k(x) = 0$ and that $\partial h^1_k(x)$ is bounded from below and directed along dz^1_k. This makes it possible to define μ^j_k and u^j_k as in the non-parametric case, and the parametric version of Theorem 12 of [10] yields a suitable perturbation depending continuously on t.

The argument of Step 3a also extends to the parametric case, using the following observation. Fix a point $x \in X$, and let $\rho_k(t) = |\partial\phi_{k,t}(x)|$. For all values of t such that $\rho_k(t)$ is small enough (smaller than a fixed constant $\alpha > 0$), we can perform the construction as in the non-parametric case, defining $u_{j,t}$ and θ_t. If $\rho'_k(t) = |\theta_t(x)|$ is small enough (smaller than α), then we can apply the same argument as in the non-parametric case to define polynomials $(P^1_{j,t}, P^2_{j,t})$ and local sections $\epsilon_{j,t}$ of $\mathcal{E}_0$. However the definition of P^1_{n+1} needs to be modified as follows. Although it is still possible after a suitable rotation depending continuously on t to assume that $\partial u_2 \wedge \partial h^2_k(x)$ is small, the choice of an index $j_0 \neq 1$ such that $dz^{j_0}_k$ lies far from the span of the $\partial u_j(x)$ may depend on t. Instead, we define $\nu_{k,t}$ as a unit vector in $\mathbb{C}^{n-1}$ depending continuously on t and such that $\sum_{j=2}^n \nu^j_{k,t}\, dz^j_k$ lies far from the span of $\partial u_j(x)$, and let $P^1_{n+1,t} = \sum_{j=2}^n \nu^j_{k,t} z^2_k z^j_k$. Then the required properties are satisfied, and we can proceed with the argument. So, provided that $\rho_k(t)$ and $\rho'_k(t)$ are both smaller than α, we can use Theorem 12 of [10] to obtain a localized perturbation $\tau_{k,t}$ depending continuously on t and such that $s_{k,t} + \tau_{k,t}$ satisfies the desired transversality property near x.

In order to obtain a well-defined perturbation for all values of t, we introduce a continuous cut-off function $\beta : \mathbb{R}_+ \to [0,1]$ which equals 1 over $[0, \alpha/2]$ and vanishes outside of $[0, \alpha]$. Then, we set $\tilde\tau_{k,t} = \beta(\rho_k(t))\beta(\rho'_k(t))\tau_{k,t}$, which is well-defined for all t and depends continuously on t. Since $s_{k,t} + \tilde\tau_{k,t}$ coincides with $s_{k,t} + \tau_{k,t}$ when $\rho_k(t)$ and $\rho'_k(t)$ are smaller than $\alpha/2$, the required transversality holds for these values of t ; moreover, for the other values of t we know that the 2-jet of $s_{k,t}$ already lies at distance more than $\alpha/2$ from the stratum S_0, and we can safely assume that $\tilde\tau_{k,t}$ is much smaller than $\alpha/2$, so the perturbation does not affect transversality. Therefore we obtain a well-defined local perturbation for all $t \in [0,1]$, and the one-parameter version of the result of Step 3a follows by the standard globalization argument.

The argument of Step 3b is extended to one-parameter families in the same way : given a point $x \in X$, the same ideas as for Step 3a yield, for all values of the parameter t such that the 2-jet of $s_{k,t}$ at x lies close to the stratum S, small localized perturbations $\tau_{k,t}$ depending continuously on t and such that $s_{k,t} + \tau_{k,t}$ satisfies the desired property over a small ball centered at x. As seen above, two different types of formulas for $\tau_{k,t}$ arise depending on which component of the stratum S is being hit; however, the result of Step

3a implies that, in any interval of parameter values such that the jet of $s_{k,t}$ remains close to S, only one of the two components of S has to be considered, so $\tau_{k,t}$ indeed depends continuously on t. The same type of cut-off argument as for Step 3a then makes it possible to extend the definition of $\tau_{k,t}$ to all parameter values and complete the proof. □

4.2. The topology of quasiholomorphic maps

We now describe the topological features of quasiholomorphic maps and the local models which characterize them near the critical points.

Proposition 4.2. *Let $f_k : X - Z_k \to \mathbb{CP}^2$ be a sequence of quasiholomorphic maps. Then the fibers of f_k are codimension 4 symplectic submanifolds, intersecting at the set of base points Z_k, and smooth away from the critical curve $R_k \subset X$. The submanifolds R_k and Z_k of X are smooth and symplectic, and the image $f_k(R_k) = D_k$ is a symplectic braided curve in $\mathbb{CP}^2$.*

Moreover, given any point $x \in R_k$, there exist local approximately holomorphic coordinates on X near x and on $\mathbb{CP}^2$ near $f_k(x)$ in which f_k is topologically conjugate to one of the two following models :

(i) $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$ *(points where $f_{k|R_k}$ is an immersion) ;*

(ii) $(z_1, \dots, z_n) \mapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$ *(near the cusp points).*

Proof. The smoothness and symplecticity properties of the various submanifolds appearing in the statement follow from the observation made by Donaldson in [8] that the zero sets of approximately holomorphic sections satisfying a uniform transversality property are smooth and approximately J-holomorphic, and therefore symplectic. In particular, the smoothness and symplecticity of the fibers of f_k away from R_k follow immediately from Definition 3.2 : since $\text{Jac}(f_k)$ is bounded from below away from R_k (because it satisfies a uniform transversality property), and since the sections s_k are asymptotically holomorphic, it is easy to check that the level sets of f_k are, away from R_k, smooth symplectic submanifolds. Symplecticity near the singular points is an immediate consequence of the local models (i) and (ii) that we will obtain later in the proof.

The corresponding properties of Z_k and R_k are obtained by the same argument : Z_k and R_k are the zero sets of asymptotically holomorphic sections, both satisfying a uniform transversality property (by conditions (1) and (3) of Definition 3.2, respectively), so they are smooth and symplectic.

We now study the local models at critical points of f_k. We start with the case of a cusp point $x \in X$. By property (2) of Definition 3.2, ∂f_k has complex rank 1 at x, so we can find local complex coordinates (Z_1, Z_2) on $\mathbb{CP}^2$ near $f_k(x)$ such that $\text{Im}\, \partial f_k(x)$ is the Z_2 axis. Pulling back Z_2 via the map f_k, we obtain, using property (4′), a $\tilde{J}_k$-holomorphic function whose differential does not vanish near x ; therefore, we can find a $\tilde{J}_k$-holomorphic coordinate chart $(z_1, \dots, z_n)$ on X at x such that $z_n = Z_2 \circ f_k$. In the chosen coordinates, we get $f_k(z_1, \dots, z_n) = (g(z_1, \dots, z_n), z_n)$, where g is holomorphic and $\partial g(0) = 0$.

Since x is by assumption a cusp point, the tangent direction to R_k at x lies in the kernel of $\partial f_k(0)$, i.e. in the span of the $n-1$ first coordinate axes ; after a suitable rotation we may assume that $T_x R_k$ is the z_1 axis. Near the origin, $\mathrm{Jac}(f_k)$ is characterized by its $n-1$ components $(\partial g/\partial z_1, \dots, \partial g/\partial z_{n-1})$, and the critical curve R_k is the set of points where these quantities vanish. Therefore, at the origin, $\partial^2 g/\partial z_1^2 = \partial^2 g/\partial z_1 \partial z_2 = \dots = \partial^2 g/\partial z_1 \partial z_{n-1} = 0$. Nevertheless, $\mathrm{Jac}(f_k)$ vanishes transversely to 0 at the origin, so the matrix of second derivatives $M = (\partial^2 g/\partial z_i \partial z_j(0))$, $2 \le i \le n$, $1 \le j \le n-1$, is non-degenerate (invertible) at the origin. In particular, the first column of M (corresponding to $j=1$) is non-zero, and therefore $\partial^2 g/\partial z_1 \partial z_n(0)$ is necessarily non-zero ; after a suitable rescaling of the coordinates we may assume that this coefficient is equal to 1. Moreover, the invertibility of M implies that the submatrix $M' = (\partial^2 g/\partial z_i \partial z_j(0))$, $2 \le i, j \le n-1$ is also invertible, i.e. it represents a non-degenerate quadratic form.

Diagonalizing this quadratic form, we can assume after a suitable linear change of coordinates that the diagonal coefficients of M' are equal to 2 and the others are zero. Therefore g is of the form $g(z_1, \dots, z_n) = z_1 z_n + \sum_{j=2}^{n-1} z_j^2 + \sum_{j=2}^{n-1} \alpha_j z_j z_n + O(z^3)$. Changing coordinates on X to replace z_j by $z_j + \frac{1}{2}\alpha_j z_n$ for all $2 \le j \le n-1$, and on $\mathbb{CP}^2$ to replace Z_1 by $Z_1 + \frac{1}{4}\sum \alpha_j^2 Z_2^2$, we can ensure that $g(z_1, \dots, z_n) = z_1 z_n + \sum_{j=2}^{n-1} z_j^2 + O(z^3)$.

Observe that R_k is described near the origin by expressing the coordinates $z_2, \dots, z_n$ as functions of z_1. By assumption the expressions of $z_2, \dots, z_n$ are all of the form $O(z_1^2)$. Substituting into the formula for $\mathrm{Jac}(f_k)$, and letting $g_{ijk} = \partial^3 g/\partial z_i \partial z_j \partial z_k(0)$, we get that local equations of R_k near the origin are $z_j = -\frac{3}{2} g_{j11} z_1^2 + O(z_1^3)$ for $2 \le j \le n-1$, and $z_n = -3 g_{111} z_1^2 + O(z_1^3)$. It follows that $f_{k|R_k}$ is locally given in terms of z_1 by the map $z_1 \mapsto (-2 g_{111} z_1^3 + O(z_1^4), -3 g_{111} z_1^2 + O(z_1^3))$. Therefore, the transverse vanishing of $\partial(f_{k|R_k})$ at the origin implies that $g_{111} \neq 0$, so after a suitable rescaling we may assume that the coefficient of z_1^3 in the power series expansion of g is equal to one.

On the other hand, suitable coordinate changes can be used to kill all other degree 3 terms in the expansion of g : if $2 \le i \le n-1$ the coefficient of $z_i z_j z_k$ can be made zero by replacing z_i by $z_i + \frac{c}{2} z_j z_k$; similarly for z_n^3 (replace Z_1 by $Z_1 + c Z_2^3$), $z_1 z_n^2$ and $z_1^2 z_n$ (replace z_1 by $z_1 + c z_n^2 + c' z_1 z_n$). So we get that $f_k(z_1, \dots, z_n) = (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2 + O(z^4), z_n)$. It is then a standard result of singularity theory that the higher order terms can be absorbed by suitable coordinate changes (see e.g. [1]).

We now turn to the case of where x is a point of R_k which does not lie close to any of the cusp points. Conditions (2) and (3′) imply that the differential of f_k at x has real rank 2 and that its image lies close to a complex line in the tangent plane to $\mathbb{CP}^2$ at $f_k(x)$. Therefore, there exist local approximately holomorphic coordinates (Z_1, Z_2) on $\mathbb{CP}^2$ such that $\mathrm{Im}\,\nabla f_k(x)$ is the Z_2 axis. Moreover, because $Z_2 \circ f_k$ is an approximately holomorphic function whose derivative at x satisfies a uniform lower bound, it remains possible to find local approximately holomorphic coordinates $z_1, \dots, z_n$ on X such that $z_n = Z_2 \circ f_k$. As before, we can write $f_k(z_1, \dots, z_n) = (g(z_1, \dots, z_n), z_n)$, where g is an approximately holomorphic function such that $\nabla g(0) = 0$.

By assumption f_k restricts to R_k as an immersion at x, so the projection to the z_n axis of T_xR_k is non-trivial. In fact, property (4) implies that, if $\partial(f_{k|R_k})$ is very small at x, then a cusp point lies nearby ; so we can assume that the z_n component of T_xR_k is larger than some fixed constant. As a consequence, one can show that R_k is locally given by equations of the form $z_j = h_j(z_n)$, where the functions h_j are approximately holomorphic and have bounded derivatives. Therefore, a suitable change of coordinates on X makes it possible to assume that R_k is locally given by the equations $z_1 = \cdots = z_{n-1} = 0$. Similarly, a suitable approximately holomorphic change of coordinates on $\mathbb{CP}^2$ makes it possible to assume that $f_k(R_k)$ is locally given by the equation $Z_1 = 0$.

As a consequence, we have that $g_{|R_k} = 0$ and, since the image of ∇f_k at a point of R_k coincides with the tangent space to $f_k(R_k)$, ∇g vanishes at all points of R_k. In particular this implies that $\partial^2 g/\partial z_j \partial z_n(0) = 0$ for all $1 \le j \le n$. Moreover, property (3) implies that $\mathrm{Jac}(f_k)$ vanishes transversely at the origin, and therefore that the matrix $(\partial^2 g/\partial z_i \partial z_j(0))$, $1 \le i, j \le n-1$ is invertible, i.e. it represents a non-degenerate quadratic form. This quadratic form can be diagonalized by a suitable change of coordinates ; because the transversality property (3) is uniform, the coefficients are bounded between fixed constants. After a suitable rescaling, we can therefore assume that $\partial^2 g/\partial z_i \partial z_j(0)$ is equal to 2 if $i = j$ and 0 otherwise.

In conclusion, we get that $g(z_1, \dots, z_n) = z_1^2 + \cdots + z_{n-1}^2 + h(z_1, \dots, z_n)$, where h is the sum of a holomorphic function which vanishes up to order 3 at the origin and of a non-holomorphic function which vanishes up to order 2 at the origin and has derivatives bounded by $O(k^{-1/2})$.

Let z be the column vector $(z_1, \dots, z_{n-1})$, and denote by $\mathbf{z}$ the vector $(z_1, \dots, z_n)$. Using the fact that g vanishes up to order 2 along R_k, we conclude that there exist matrix-valued functions α, β and γ with the following properties :

(*a*) $g(\mathbf{z}) = {}^t z\alpha(\mathbf{z})z + {}^t\bar{z}\beta(\mathbf{z})z + {}^t\bar{z}\gamma(\mathbf{z})\bar{z}$; (α and γ are symmetric) ;

(*b*) α is approximately holomorphic and has uniformly bounded derivatives ; $\alpha(0) = I$;

(*c*) β and γ and their derivatives are bounded by fixed multiples of $k^{-1/2}$.

The implicit function theorem then makes it possible to construct a C^∞ approximately holomorphic change of coordinates of the form $z \mapsto \lambda(\mathbf{z})z + \mu(\mathbf{z})\bar{z}$ (with $\lambda(0)$ orthogonal, λ approximately holomorphic, $\mu = O(k^{-1/2})$), such that g becomes of the form $g(\mathbf{z}) = {}^t zz + {}^t\bar{z}\tilde{\gamma}(\mathbf{z})\bar{z}$.

Unfortunately, smooth coordinate changes are not sufficient to further simplify this expression; instead, in order to obtain the desired local model one must use as coordinate change an "approximately holomorphic homeomorphism", which is smooth away from R_k but admits only directional derivatives at the points of R_k. More precisely, starting from $g = {}^t zz + h$ and using that $h/|z|^2$ is bounded by $O(k^{-1/2}) + O(\mathbf{z})$, we can write

$$g(\mathbf{z}) = \sum_{j=1}^{n-1} \tilde{z}_j^2, \qquad \tilde{z}_j = z_j \Big(1 + \frac{\bar{z}_j}{z_j} \frac{h(\mathbf{z})}{|z|^2}\Big)^{1/2}.$$

This gives the desired local model and ends the proof. □

Remark 4.1. The local model at points of R_k only holds topologically (up to an approximately holomorphic homeomorphism), which is not fully satisfactory. However, by replacing ($3'$) by a stronger condition, it is possible to obtain the same result in smooth approximately holomorphic coordinates. This new condition can be formulated as follows. Away from the cusp points, the complex lines $(\mathrm{Im}\,\partial f_k)^\perp$ define a line bundle $V \subset T\mathbb{CP}^2_{|D_k}$, everywhere transverse to TD_k. A neighborhood of the zero section in V can be sent via the exponential map of the Fubini-Study metric onto a neighborhood of D_k (away from the cusps), in such a way that each fiber V_x is mapped holomorphically to a subset $\mathcal{V}_x$ contained in a complex line in $\mathbb{CP}^2$.

Lifting back to a neighborhood of R_k in X, we can define slices $\mathcal{W}_x = f_k^{-1}(\mathcal{V}_{f_k(x)})$ for all $x \in R_k$ lying away from $\mathcal{C}_k$. It is then possible to identify a neighborhood of R_k (away from $\mathcal{C}_k$) with a neighborhood of the zero section in the vector bundle W whose fiber at $x \in R_k$ is $\mathrm{Ker}\,\partial f_k(x)$, in such a way that each fiber W_x gets mapped to $\mathcal{W}_x$. Observe moreover that, since W_x is a complex subspace in $(T_xX, \tilde{J}_k)$, W is endowed with a natural complex structure induced by $\tilde{J}_k$. It is then possible to ensure that the "exponential map" from W_x to $\mathcal{W}_x$ is approximately $\tilde{J}_k$-holomorphic for every x, and, using condition ($4'$), holomorphic when x lies at distance less than $\delta/2$ from a cusp point.

With this setup understood, and composing on both sides with the exponential maps, f_k induces a fiber-preserving map ψ_k between the bundles W and V ; this map is approximately holomorphic everywhere, and holomorphic at distance less than $\delta/2$ from $\mathcal{C}_k$. The condition which we impose as a replacement of ($3'$) is that ψ_k should be fiberwise holomorphic over a neighborhood of the zero section in W.

The proof of existence of quasiholomorphic maps satisfying this strengthened condition follows a standard argument : trivializing locally V and W for each value of k, and given asymptotically holomorphic maps ψ_k, Lemma 8 of [3] (see also [8]) implies the existence of a fiberwise holomorphic map $\tilde{\psi}_k$ differing from ψ_k by $O(k^{-1/2})$ over a neighborhood of the zero section. It is moreover easy to check that $\tilde{\psi}_k = \psi_k$ near the cusp points. So, in order to obtained the desired property, we introduce a smooth cut-off function and define a map $\hat{\psi}_k$ which equals $\tilde{\psi}_k$ near the zero section and coincides with ψ_k beyond a certain distance. Going back through the exponential maps, we obtain a map $\hat{f}_k$ which differs from f_k by $O(k^{-1/2})$ and coincides with f_k outside a small neighborhood of R_k and near the cusp points. The corresponding perturbations of the asymptotically holomorphic sections $s_k \in \Gamma(\mathbb{C}^3 \otimes L^{\otimes k})$ are easy to construct. Moreover, we can always assume that $\tilde{\psi}_k$ and ψ_k coincide at order 1 along the zero section, i.e. that $\hat{f}_k$ and f_k coincide up to order 1 along the branch curve ; therefore, the branch curve of $\hat{f}_k$ and its image are the same as for f_k, and so all properties of Definition 3.2 hold for $\hat{f}_k$.

Once this condition is satisfied, getting the correct local model at a point $x \in R_k$ in smooth approximately holomorphic coordinates is an easy task. Namely, we can define, near $f_k(x)$, local approximately holomorphic coordinates Z_2 on D_k and Z_1 on the fibers of V (Z_1 is a complex linear function on each fiber, depending approximately holomorphically on Z_2). Using the exponential map, we can use (Z_1, Z_2) as local coordinates on $\mathbb{CP}^2$. Lifting Z_2 via $\hat{f}_k$ yields a local coordinate z_n on R_k near x. Moreover, we can locally define complex linear coordinates $z_1, \dots, z_{n-1}$ in the fibers of W, depending approximately holomorphically on z_n. Using again the exponential map, $(z_1, \dots, z_n)$ define local approximately holomorphic coordinates on X. Then, by construction, local equations are $z_1 = \dots = z_{n-1} = 0$ for R_k and $Z_1 = 0$ for D_k, and f_k is given by $f_k(z_1, \dots, z_n) = (\psi_k(z_1, \dots, z_n), z_n)$. Moreover, we know that ψ_k is, for each value of z_n, a holomorphic function of $z_1, \dots, z_{n-1}$, vanishing up to order 2 at the origin. We can then use the argument in the proof of Proposition 4.2 to obtain the expected local model in smooth approximately holomorphic coordinates.

4.3. Monodromy invariants of quasiholomorphic maps

We now look at the monodromy invariants naturally arising from quasiholomorphic maps to $\mathbb{CP}^2$. Let $f : X - Z \to \mathbb{CP}^2$ be one of the maps constructed in Theorem 3.1 for large enough k. The fibers of f are singular along the smooth symplectic curve $R \subset X$, whose image in $\mathbb{CP}^2$ is a symplectic braided curve. Therefore, we obtain a first interesting invariant by considering the critical curve $D \subset \mathbb{CP}^2$.

As in the four-dimensional case, using the projection $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ we can describe the topology of D by a braid monodromy map

$$\rho_n : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d, \tag{6}$$

where $p_1, \dots, p_r$ are the images by π of the cusps, nodes and tangency points of D, and $d = \deg D$. Alternately, we can also express this monodromy as a braid group factorization

$$\Delta^2 = \prod_{j=1}^{r} Q_j^{-1} X_1^{r_j} Q_j. \tag{7}$$

Like in the four-dimensional case, this braid factorization completely characterizes the curve D up to isotopy, but it is only well-defined up to simultaneous conjugation and Hurwitz equivalence.

We now turn to the second part of the problem, namely describing the topology of the map $f : X - Z \to \mathbb{CP}^2$ itself. As in the case of Lefschetz pencils, we blow up X along Z in order to obtain a well-defined map $\hat{f} : \hat{X} \to \mathbb{CP}^2$. The fibers of $\hat{f}$ are naturally identified with those of f, made mutually disjoint by the blow-up process.

Denote by Σ^{2n-4} the generic fiber, i.e. the fiber above a point of $\mathbb{CP}^2 - D$. The structure of the singular fibers of $\hat{f}$ can be easily understood by looking at the local models obtained in Proposition 4.2. The easiest case is that of the fiber above a smooth point of D. This fiber intersects R transversely in one point, where the local model is $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$, which can be thought of as a one-parameter version of the model map for the singularities of a Lefschetz pencil in dimension $2n-2$. Therefore, as in that case, the singular fiber is obtained by collapsing a vanishing cycle, namely a Lagrangian sphere S^{n-2}, in the generic fiber Σ, and the monodromy of $\hat{f}$ maps a small loop around D to a positive Dehn twist along the vanishing cycle.

The fiber of $\hat{f}$ above a nodal point of D intersects R transversely in two points, and is similarly obtained from Σ by collapsing two disjoint Lagrangian spheres. In fact, the nodal point does not give rise to any specific local model in X, as it simply corresponds to the situation where two points of R happen to lie in the same fiber.

Finally, in the case of a cusp point of D, the local model $(z_1, \dots, z_n) \mapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$ can be used to show that the singular fiber is a "fishtail" fiber, obtained by collapsing two Lagrangian spheres which intersect transversely in one point.

With this understood, the topology of $\hat{f}$ is described by its monodromy around the singular fibers. As in the case of Lefschetz fibrations, the monodromy consists of symplectic automorphisms of Σ preserving the submanifold Z. However, as in §2, defining a monodromy map with values in $\mathrm{Map}^\omega(\Sigma, Z)$ requires a trivialization of the normal bundle of Z, which is only possible over an affine subset $\mathbb{C}^2 \subset \mathbb{CP}^2$. So, the monodromy of $\hat{f}$ is described by a group homomorphism

$$\psi_n : \pi_1(\mathbb{C}^2 - D) \to \mathrm{Map}^\omega(\Sigma, Z). \tag{8}$$

A simpler description can be obtained by restricting oneself to a generic line $L \subset \mathbb{CP}^2$ which intersects D transversely in d points $q_1, \dots, q_d$. In fact, Definition 3.2 implies that we can use the fiber of π above $(0:1)$ for this purpose. As in §3.2, the inclusion $i : \mathbb{C} - \{q_1, \dots, q_d\} \to \mathbb{C}^2 - D$ induces a surjective homomorphism on fundamental groups. The relations between the geometric generators $\gamma_1, \dots, \gamma_d$ of $\pi_1(\mathbb{C}^2 - D)$ are again given by the braid factorization (one relation for each factor) in the same manner as in §3.2. Note that the relation $\gamma_1 \dots \gamma_d = 1$ only holds in $\pi_1(\mathbb{CP}^2 - D)$, not in $\pi_1(\mathbb{C}^2 - D)$.

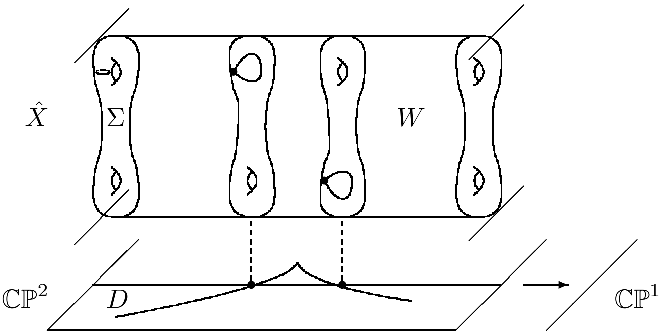
It follows from these observations that the monodromy of $\hat{f}$ can be described by the monodromy morphism

$$\theta_{n-1} : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to \mathrm{Map}^\omega(\Sigma, Z) \tag{9}$$

defined by $\theta_{n-1} = \psi_n \circ i_*$. We know from the above discussion on the structure of $\hat{f}$ near its critical points that θ_{n-1} maps the geometric generators of $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ to positive Dehn twists. Moreover, by considering the normal bundle to the exceptional divisor in $\hat{X}$ one easily checks that the monodromy around infinity is again a twist along Z in Σ, i.e. $\theta_{n-1}(\gamma_1 \dots \gamma_d) = \delta_Z$.

These properties of θ_{n-1} are strikingly similar to those of the monodromy of a symplectic Lefschetz pencil. In fact, let $W = f^{-1}(L)$ be the preimage of a complex line $L = \mathbb{CP}^1 \subset \mathbb{CP}^2$ intersecting D transversely. Then the restriction of f to the smooth symplectic hypersurface $W \subset X$ endows it with a structure of symplectic Lefschetz pencil with generic fiber Σ and base set Z ; for example, if one chooses $L = \pi^{-1}(0:1)$, then W is the zero set of s_k^0 and the restricted pencil $f_{|W} : W - Z \to \mathbb{CP}^1$ is defined by the two sections s_k^1 and s_k^2. The monodromy of the restricted pencil is, by construction, given by the map θ_{n-1}.

The situation is summarized in the following picture :



Remark 4.2. If a cusp point of D happens to lie close to the chosen line L, then two singular points of the restricted pencil $f_{|W}$ lie close to each other. This is not a problem here, but in general if we want to avoid this situation we need to impose one additional transversality condition on f. Namely, we must require the uniform transversality to 0 of $\partial(f_{|W})$, which is easily obtained by imitating Donaldson's argument from [10]. Another situation in which this property naturally becomes satisfied is the one described in §5.

Given a braided curve $D \subset \mathbb{CP}^2$ of degree d described by a braid factorization as in (7), and given a monodromy map θ_{n-1} as in (9), certain compatibility conditions need to hold between them in order to ensure the existence of a $\mathbb{CP}^2$-valued map with critical curve D and monodromy θ_{n-1}. Namely, θ_{n-1} must factor through $\pi_1(\mathbb{C}^2 - D)$, and the fibration must behave in accordance with the expected models near the special points of D. We introduce the following definition summarizing these compatibility properties :

Definition 4.1. A geometric $(n-1)$-dimensional monodromy representation associated to a braided curve $D \subset \mathbb{CP}^2$ is a group homomorphism θ_{n-1} from the free group $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) = F_d$ to a symplectic mapping class group $\mathrm{Map}^\omega(\Sigma^{2n-4}, Z^{2n-6})$, mapping the geometric generators γ_i (and thus also the $\gamma_i * Q_j$) to positive Dehn twists and such that

$\theta_{n-1}(\gamma_1 \dots \gamma_d) = \delta_Z$,
$\theta_{n-1}(\gamma_1 * Q_j) = \theta_{n-1}(\gamma_2 * Q_j)$ if $r_j = 1$,
$\theta_{n-1}(\gamma_1 * Q_j)$ and $\theta_{n-1}(\gamma_2 * Q_j)$ are twists along disjoint Lagrangian spheres if $r_j = \pm 2$,
$\theta_{n-1}(\gamma_1 * Q_j)$ and $\theta_{n-1}(\gamma_2 * Q_j)$ are twists along Lagrangian spheres transversely intersecting in one point if $r_j = 3$.

As in the four-dimensional case, θ_{n-1} remains unchanged and the compatibility conditions are preserved when the braid factorization defining D is affected by a Hurwitz move. However, when all factors in the braid factorization are simultaneously conjugated by a certain braid $Q \in B_d$, the system of geometric generators $\gamma_1, \dots, \gamma_d$ changes accordingly, and so the geometric monodromy representation θ_{n-1} should be replaced by $\theta_{n-1} \circ Q_*$, where Q_* is the automorphism of F_d induced by the braid Q. For example, conjugating the braid factorization by one of the generating half-twists in B_d affects the monodromy θ_{n-1} of the restricted pencil by a Hurwitz move.

One easily checks that, given a symplectic braided curve $D \subset \mathbb{CP}^2$ and a compatible monodromy representation $\theta_{n-1} : F_d \to \mathrm{Map}^\omega(\Sigma, Z)$, it is possible to recover a compact $2n$-manifold X and a map $f : X - Z \to \mathbb{CP}^2$ in a canonical way up to smooth isotopy. Moreover, it is actually possible to endow X with a symplectic structure, canonically up to symplectic isotopy. Indeed, by first applying Theorem 2.2 to the monodromy map θ_{n-1} we can recover a canonical symplectic structure on the total space W of the restricted Lefschetz pencil ; furthermore, as will be shown in §4.4 below, the braid monodromy of D and the compatible monodromy representation θ_{n-1} determine on X a structure of Lefschetz pencil with generic fiber W and base set Σ, which implies by a second application of Theorem 2.2 that X carries a canonical symplectic structure. The same result can also be obtained more directly, by adapting the statement and proof of Theorem 2.2 to the case of $\mathbb{CP}^2$-valued maps.

As in the four-dimensional case, we can naturally define symplectic invariants arising from the quasiholomorphic maps constructed in Theorem 4.1. However, we again need to take into account the possible presence of negative self-intersections in the critical curves of these maps. Therefore, the braid factorizations we obtain are only canonical up to global conjugation, Hurwitz equivalence, and pair cancellations or creations. As in the four-dimensional case, a pair creation operation (inserting two mutually inverse factors anywhere in the braid factorization) is only allowed if the new factorization remains compatible with the monodromy representation θ_{n-1}, i.e. if θ_{n-1} maps the two corresponding geometric generators to Dehn twists along disjoint Lagrangian spheres.

With this understood, we can introduce a notion of m-equivalence as in Definition 3.5. The following result then holds :

Theorem 4.3. *The braid factorizations and geometric monodromy representations associated to the quasiholomorphic maps to $\mathbb{CP}^2$ obtained in Theorem 4.1 are, for $k \gg 0$, canonical up to m-equivalence (up to a choice of line bundle L when the cohomology class $[\omega]$ is not integral), and define symplectic invariants of (X^{2n}, ω).*

Conversely, the data consisting of a braid factorization and a geometric $(n-1)$-dimensional monodromy representation, or a m-equivalence class of such data, determines a symplectic $2n$-manifold in a canonical way up to symplectomorphism.

Remark 4.3. The invariants studied in this section are a very natural generalization of those defined in §3.2 for 4-manifolds. Namely, when $\dim X = 4$, we naturally get that $Z = \emptyset$ and $\dim \Sigma = 0$, i.e. the generic fiber Σ consists of a finite number of points, as expected for a branched covering map. In particular, the mapping class group $\mathrm{Map}(\Sigma)$ of the 0-manifold Σ is in fact the symmetric group of order $\mathrm{card}(\Sigma)$. Finally, a Lagrangian 0-sphere in Σ is just a pair of points of Σ, and the associated Dehn twist is simply the corresponding transposition. With this correspondence, the results of §3 are the exact four-dimensional counterparts of those described here.

4.4. Quasiholomorphic maps and symplectic Lefschetz pencils

Consider again a symplectic manifold (X^{2n}, ω) and let $f : X - Z \to \mathbb{CP}^2$ be a map with the same topological properties as those obtained by Theorem 4.1 from sections of $L^{\otimes k}$ for k large enough. As in the four-dimensional case, the $\mathbb{CP}^1$-valued map $\pi \circ f$ defines a Lefschetz pencil structure on X, obtained by lifting via f a pencil of lines on $\mathbb{CP}^2$. The base set of this pencil is the fiber of f above the pole $(0\!:\!0\!:\!1)$ of the projection π.

In fact, starting from the quasiholomorphic maps f_k given by Theorem 4.1, the symplectic Lefschetz pencils $\pi \circ f_k$ coincide for $k \gg 0$ with those obtained by Donaldson in [10] and described in §2 ; calling s_k^0, s_k^1, s_k^2 the sections of $L^{\otimes k}$ defining f_k, the Lefschetz pencil $\pi \circ f_k$ is the one induced by the sections s_k^0 and s_k^1.

Therefore, as in the case of a 4-manifold, the invariants described in §4.3 (braid factorization and $(n-1)$-dimensional geometric monodromy representation) completely determine those discussed in §2 (factorizations in mapping class groups). Once again, the topological description of the relation between quasiholomorphic maps and Lefschetz pencils involves a subgroup of θ_{n-1}*-liftable braids* in the braid group, and a group homomorphism from this subgroup to a mapping class group.

Consider a symplectic braided curve $D \subset \mathbb{CP}^2$, described by its braid monodromy $\rho_n : \pi_1(\mathbb{C} - \{p_1, \ldots, p_r\}) \to B_d$, and a compatible $(n-1)$-dimensional monodromy representation $\theta_{n-1} : F_d = \pi_1(\mathbb{C} - \{q_1, \ldots, q_d\}) \to \mathrm{Map}^\omega(\Sigma^{2n-4}, Z^{2n-6})$. Then we can make the following definition :

Definition 4.2. The subgroup $B_d^0(\theta_{n-1})$ of liftable braids is the set of all braids $Q \in B_d$ such that $\theta_{n-1} \circ Q_* = \theta_{n-1}$, where $Q_* \in \mathrm{Aut}(F_d)$ is the automorphism induced by the braid Q on $\pi_1(\mathbb{C} - \{q_1, \ldots, q_d\})$.

A topological definition of $B_d^0(\theta_{n-1})$ can also be given in terms of universal fibrations and coverings of configuration spaces, similarly to the description in §3.3.

More importantly, denote by W the total space of the symplectic Lefschetz pencil $LP(\theta_{n-1})$ with generic fiber Σ and monodromy θ_{n-1}. For example, if ρ_n and θ_{n-1} are the monodromy morphisms associated to a quasiholomorphic map given by sections s_k^0, s_k^1, s_k^2 of $L^{\otimes k}$ over X, then W is the smooth symplectic hypersurface in X given by the equation $s_k^0 = 0$; indeed, as seen in §4.3, this hypersurface carries a Lefschetz pencil structure with generic fiber Σ, induced by s_k^1 and s_k^2, and the monodromy of this restricted pencil is precisely θ_{n-1}. A braid $Q \in B_d$ can be viewed as a motion of the critical set $\{q_1, \ldots, q_d\}$ of the Lefschetz pencil $LP(\theta_{n-1})$; after this motion we obtain a new Lefschetz pencil with monodromy $\theta_{n-1} \circ Q_*$. So the subgroup $B_d^0(\theta_{n-1})$ precisely consists of those braids which preserve the monodromy of the Lefschetz pencil $LP(\theta_{n-1})$.

Viewing braids as compactly supported symplectomorphisms of the plane preserving $\{q_1, \ldots, q_d\}$, the fact that Q belongs to $B_d^0(\theta_{n-1})$ means that it can be lifted via the Lefschetz pencil map $W - Z \to \mathbb{CP}^1$ to a symplectomorphism of W. Since the monodromy of the pencil $LP(\theta_{n-1})$ preserves a neighborhood of the base set Z, the lift to W of the braid Q coincides with the identity over a neighborhood of Z. Even better, because Q

is compactly supported, its lift to W coincides with Id near the fiber above the point at infinity in $\mathbb{CP}^1$, which can be identified with Σ. Therefore, the lift of Q to W is a well-defined element of the mapping class group $\mathrm{Map}^\omega(W,\Sigma)$, which we call $(\theta_{n-1})_*(Q)$. This construction defines a group homomorphism

$$(\theta_{n-1})_* : B_d^0(\theta_{n-1}) \to \mathrm{Map}^\omega(W^{2n-2},\Sigma^{2n-4}).$$

Since the geometric monodromy representation θ_{n-1} is compatible with the braided curve $D \subset \mathbb{CP}^2$, the image of the braid monodromy homomorphism $\rho_n : \pi_1(\mathbb{C}-\{p_1,\dots,p_r\}) \to B_d$ describing D is entirely contained in $B_d^0(\theta_{n-1})$. Indeed, it follows from Definition 4.1 that θ_{n-1} factors through $\pi_1(\mathbb{C}^2-D)$, on which the braids of $\mathrm{Im}\,\rho_n$ act trivially. As a consequence, we can use the group homomorphism $(\theta_{n-1})_*$ in order to obtain, from the braid monodromy ρ_n, a group homomorphism

$$\theta_n = (\theta_{n-1})_* \circ \rho_n : \pi_1(\mathbb{C}-\{p_1,\dots,p_r\}) \to \mathrm{Map}^\omega(W,\Sigma).$$

If ρ_n and θ_{n-1} describe the monodromy of a $\mathbb{CP}^2$-valued map f, then θ_n is by construction the monodromy of the corresponding Lefschetz pencil $\pi\circ f$. Therefore, the following result holds :

Proposition 4.4. *Let $f : X-Z \to \mathbb{CP}^2$ be one of the quasiholomorphic maps of Theorem 4.1. Let $D \subset \mathbb{CP}^2$ be its critical curve, and denote by $\rho_n : \pi_1(\mathbb{C}-\{p_1,\dots,p_r\}) \to B_d^0(\theta_{n-1})$ and $\theta : F_d \to \mathrm{Map}^\omega(\Sigma,Z)$ be the corresponding monodromies. Then the monodromy map $\theta_n : \pi_1(\mathbb{C}-\{p_1,\dots,p_r\}) \to \mathrm{Map}^\omega(W,\Sigma)$ of the Lefschetz pencil $\pi\circ f$ is given by the identity $\theta_n = (\theta_{n-1})_* \circ \rho_n$.*

In particular, for $k \gg 0$ the symplectic invariants given by Theorem 2.1 are obtained in this manner from those defined in Theorem 4.3.

As in the four-dimensional case, all the factors of degree ± 2 or 3 in the braid monodromy (corresponding to the cusps and nodes of D) lie in the kernel of $(\theta_{n-1})_*$; the only terms which contribute non-trivially to the pencil monodromy θ_n are those arising from the tangency points of the branch curve D, and each of these contributions is a Dehn twist.

More precisely, the image in $\mathrm{Map}^\omega(W,\Sigma)$ of a half-twist $Q \in B_d^0(\theta_{n-1})$ arising as the braid monodromy around a tangency point of D can be constructed as follows. Consider the Lefschetz pencil $LP(\theta_{n-1})$ with total space W, generic fiber Σ, critical levels $q_1,\dots,q_d$ and monodromy θ_{n-1}. Call γ the path joining two of the points $q_1,\dots,q_d$ (e.g., q_{i_1} and q_{i_2}) and naturally associated to the half-twist Q (the path along which the twisting occurs). By Definition 4.1, the monodromies of $LP(\theta_{n-1})$ around the two end points q_{i_1} and q_{i_2} are the same Dehn twists (using γ to identify the two singular fibers). Even better, in this context one easily shows that the vanishing cycles at the two end points of γ are isotopic Lagrangian spheres in Σ. Then it follows from the work of Donaldson and Seidel that, above the path γ, one can find a Lagrangian sphere $L = S^{n-1} \subset W$, joining the singular points of the fibers above q_{i_1} and q_{i_2}, and intersecting each fiber inbetween in a Lagrangian sphere S^{n-2} (there is in fact a hidden subtlety in the argument, but working on pencils rather than fibrations it can be seen that the isotopy of the two vanishing cycles is sufficient). The element $(\theta_{n-1})_*(Q)$ in $\mathrm{Map}^\omega(W,\Sigma)$ is the positive Dehn twist along the Lagrangian sphere L.

Remark 4.4. Let (X^{2n},ω) be a compact symplectic manifold, and consider the symplectic Lefschetz pencils given by Donaldson's result (Theorem 2.1) from pairs of sections of $L^{\otimes k}$ for $k \gg 0$; the monodromy of these Lefschetz pencils consists of generalized Dehn twists around Lagrangian $(n-1)$-spheres in the generic fiber W_k. It follows from Proposition 4.4 that these Lagrangian spheres are not arbitrary. Indeed, they can all be obtained by endowing W_k with a structure of symplectic Lefschetz pencil induced by two sections of $L^{\otimes k}$ (the existence of such a structure follows from the results of this section), and by looking for Lagrangian $(n-1)$-spheres which join two mutually isotopic vanishing cycles of this pencil above a path in the base.

As observed by Seidel, this remarkable structure of vanishing cycles makes it possible to hope for a purely combinatorial description of Lagrangian Floer homology, at least for Lagrangian spheres : one can try to use the structure of vanishing cycles in a $2n$-dimensional Lefschetz pencil to reduce things first to the $2n-2$-dimensional case, and then by induction eventually to the case of 0-manifolds, in which the calculations are purely combinatorial.

5. Complete linear systems and dimensional induction

We now show how the results of §4 can be used in order to reduce in principle the classification of compact symplectic manifolds to a purely combinatorial problem.

The idea behind this approach is to consider a linear system of rank greater than 3, using partial monodromy data to define invariants which allow a dimensional reduction process. This strategy is somewhat complementary to the result obtained by Gompf in [11], showing that the total space of a "hyperpencil" (a rank $n-1$ linear system) carries a canonical symplectic structure.

Definition 5.1. Let (X^{2n},ω) be a compact symplectic manifold. We say that asymptotically holomorphic $(n+1)$-tuples of sections of $L^{\otimes k}$ define *braiding complete linear systems* on X if, for large values of k, these sections $s_0,\dots,s_n \in \Gamma(L^{\otimes k})$ satisfy the following properties :

(a) for $0 \le r \le n-1$, the section $(s_{r+1},\dots,s_n)$ of $\mathbb{C}^{n-r}\otimes L^{\otimes k}$ satisfies a uniform transverslity property, and its zero set $\Sigma_r = \{s_{r+1}=\dots=s_n=0\}$ is a smooth symplectic submanifold of dimension $2r$ in X. We also define $\Sigma_n = X$ and $\Sigma_{-1}=\emptyset$;

(b) for $1 \le r \le n$, the pair of sections $(s_r, s_{r-1}) \in \Gamma(\mathbb{C}^2\otimes L^{\otimes k})$ defines a structure of symplectic Lefschetz pencil on Σ_r, with generic fiber Σ_{r-1} and base set Σ_{r-2} ;

(c) for $2 \le r \le n$, the triple of sections $(s_r,s_{r-1},s_{r-2}) \in \Gamma(\mathbb{C}^3\otimes L^{\otimes k})$ defines a quasiholomorphic map from Σ_r to $\mathbb{CP}^2$, with generic fiber Σ_{r-2} and base set Σ_{r-3}.

One can think of a braiding complete linear system in the following way. First, the two sections s_n and s_{n-1} define a Lefschetz pencil structure on X. By adding the section s_{n-2}, this structure is refined into a quasiholomorphic map to $\mathbb{CP}^2$. As observed in §4, by restricting to the hypersurface Σ_{n-1} we get a symplectic Lefschetz pencil defined by

s_{n-1} and s_{n-2}. This structure is in turn refined into a quasiholomorphic map by adding the section s_{n-3} ; and so on.

Note that, except for the case $r = 1$, part (b) of Definition 5.1 is actually an immediate consequence of part (c), because by composing $\mathbb{CP}^2$-valued quasiholomorphic maps with the projection $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ one always obtains Lefschetz pencils. Also note that, in order to make sense out of these properties, one implicitly needs to endow the submanifolds Σ_r with ω-compatible almost-complex structures ; these restricted almost-complex structures can be chosen to differ from the almost-complex structure J on X by $O(k^{-1/2})$, so that asymptotic holomorphicity and transversality properties are not affected by this choice.

Theorem 5.1. *Let (X^{2n}, ω) be a compact symplectic manifold. Then for all large enough values of k it is possible to find asymptotically holomorphic sections of $\mathbb{C}^{n+1} \otimes L^{\otimes k}$ determining braiding complete linear systems on X. Moreover, for large k these structures are canonical up to isotopy and up to cancellations of pairs of nodes in the critical curves of the quasiholomorphic $\mathbb{CP}^2$-valued maps.*

Proof. We only give a sketch of the proof of Theorem 5.1. As usual, we need to obtain two types of properties : uniform transversality conditions, which we ensure in the first part of the argument, and compatibility conditions, which are obtained by a subsequent perturbation. As in previous arguments, the various uniform transversality properties are obtained successively, using the fact that, because transversality is an open condition, it is preserved by any sufficiently small subsequent perturbations.

The first transversality properties to be obtained are those appearing in part (a) of Definition 5.1, i.e. the transversality to 0 of $(s_{r+1}, \dots, s_n)$ for all $0 \le r \le n-1$; this easy case is e.g. covered by the main result of [2].

One next turns to the transversality conditions arising from the requirement that the three sections (s_n, s_{n-1}, s_{n-2}) define quasiholomorphic maps from X to $\mathbb{CP}^2$: it follows immediately from the proof of Theorem 4.1 that these properties can be obtained by suitable small perturbations.

Next, we try to modify s_{n-1}, s_{n-2} and s_{n-3} in order to ensure that the restrictions to $\Sigma_{n-1} = s_n^{-1}(0)$ of these three sections satisfy the transversality properties of Definition 3.2. A general strategy to handle this kind of situation is to use the following remark (Lemma 6 of [3]) : if ϕ is a section of a vector bundle $\mathcal{F}$ over X, satisfying a uniform transversality property, and if $W = \phi^{-1}(0)$, then the uniform transversality to 0 over W of a section ξ of a vector bundle $\mathcal{E}$ is equivalent to the uniform transversality to 0 over X of the section $\xi \oplus \phi$ of $\mathcal{E} \oplus \mathcal{F}$, up to a change in transversality estimates. This makes it possible to replace all transversality properties to be satisfied over submanifolds of X by transversality properties to be satisfied over X itself ; each property can then be ensured by the standard type of argument, using the globalization principle to combine suitably chosen local perturbations (see [4] for more details).

However, in our case the situation is significantly simplified by the fact that, no matter how we perturb the sections s_{n-1}, s_{n-2} and s_{n-3}, the submanifold Σ_{n-1} itself is not affected. Moreover, the geometry of Σ_{n-1} is controlled by the transversality properties obtained on s_n ; for example, a suitable choice of the constant $\rho > 0$ (independent of k) ensures that the intersection of Σ_{n-1} with any ball of g_k-radius ρ centered at one of its points is topologically a ball (see e.g. Lemma 4 of [2]). Therefore, we can actually imitate all steps of the argument used to prove Theorem 4.1, working with sections of $L^{\otimes k}$ over Σ_{n-1}. The localized reference sections of $L^{\otimes k}$ over Σ_{n-1} that we use in the arguments are now chosen to be the restrictions to Σ_{n-1} of the localized sections $s^{\mathrm{ref}}_{k,x}$ of $L^{\otimes k}$ over X ; similarly, the approximately holomorphic local coordinates over Σ_{n-1} in which we work are obtained as the restrictions to Σ_{n-1} of local coordinate functions on X. With these two differences understood, we can still construct localized perturbations by the same algorithms as in §4.1 and, using the standard globalization argument, achieve the desired transversality properties over Σ_{n-1}. Moreover, all these local perturbations are obtained as products of the localized reference sections by polynomial functions of the local coordinates. Therefore, they naturally arise as restrictions to Σ_{n-1} of localized sections of $L^{\otimes k}$ over X, and so we actually obtain well-defined perturbations of the sections s_{n-1}, s_{n-2} and s_{n-3} over X which yield the desired transversality properties over Σ_{n-1}.

We can continue similarly by induction on the dimension, until we obtain the transversality properties required of s_2, s_1 and s_0 over Σ_2, and finally the transversality properties required of s_1 and s_0 over Σ_1. Observe that, even though the perturbations performed over each Σ_r result in modifications of the submanifolds Σ_j $(j < r)$ lying inside them, these perturbations preserve the transversality properties of $(s_{j+1}, \dots, s_n)$, and so the submanifolds Σ_j retain their smoothness and symplecticity properties.

We now turn to the second part of the argument, i.e. obtaining the desired compatibility conditions. First observe that the proof of Theorem 4.1 shows how, by a perturbation of s_n, s_{n-1} and s_{n-2} smaller than $O(k^{-1/2})$, we can ensure that the various compatibility properties of Definition 3.2 are satisfied by the $\mathbb{CP}^2$-valued map f_n defined by these three sections.

Next, we proceed to perturb $f_{n-1} = (s_{n-1} : s_{n-2} : s_{n-3})$ over a neighborhood of its ramification curve $R_{n-1} \subset \Sigma_{n-1}$, in order to obtain the required compatibility properties for f_{n-1}, but without losing those previously achieved for f_n near its ramification curve $R_n \subset X$. For this purpose, we first show that the curve R_n satisfies a uniform transversality property with respect to the hypersurface Σ_{n-1} in X.

The only way in which R_n can fail to be uniformly transverse to Σ_{n-1} is if $\partial(\pi \circ f_{n|R_n})$ becomes small at a point of R_n near Σ_{n-1}. Because f_n satisfies property (6) in Definition 3.2, this can only happen if a cusp point or a tangency point of f_n lies close to Σ_{n-1}. However, property (7) of Definition 3.2 implies that this point cannot belong to Σ_{n-1}. Therefore, two of the intersection points of R_n with Σ_{n-1} must lie close to each other. Observe that the points of $R_n \cap \Sigma_{n-1}$ are precisely the critical points of the Lefschetz pencil induced on Σ_{n-1} by s_{n-1} and s_{n-2}, i.e. the tangency points of the map f_{n-1}. The transversality properties already obtained for f_{n-1} imply that two tangency points cannot lie close to each other ; we get a contradiction, so the cusps and tangencies of f_n must lie far away from Σ_{n-1}, and R_n and Σ_{n-1} are mutually transverse.

This implies in particular that a small perturbation of s_{n-1}, s_{n-2} and s_{n-3} localized near Σ_{n-1} cannot affect properties (4′) and (6′) for f_n, and also that the only place where perturbing f_{n-1} might affect f_n is near the tangency points of f_{n-1}.

We now consider the set $\mathcal{C}_{n-1} \cup \mathcal{T}_{n-1} \cup \mathcal{I}_{n-1}$ of points where we need to ensure properties (4′), (6′) and (8′) for f_{n-1}. The first step is as usual to perturb J into an almost-complex structure which is integrable near these points ; once this is done, we perturb f_{n-1} to make it locally holomorphic with respect to this almost-complex structure.

We start by considering a point $x \in \mathcal{C}_{n-1} \cup \mathcal{I}_{n-1}$, where the issue of preserving properties of f_n does not arise. We follow the argument in §4.1 of [3]. First, it is possible to perturb the almost-complex structure J over a neighborhood of x in X in order to obtain an almost-complex structure $\tilde{J}$ which differs from J by $O(k^{-1/2})$ and is integrable over a small ball centered at x. Recall from [3] that $\tilde{J}$ is obtained by choosing approximately holomorphic coordinates on X and using them to pull back the standard complex structure of $\mathbb{C}^n$; a cut-off function is used to splice J with this locally defined integrable structure. Since we can choose the local coordinates in such a way that a local equation of Σ_{n-1} is $z_n = 0$, we can easily ensure that Σ_{n-1} is, over a small neighborhood of x, a $\tilde{J}$-holomorphic submanifold of X. Next, we can perturb the sections $s_{n-1}, s_{n-2}, s_{n-3}$ of $L^{\otimes k}$ by $O(k^{-1/2})$ in order to make the projective map defined by them $\tilde{J}$-holomorphic over a neighborhood of x in X (see [3]). This holomorphicity property remains true for the restrictions to the locally $\tilde{J}$-holomorphic submanifold Σ_{n-1}. So, we have obtained the desired compatibility property near x.

We now consider the case of a point $x \in \mathcal{T}_{n-1}$, where we need to obtain property (6′) for f_{n-1} while preserving property (8′) for f_n. We first observe that, by the construction of the previous step (getting property (8′) for f_n at x), we have a readily available almost-complex structure $\tilde{J}$ integrable over a neighborhood of x in X. In particular, by construction f_n is locally $\tilde{J}$-holomorphic and Σ_{n-1} is locally a $\tilde{J}$-holomorphic submanifold of X. We next try to make the projective map f_{n-1} holomorphic over a neighborhood of x, using once again the argument of [3]. The key observation here is that, because one of the sections s_{n-1} and s_{n-2} is bounded from below at x, we can reduce to a $\mathbb{C}^2$-valued map whose first component is already holomorphic. Therefore, the perturbation process described in [3] only affects s_{n-3}, while the two other sections are preserved. This means that we can ensure the local $\tilde{J}$-holomorphicity of f_{n-1} without affecting f_n.

It is easy to combine the various localized perturbations performed near each point of $\mathcal{C}_{n-1} \cup \mathcal{T}_{n-1} \cup \mathcal{I}_{n-1}$; this yields properties (4′), (6′) and (8′) of Definition 3.2 for f_{n-1}.

We now use a generically chosen small perturbation of s_{n-1}, s_{n-2} and s_{n-3} in order to ensure property (7), i.e. the self-transversality of the critical curve of f_{n-1}. It is important to observe that, because f_n satisfies property (7), the images by the projective map $(s_{n-1} : s_{n-2})$ of the points of $R_n \cap \Sigma_{n-1} = \mathcal{I}_n = \mathcal{T}_{n-1}$ are all distinct from each other, and because f_n satisfies property (5) they are also distinct from $(0:1)$. Therefore, we can choose a perturbation which vanishes identically over a neighborhood of $\mathcal{T}_{n-1}$; this makes it possible to obtain property (7) for f_{n-1} without losing any property of f_n.

Finally, by the process described in §4.2 of [3] we construct a perturbation yielding property ($3'$) along the critical curve of f_{n-1} ; this perturbation is originally defined only for the restrictions to Σ_{n-1} but it can easily be extended outside of Σ_{n-1} by using a cut-off function. The two important properties of this perturbation are the following : first, it vanishes identically near the points where f_{n-1} has already been made $\tilde{J}$-holomorphic, and in particular near the points of $\mathcal{T}_{n-1}$; therefore, none of the properties of f_n are affected, and properties ($4'$), ($6'$) and ($8'$) of f_{n-1} are not affected either. Secondly, this perturbation does not modify the critical curve of f_{n-1} nor its image, so property (7) is preserved. We have therefore obtained all desired properties for f_{n-1}.

We can continue similarly by induction on the dimension, until all required compatibility properties are satisfied. Observe that, because the ramification curve of f_r remains away from its fiber at infinity Σ_{r-2}, we do not need to worry about the possible effects on f_r of perturbations of f_{r-2}. Therefore, the argument remains the same at each step, and we can complete the proof of the existence statement in Theorem 5.1 in this way.

The proof of the uniqueness statement relies, as usual, on the extension of the whole construction to one-parameter families ; this is easily done by following the same ideas as in previous arguments. □

The structures of braiding complete linear systems given by Theorem 5.1 are extremely rich, and lead to interesting invariants of compact symplectic manifolds. Indeed, recall from Definition 5.1 that, for $1 \le r \le n$, the sections s_r and s_{r-1} define a symplectic Lefschetz pencil structure on Σ_r, with generic fiber Σ_{r-1} and base set Σ_{r-2}. The monodromy of this pencil is given by a group homomorphism

$$\theta_r : \pi_1(\mathbb{C} - \{p_1, \dots, p_{d_r}\}) \to \mathrm{Map}^\omega(\Sigma_{r-1}, \Sigma_{r-2}). \tag{10}$$

Moreover, for $2 \le r \le n$, the sections s_r, s_{r-1} and s_{r-2} define a quasiholomorphic map from $\Sigma_r - \Sigma_{r-3}$ to $\mathbb{CP}^2$, with generic fiber Σ_{r-2}. Denote by $D_r \subset \mathbb{CP}^2$ the critical curve of this map, and let $d_{r-1} = \deg D_r$. As shown in §4.3, we obtain two monodromy morphisms : on one hand, the braid monodromy homomorphism characterizing D_r,

$$\rho_r : \pi_1(\mathbb{C} - \{p_1, \dots, p_{s_r}\}) \to B_{d_{r-1}}, \tag{11}$$

and on the other hand, a compatible $(r-1)$-dimensional monodromy representation, which was shown in §4.3 to be none other than

$$\theta_{r-1} : \pi_1(\mathbb{C} - \{p_1, \dots, p_{d_{r-1}}\}) \to \mathrm{Map}^\omega(\Sigma_{r-2}, \Sigma_{r-3}).$$

Finally, it was shown in §4.4 that $\mathrm{Im}(\rho_r) \subseteq B^0_{d_{r-1}}(\theta_{r-1})$, and that the various monodromies are related to each other by the identity

$$\theta_r = (\theta_{r-1})_* \circ \rho_r. \tag{12}$$

In particular, the manifold X is completely characterized by the braid monodromies $\rho_2, \dots, \rho_n$ and by the map θ_1 with values in $\mathrm{Map}^\omega(\Sigma_0, \emptyset)$, which is a symmetric group ; this data is sufficient to successively reconstruct all morphisms θ_r and all submanifolds Σ_r by inductively using equation (12).

In other words, a symplectic $2n$-manifold is characterized by $n-2$ braid factorizations and a word in a symmetric group ; or, stopping at θ_2, we can also consider $n-3$ braid factorizations and a word in the mapping class group of a Riemann surface.

These results can be summarized by the following theorem :

Theorem 5.2. *The braid monodromies $\rho_2, \ldots, \rho_n$ and the symmetric group representation θ_1 associated to the braiding complete linear systems obtained in Theorem 5.1 are, for $k \gg 0$, canonical up to m-equivalence, and define symplectic invariants of (X^{2n}, ω).*

Conversely, the data consisting of several braid factorizations and a symmetric group representation satisfying suitable compatibility conditions, or a m-equivalence class of such data, determines a symplectic $2n$-manifold in a canonical way up to symplectomorphism.

In principle, this result reduces the study of compact symplectic manifolds to purely combinatorial questions about braid groups and symmetric groups ; however, the invariants it introduces are probably quite difficult to compute as soon as one considers examples which are not complex algebraic. Nevertheless, it seems that this construction should be very helpful in improving our understanding of the topology of Lefschetz pencils in dimensions greater than 4.

References

[1] V. I. Arnold, S. M. Gusein-Zadeh, A. N. Varchenko, *Singularities of smooth mappings*, Monographs in Mathematics, Birkhäuser, Boston (1982).

[2] D. Auroux, *Asymptotically Holomorphic Families of Symplectic Submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[3] D. Auroux, *Symplectic 4-manifolds as Branched Coverings of $\mathbb{CP}^2$*, Invent. Math. **139** (2000), 551–602.

[4] D. Auroux, *Estimated transversality in symplectic geometry and projective maps*, to appear in Proc. KIAS International Conference on Symplectic Geometry, Seoul (2000), World Scientific.

[5] D. Auroux, L. Katzarkov, *Branched coverings of $\mathbb{CP}^2$ and invariants of symplectic 4-manifolds*, Invent. Math. **142** (2000), 631–673.

[6] J. Birman, *Braids, Links and Mapping class groups*, Annals of Math. Studies **82**, Princeton Univ. Press, Princeton, 1974.

[7] J. Birman, B. Wajnryb, *3-fold branched coverings and the mapping class group of a surface*, Geometry and Topology (College Park, 1983/84), Lecture Notes in Math. **1167**, Springer, 1985, 24–46.

[8] S.K. Donaldson, *Symplectic Submanifolds and Almost-complex Geometry*, J. Differential Geom. **44** (1996), 666–705.

[9] S.K. Donaldson, *Lefschetz Fibrations in Symplectic Geometry*, Documenta Math., Extra Volume ICM 1998, II, 309–314.

[10] S.K. Donaldson, *Lefschetz Pencils on Symplectic Manifolds*, preprint (1999).

[11] R.E. Gompf, *The Topology of Symplectic Manifolds*, Proc. 7th Gökova Geometry-Topology Conference (2000), International Press.

[12] R.E. Gompf, A.I. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Math. **20**, Amer. Math. Soc., Providence, 1999.

[13] B. Moishezon, *Stable Branch Curves and Braid Monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, 1981, 107–192.

[14] B. Moishezon, *The Arithmetic of Braids and a Statement of Chisini*, Geometric Topology (Haifa, 1992), Contemp. Math. **164**, Amer. Math. Soc., Providence, 1994, 151–175.

[15] V. Muñoz, F. Presas, I. Sols, *Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds*, preprint (1999).
[16] P. Seidel, Ph.D thesis, Oxford University.
[17] P. Seidel, *Vanishing cycles and mutation*, preprint, `math.SG/0007115`.
[18] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math., **62** (part 1), Amer. Math. Soc., 1997, 127–150.

Centre de Mathématiques, Ecole Polytechnique, F-91128 Palaiseau, France
E-mail address: `auroux@math.polytechnique.fr`