Turk J Math
25 (2001) , 169 – 194

Topological quantum field theory and hyperkähler geometry

Justin Sawon

1. Introduction

Rozansky and Witten [17] proposed a 3-dimensional sigma-model whose target space is a hyperkähler manifold X. For compact X, they conjectured that this theory has an associated topological quantum field theory (TQFT) with Hilbert spaces[1] given by certain cohomology groups of X. In particular, the vector space $\mathcal{H}_g$ for a genus g Riemann surface should be

$$\mathcal{H}_g := \bigoplus_q \mathrm{H}^q(X, (\Lambda^\bullet T)^{\otimes g}),$$

where we regard X as a complex manifold with respect to some choice of complex structure compatible with the hyperkähler metric (precisely how these spaces depend on this choice is a subtle matter). For X a K3 surface, Rozansky and Witten investigated the cases $g = 0$ and $g = 1$, and exhibited an action of the mapping class group in the latter case.

There is a modified TQFT constructed by Murakami and Ohtsuki [16] using the universal quantum invariant. The vector spaces in this theory are certain spaces of diagrams, which are graded modules over a certain commutative ring (we shall make this precise in due course). This diagrammatic TQFT satisfies a modified version of the usual TQFT axioms. Let us give some background on the construction of the Murakami-Ohtsuki TQFT.

The Kontsevich integral [9] was the first construction of a universal finite-type invariant of links. Le, Murakami, and Ohtsuki later used the Kontsevich integral to construct an invariant of closed 3-manifolds, the LMO invariant [12]. This invariant is universal among finite-type invariants of integral homology spheres; this is also the case for rational homology spheres if we use the Goussarov-Habiro theory of finite-type invariants. The Murakami-Ohtsuki TQFT is based on a generalization of the LMO invariant to 3-manifolds with boundary. Hence we believe that, in some sense, the Murakami-Ohtsuki TQFT should also be regarded as some kind of universal finite-type object. By applying weight systems to this "universal finite-type TQFT" it should be possible to obtain particular TQFTs.

The author is supported by a Junior Research Fellowship at New College, Oxford.

[1] Although physicists would call these Hilbert spaces, to a mathematician they are really just vector spaces, and we will use the latter terminology henceforth.

Rozansky-Witten theory naturally leads to a weight system on graph cohomology built from a hyperkähler manifold. In [20] the author constructed a generalization of this weight system to chord diagrams on circles by adding vector bundles over the hyperkähler manifold (this construction was also discovered independently by Thompson [22]). In this article we extend these ideas in order to apply a “hyperkähler weight system” to the Murakami-Ohtsuki TQFT. There are still some difficulties with this construction (in particular, it is not clear how to apply the weight system to connected 3-manifolds with disconnected boundaries). Nevertheless, we are led to a hyperkähler TQFT with the same vector spaces as Rozansky and Witten’s. Presumably these are the same TQFT - given the close relation between the LMO invariant and the Rozansky-Witten invariant, as investigated by Habegger and Thompson [6], this seems like a natural conjecture to make.

Let us outline the contents of this article. We begin in Section 2 by reviewing the construction of the Murakami-Ohtsuki TQFT. In Section 3 we describe hyperkähler manifolds and how they may be used to construct weight systems. We then apply such a weight system to the Murakami-Ohtsuki TQFT and describe the hyperkähler TQFT so obtained. In Section 4 we reinterpret the observables of Rozansky-Witten theory in the context of this TQFT. The main result here is Proposition 4.1 - in effect we see that all of the observables can be obtained by pairing vectors from our TQFT with cohomology classes. Section 5 is an appendix containing a technical diagrammatic result required in the construction of the hyperkähler weight system.

Although these ideas should lead to a better understanding of Rozansky-Witten theory, there are many new questions to explore. For example, Murakami [15] described the actions of the mapping class groups on the Murakami-Ohtsuki TQFT. From this we may deduce the actions of the mapping class groups on the vector spaces $\mathcal{H}_g$ of the hyperkähler TQFT. This will appear in a future article.

The author would like to express his gratitude to the organizers for the invitation to speak at the Gokova conference. He would also like to thank Nigel Hitchin, Thang Le, Christine Lescop, Jun Murakami, Tomotada Ohtsuki, Justin Roberts, and Simon Willerton for their comments and many useful discussions.

2. The Murakami-Ohtsuki TQFT

In this first section we will describe the modified TQFT of Murakami and Ohtsuki [16]. To do this we need to understand the Kontsevich integral of links in S^3 and the LMO invariant of 3-manifolds, and we outline the basic ideas behind their construction. Then we can define the generalizations of these objects used to construct the TQFT. First we describe the spaces of diagrams to which these objects belong.

Let P be an oriented 1-dimensional space. In fact, we shall only be interested in the cases that P is a collection of circles and trivalent graphs, possible empty. An orientation of a trivalent graph in this instance is given by an orientation of each edge. A *chord diagram* D on P is the union of P and a *chord graph* Q - an oriented unitrivalent graph whose univalent vertices lie on the non-singular part of P. For these unitrivalent graphs, an orientation shall mean an equivalence class of cyclic orderings of the edges at each trivalent vertex, with two such being equivalent if they differ at an even number of vertices. We allow Q to be disconnected, and even to contain connected components with no univalent vertices. We usually distinguish P from Q in a chord diagram D, but if we do not then D itself may be regarded as a trivalent graph. Note that those vertices which occur when a univalent vertex of Q meets P have a canonical cyclic ordering of edges, induced from the orientation of P: for example, we can take the ordering outgoing edge of P, followed by incoming edge of P, followed by the edge belonging to Q. In our diagrams we shall draw P with bold lines to distinguish it from Q[2].

We will consider the space of rational linear combinations of chord diagrams on P. We factor out certain equivalence relations, known as the AS, IHX, STU, and branching relations. The first of these says that reversing the orientation of the chord graph Q (i.e. reversing the cyclic ordering of edges at an odd number of trivalent vertices) is the same as multiplying by -1. The remaining relations are as shown in Figure 1, with each diagram denoting some part of a chord diagram. Note that in these and all future diagrams tetravalent vertices are not vertices at all - they are simply crossings of edges. Chord diagrams are abstract objects, meaning they are not embedded in any ambient space. However, since we only have a sheet of paper on which to draw them we are inevitably hampered by the dimensional deficiencies of this environment.

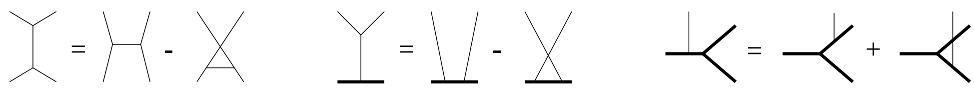

FIGURE 1. The IHX, STU, and branching relations respectively.

When drawn in the plane, we shall assume that our diagrams have the canonical orientation given by ordering the edges at each vertex clockwise. In particular, this implies that in the above diagrams the edges of P (the bold lines) should all be oriented to the right. We won't always mark the orientation of P on our diagrams as it will usually be clear from these conventions.

Furthermore, the operation which reverses the orientation of an edge of P can be made to act on diagrams in a compatible way. Henceforth we shall generally refrain from discussing questions of orientation of chord diagrams. The dedicated reader is encouraged to check compatibility at various stages.

There is a grading on chord diagrams given by half the number of vertices (univalent and trivalent) of the chord graph Q. We denote the graded completion of the space of chord diagrams on P modulo the above relations by $\mathcal{A}(P)$. For example, if P is the empty set then $\mathcal{A}(\emptyset)$ is the graded completion of the space of rational linear combinations

[2] An alternative convention common in the literature is to draw P with solid lines and Q with dashed lines.

of oriented trivalent graphs modulo the AS and IHX relations, also known as *graph cohomology*. This space is a graded commutative ring, with multiplication given by disjoint union of graphs. In general $\mathcal{A}(P)$ is a graded module over $\mathcal{A}(\emptyset)$, with the action given by disjoint union of trivalent graphs with chord diagrams on P.

2.1. The Kontsevich integral

The Kontsevich integral Z is an invariant of framed oriented links in S^3 [9]. For a link L with l components, $Z(L)$ takes values in the space of chord diagrams on the disjoint union of l oriented circles

$$\mathcal{A}(\coprod_{i=1}^{l} S^1).$$

The Kontsevich integral is a universal finite-type invariant of links, which means that all finite-type invariants are obtained by applying an appropriate weight system to Z (see [2] for details).

There are various ways to define Z; we shall outline the main ideas behind one way of constructing it. Suppose we have a framed oriented link L in S^3. Then by an ambient isotopy we can "stretch it out" in such a way that taking horizontal slices results in only fairly simple pieces. Figure 2 shows this for the trefoil. In fact what we are really using is a projection of L to the plane, known as a *link diagram* (not to be confused with a chord diagram). In such a projection the framing is given by the blackboard framing. We can define how Z acts on each of these pieces and then recombine to get complete (linear combinations of) chord diagrams.

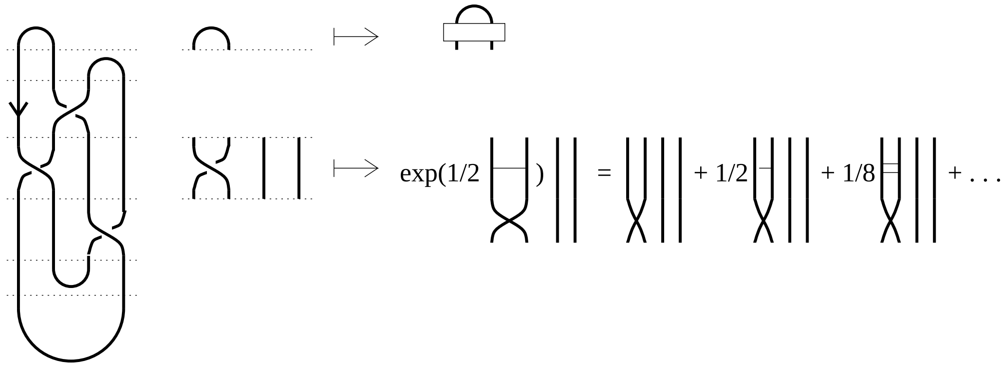

FIGURE 2. Constructing the Kontsevich integral of a trefoil in S^3.

Once we suspect that it may be possible to define Z in this way, then there is really not much choice in what the individual pieces should be mapped to, as Z must be invariant under Reidemeister moves and isotopy of the plane. In particular, Z of the under-crossing and over-crossing are completely determined. In Figure 2 we see that the over-crossing is mapped to the exponential of a single chord multiplied by $1/2$, with additional strands simply carried through the calculation. As usual, the tetravalent vertices are not really vertices. We include them so that when we reassemble the pieces we get a circle with chords on it (or a collection of circles with chords on them, in the case of a link). The under-crossing is similar, but with a factor of $-1/2$ instead of $1/2$. There is some freedom in choosing Z of the cap and cup, which we have indicated in Figure 2 by mapping the cap to a box denoting some unspecified collection of chords. However, these choices only affect the Kontsevich integral up to an overall normalization. Furthermore, if we assume Z is the original invariant as defined by Kontsevich [9] then the normalization is also determined. There is also an additional piece which moves the endpoints of strands near or away from each other, and which is mapped to the *associator*. We do not wish to elaborate on this but instead refer the reader to [9] for the details.

2.2. The LMO invariant

Le, Murakami, and Ohtsuki [12] were able to use the Kontsevich integral to construct an invariant of 3-manifolds, which is a universal finite-type invariant of rational homology 3-spheres. By a theorem of Lickorish and Wallace any 3-manifold M can be obtained by surgery on a framed oriented link L in S^3, and we say L *presents* M. The link L is not uniquely determined, but any other such link will be obtained from L by a sequence of Kirby moves. This means that a link invariant which does not distinguish between links related by Kirby moves can be used to define a 3-manifold invariant.

Given a 3-manifold M, Le, Murakami, and Ohtsuki take the Kontsevich integral $Z(L)$ of some link presenting M, and apply to it an operation ι_n which removes the circles and replaces them by part of a unitrivalent graph. If a circle has m legs on it, we replace it by a an n component tree T^n_m with m legs, and join these legs to those which were on the circle (T^n_m satisfies a symmetry property which ensures that the way the legs are joined is irrelevant). It is not too difficult to show that T^n_m does not exist for $m < 2n$ and therefore if a chord diagram has fewer than $2n$ legs on some circle it gets mapped to zero by ι_n. When $m = 2n$, ι_n is given by removing the circle and then summing over all ways of connecting in pairs the $2n$ legs. Figure 3 shows this for $n = 2$. Le, Murakami, and Ohtsuki further show that any chord diagram can be expressed as a sum of chord diagrams with at most $2n$ legs on each circle and some additional terms which are killed by ι_n (Lemma 3.1 in [12]). So in effect we only need to use this 'connecting legs in pairs' operation.

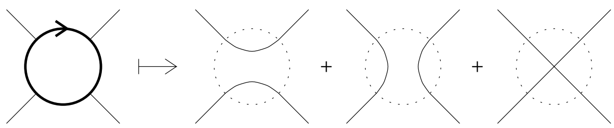

FIGURE 3. Removing a circle using the operation ι_2.

After some additional normalization, we get

$$\Omega_n(L) = (\text{normalizing term})\iota_n(Z(L)) \in \mathcal{A}^{\leq n}(\emptyset)$$

where the superscript $\leq n$ is used to denote all diagrams up to and including degree n. The crucial point is that these terms are invariant under Kirby moves, and hence can be used to build an invariant of 3-manifolds. In fact, we only need the degree n term in $\Omega_n(L)$ as lower degree terms are contained in $\Omega_{n-1}(L)$. Thus we get the LMO invariant

$$\Omega(M) = 1 + \sum_{n=1}^{\infty} \Omega_n(L)^{(n)} \in \mathcal{A}(\emptyset)$$

of M. For rational homology 3-spheres we take a slightly different normalization $\hat{\Omega}(M)$, which is given by rescaling terms by powers of the order of the torsion group $\mathrm{H}_1(M, \mathbb{Z})$. This gives us a universal finite-type invariant of rational homology 3-spheres.

As an aside, consider graph cohomology classes of degree one, which are unique up to scale. They are represented by the *theta graph* Θ for example, and for general M the coefficient of $\Theta \in \mathcal{A}(\emptyset)$ in $\Omega(M)$ is Lescop's generalization of the Casson-Walker invariant [13]. Thus the LMO invariant is every bit as powerful as the Casson invariant.

Using these ideas, we can also extend the Kontsevich integral to an invariant of framed oriented links in arbitrary 3-manifolds as follows. Suppose J is a link in M, and that L presents M. Then there is a link J' in S^3 such that surgery on $L \subset S^3$ takes J' to $J \subset M$. We take the Kontsevich integral of $J' \cup L$ in S^3, and then use the operation ι_n as above to remove the circles corresponding to the components of the link L. We can combine the results to get (after some normalizing) an invariant of $J \subset M$, taking values in

$$\mathcal{A}(\coprod_{i=1}^{j} S^1)$$

where j is the number of components of J.

2.3. The universal quantum invariant for embedded graphs

Now suppose that instead of a link in S^3, we have an embedded framed oriented trivalent graph. Call the abstract graph Γ, and denote its embedding in S^3 by G. Then one can construct an invariant of $G \subset S^3$ taking values in $\mathcal{A}(\Gamma)$ in the same way as we constructed the Kontsevich integral above. The only new feature is that we need to define the invariant on a piece of G containing a trivalent vertex, as shown in Figure 4.

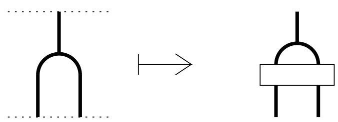

FIGURE 4. The Kontsevich integral of a trivalent vertex.

As before the box denotes some collection of chords, uniquely determined by requiring that the overall invariant thus obtained is invariant under isotopy of the plane and by the Reidemeister-type move shown in Figure 5.

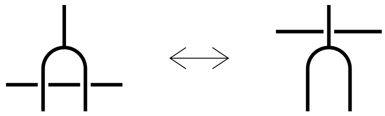

FIGURE 5. Reidemeister move for trivalent graphs.

This is precisely how Murakami and Ohtsuki [16] define what they call the *universal Vassiliev-Kontsevich invariant* of the graph G in S^3, and denote by $\hat{Z}(G)$. As with the Kontsevich integral, we can extend this to an invariant of framed oriented trivalent graphs embedded in an arbitrary 3-manifold by combining with the LMO invariant. Murakami and Ohtsuki call this generalization the *universal quantum invariant* of the graph G in M, denoted $\Omega(M, G)$. When M is a rational homology 3-sphere we once again have a slightly different normalization, denoted $\hat{\Omega}(M, G)$.

2.4. The LMO invariant for 3-manifolds with boundary

So far we have only discussed closed 3-manifolds. Now let M be a 3-manifold with boundary ∂M. The following construction works equally well when ∂M has several connected components, but for ease of exposition we shall assume that ∂M consists of a single Riemann surface of genus g.

Let Γ_g be the *chain graph* with g loops as shown in Figure 6. It is given the blackboard framing and its orientation is as shown. If Γ_g is embedded in some ambient 3-dimensional space then its neighbourhood $N(G)$ is a 3-manifold whose boundary is a genus g surface Σ_g.

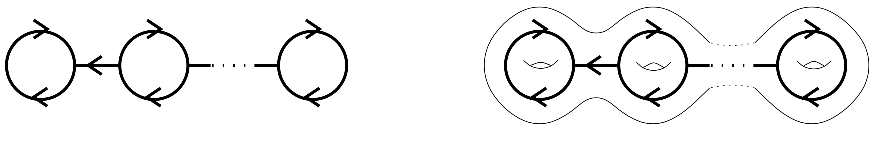

FIGURE 6. The chain graph Γ_g and embedded in a neighbourhood $N(G)$.

Suppose we are given an identification of Σ_g with ∂M. This requires choosing meridians and longitudes on ∂M and identifying them with standard ones on Σ_g. Once we've done this, we can glue the neighbourhood $N(G)$ of the embedded Γ_g into M, identifying the boundaries Σ_g and ∂M. The result is a closed 3-manifold $\hat{M}$, which contains Γ_g as an embedded graph G. We can regard the universal quantum invariant $\Omega(\hat{M}, G)$ (or

$\hat{\Omega}(\hat{M},G)$ when $\hat{M}$ is a rational homology 3-sphere) of $G\subset\hat{M}$ as the LMO invariant of the 3-manifold M with boundary. In the next subsection we shall use this to construct a modified TQFT.

2.5. The modified TQFT axioms

A 3-dimensional TQFT is a functor from the category of oriented 3-cobordisms to a category of modules over a commutative ring k. In concrete terms, we associate to each Riemann surface Σ a module $V(\Sigma)$ and to each 3-manifold M with boundary $\Sigma=\partial M$ an element $Z(M)\in V(\Sigma)$. We require the following axioms to hold.

(A1) For surfaces Σ_1 and Σ_2, $V(\Sigma_1\sqcup\Sigma_2)=V(\Sigma_1)\otimes V(\Sigma_2)$[3].

(A2) Reversing the orientation of Σ gives the dual module $V(-\Sigma)=V(\Sigma)^*$.

(A3) For the empty surface $V(\emptyset)=k$. In particular, $Z(M)\in k$ for a closed 3-manifold.

(A4) Suppose M_1 and M_2 have boundaries $\partial M_1=\Sigma\sqcup\Sigma_1$ and $\partial M_2=(-\Sigma)\sqcup\Sigma_2$ respectively. We can glue M_1 and M_2 along Σ to get $M=M_1\cup_\Sigma M_2$. Then $Z(M)=\langle Z(M_1),Z(M_2)\rangle_\Sigma$, where

$$\langle\quad,\quad\rangle_\Sigma : V(\Sigma)\otimes V(\Sigma_1)\otimes V(\Sigma)^*\otimes V(\Sigma_2)\longrightarrow V(\Sigma_1)\otimes V(\Sigma_2)$$

denotes the contraction mapping.

In [16] Murakami and Ohtsuki construct the following modified TQFT. Firstly, we take a sub-category of the category of 3-cobordisms such that all the closed 3-manifolds involved will be rational homology 3-spheres (this amounts to some condition on the homology groups of the cobordisms). For a genus g Riemann surface Σ_g we define

$$V(\Sigma_g):=\mathcal{A}(\Gamma_g),$$

which is a module over the commutative ring $\mathcal{A}(\emptyset)$. For disconnected Riemann surfaces we define

$$V(\Sigma_1\sqcup\Sigma_2):=\mathcal{A}(\Gamma_1\sqcup\Gamma_2),$$

and in this way $V(\Sigma)$ is defined for all Riemann surfaces Σ.

Let M be a 3-manifold with boundary ∂M. Suppose the boundary is isomorphic to the genus g Riemann surface Σ_g. Then $V(\partial M)\cong V(\Sigma_g)$, and more specifically, *for each choice of a set of longitudes and meridians on* ∂M there is an isomorphism between $V(\partial M)$ and $V(\Sigma_g)=\mathcal{A}(\Gamma_g)$ (we assume Σ_g comes equipped with a standard set of longitudes and meridians). As in the last subsection, we can glue a neighbourhood of a graph to M and obtain a closed 3-manifold $\hat{M}$ containing Γ_g as an embedded graph G. Because of our choice of sub-category of 3-cobordisms, $\hat{M}$ is a rational homology 3-sphere, and hence we can define

$$Z(M):=\hat{\Omega}(\hat{M},G)\in\mathcal{A}(\Gamma_g)\cong V(\partial M).$$

[3]The subscripts here are labels and do not indicate surfaces of genus one and two. Despite this clash of notation, it should be clear what is meant from the context.

It is important to remember that the isomorphism on the right depends on a choice of meridians and longitudes for ∂M. This definition clearly extends to arbitrary (disconnected) Riemann surfaces ∂M.

The first thing to observe is that axiom (A1) is not satisfied; instead we have

(A1)′ There is an inclusion $V(\Sigma_1) \otimes V(\Sigma_2) \hookrightarrow V(\Sigma_1 \sqcup \Sigma_2)$.

This inclusion is given by disjoint union of chord diagrams on Γ_1 and Γ_2. General chord diagrams on $\Gamma_1 \sqcup \Gamma_2$ may contain chord graphs which connect Γ_1 and Γ_2, whereas the image of the above inclusion only contains chord diagrams for which Γ_1 and Γ_2 are in distinct connected components.

Axiom (A2) is replaced by

(A2)′ There is a pairing

$$\langle \quad , \quad \rangle_\Sigma : V(\Sigma) \otimes V(-\Sigma) \longrightarrow \mathcal{A}(\emptyset).$$

We now describe this pairing; it suffices to do so for a genus g surface Σ_g. In other words, we have a pair of chord diagrams on Γ_g and $-\Gamma_g$ respectively, which we align as shown in Figure 7. Note that the orientations may be arranged as shown by applying the operation which reverses the orientations of edges of Γ and acts on chord diagrams in a compatible way.

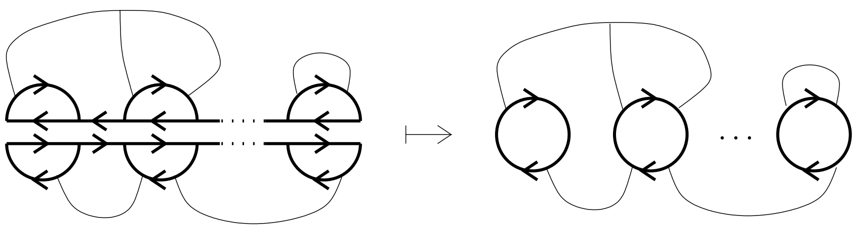

Figure 7. Pairing of chord diagrams on Γ_g and $-\Gamma_g$.

Using the branching relations, we may assume that our chord diagrams have no chords attached to the straight edges of the graphs, i.e. all legs of the unitrivalent graphs end on the curved edges of Γ_g and $-\Gamma_g$. Now we remove the straight edges and connect the curved edges to form g oriented circles. Finally, we remove the circles using the same argument as for the LMO invariant. Thus we are left with an element of $\mathcal{A}(\emptyset)$, which is precisely what we wanted. Note that by identifying $V(\Sigma)$ with the space of chord diagrams on Γ we have implicitly assumed that we have chosen a set of meridians and longitudes on Σ. The pairing depends on this choice.

Axiom (A3) needs no modification. The commutative ring k is $\mathcal{A}(\emptyset)$. Multiplication in $\mathcal{A}(\emptyset)$ and the action on $\mathcal{A}(\Gamma)$ are given by disjoint union of diagrams.

Suppose we have 3-manifolds M_1 and M_2 with boundaries $\partial M_1 \cong \Sigma$ and $\partial M_2 \cong -\Sigma$ respectively. Then we can pair $Z(M_1) \in \mathcal{A}(\Gamma)$ with $Z(M_2) \in \mathcal{A}(-\Gamma)$ as described above, to get an element

$$\langle Z(M_1), Z(M_2)\rangle_\Sigma \in \mathcal{A}(\emptyset).$$

As usual, this requires a choice of longitudes and meridians on ∂M_1 and ∂M_2, so that we can identify them with Σ and $-\Sigma$ respectively. This induces a homeomorphism f of the boundaries of our 3-manifolds, with which we can glue M_1 and M_2 to get a closed 3-manifold $M = M_1 \cup_f M_2$. Murakami and Ohtsuki [16] prove that $Z(M) \in \mathcal{A}(\emptyset)$ is essentially the same as the pairing of $Z(M_1)$ and $Z(M_2)$.

More generally, M_1 and M_2 may have additional boundary components Σ_1 and Σ_2, respectively. Then we have the following modification of axiom (A4).

(A4)′ Under the above hypotheses on M_1 and M_2, we have

$$Z(M) = (\text{normalizing terms})(\text{framing anomaly correction})\langle Z(M_1), Z(M_2)\rangle_\Sigma$$

as elements of $V(\Sigma_1 \sqcup \Sigma_2)$.

As usual the normalizing terms involve the orders of the integral homology groups of $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}$ (which are finite groups since these are rational homology 3-spheres). The *framing anomaly* is a subject we'd rather avoid here. Murakami and Ohtsuki expect that this correction term can be removed by defining a suitable framing of 3-manifolds (the interested reader should consult [16]).

The key to proving that the Murakami-Ohtsuki TQFT satisfies axiom (A4)′ is the following result (Lemma 4.3 in [16]).

Lemma 2.1. *Let M_1 and M_2 be as given above. Suppose that $\hat{M}_1$ and $\hat{M}_2$ are given by surgery on links L_1 and L_2 in S^3, respectively. Recall that these closed 3-manifolds contain embedded graphs G_1 and G_2. We use the same notation to denote the graphs in S^3 which become G_1 and G_2 after the surgeries on L_1 and L_2. Let L be the link in S^3 obtained from $L_1 \cup G_1$ and $L_2 \cup G_2$ by removing the straight edges of G_1 and G_2, and connecting the curved edges to form embedded circles (we may need to first isotope the trivalent graphs into adjacent positions). Then M is homeomorphic to the 3-manifold given by surgery on L.*

In calculating the LMO invariant $Z(M)$ of M we perform the circle removing operation on L, so we have really done the same operations as we would to calculate the pairing $\langle Z(M_1), Z(M_2)\rangle_\Sigma$. Thus axiom (A4)′ is satisfied.

Note that for a closed 3-manifold M, $Z(M)$ is the LMO invariant $\hat{\Omega}(M) \in \mathcal{A}(\emptyset)$, with the rational homology 3-sphere normalization. If M has boundary $\partial M \cong S^2$, we also take $Z(M) \in \mathcal{A}(\emptyset)$. In other words, the genus zero chain graph Γ_0 is the empty graph, but we think of a neighbourhood of Γ_0 as being a solid ball with boundary S^2. The reader can think of this as simply a convention, though in some ways it is the only sensible choice.

3. Hyperkähler geometry

It is common in the theory of knot and 3-manifold invariants to apply Lie algebra weight systems to the universal diagrammatic invariants - this is one way of generating quantum invariants. Indeed the words 'universal quantum invariant' mean that all quantum invariants arise in this way. For example, Jones' polynomial arises by applying an $\mathfrak{su}(2)$ weight system to the Kontsevich integral. In this section we will describe how a *hyperkähler weight system* can be applied to the Murakami-Ohtsuki TQFT. In some ways this is a continuation of the author's work in [20], where the genus one case was described (see also Thompson [22]). The argument presented here works only for 3-manifolds with connected boundaries, and in this sense our programme is still incomplete. Nevertheless, we expect that it should be possible to extend the results to the disconnected case.

3.1. Hyperkähler and holomorphic symplectic manifolds

Let X be a compact[4] irreducible[5] hyperkähler manifold of real-dimension $4k$. In other words, there is a metric on X whose Levi-Civita connection has holonomy $\mathrm{Sp}(k)$. Such a manifold admits a triple of complex structures I, J, and K, which act like the quaternions on the tangent bundle TX. A *hyperkähler metric* is a metric g which is Kählerian with respect to all of these complex structures. Let us call the corresponding Kähler forms ω_1, ω_2, and ω_3 respectively.

There is no canonical choice of complex structure on X compatible with the metric, but since we wish to use the methods of complex geometry we shall fix a structure I and henceforth regard X as a complex manifold of complex-dimension $2k$. Then ω_2 and ω_3 can be combined to give a two-form

$$\omega := \omega_2 + i\omega_3 \in \mathrm{H}^0(X, \Lambda^2 T^*)$$

which is holomorphic with respect to I (T now denotes the holomorphic tangent bundle).

Rozansky-Witten theory [17] was originally built around hyperkähler manifolds, though Kontsevich [10] and Kapranov [8] later showed that all one requires is a complex manifold with a holomorphic symplectic form, otherwise known as a *holomorphic symplectic manifold*. In particular, if it is Kähler then it must be a hyperkähler manifold (as we are working in the compact setting), but there are also non-Kähler examples due to Guan [5]. However, we suspect that there may be further interesting properties for hyperkähler X which are not present for general holomorphic symplectic manifolds. For example, it may be interesting to observe the way that the vector spaces of the TQFT change under a variation of the compatible complex structure on X. So although in this article the reader may assume that X is merely a holomorphic symplectic manifold, we will stick to the original "hyperkähler" terminology nonetheless.

The key to Kapranov's version of the Rozansky-Witten weight system is to use the Atiyah class [1] instead of the curvature of the hyperkähler manifold. The Atiyah class

$$\alpha_E \in \mathrm{H}^1(X, T^* \otimes \mathrm{End}E)$$

[4]To construct Rozansky-Witten invariants, as in [17], the assumption that the manifold is compact may be dropped provided the appropriate asymptotic decay conditions are satisfied by the curvature. For the construction of the TQFT we assume X is compact, and are uncertain whether this can be generalized to non-compact manifolds.

[5]The theory easily extends to reducible manifolds.

of a complex vector bundle E on X is the obstruction to the existence of a global holomorphic connection on X. The case $E = T$ will be the only one of interest to us, for which the Atiyah class lies in

$$\mathrm{H}^1(X, T^* \otimes T^* \otimes T).$$

Observe that we can identify T and T^* using the holomorphic symplectic form ω. Kapranov showed that the element α_T is totally symmetric (Proposition 5.1.1 in [8]), i.e. it lies in

$$\mathrm{H}^1(X, \mathrm{Sym}^3 T^*).$$

In local complex coordinates, we write α_{ijk} for α_T, where the subscripts refer to the three copies of T^*.

The last thing we shall need in the construction is the dual of the holomorphic symplectic form

$$\tilde{\omega} \in \mathrm{H}^0(X, \Lambda^2 T).$$

In local complex coordinates, $\tilde{\omega}$ has matrix ω^{ij}. Note that this is *minus* the inverse of the matrix ω_{ij} of ω.

3.2. Some diagrammatic preliminaries

Suppose we have a 3-manifold M with boundary ∂M a genus g Riemann surface. Then in the Murakami-Ohtsuki TQFT we get, after a choice of longitudes and meridians in ∂M, an element

$$Z(M) \in \mathcal{A}(\Gamma_g).$$

By applying a hyperkähler weight system to $Z(M)$ we would like to obtain something in

$$\mathcal{H}_g = \bigoplus_q \mathrm{H}^q(X, (\Lambda^\bullet T)^{\otimes g}).$$

First we need to rewrite elements of $\mathcal{A}(\Gamma_g)$ in a nicer way.

A *marked unitrivalent graph* is an oriented unitrivalent graph D whose univalent vertices (or *legs*) are labelled by the integers 1 to g. An orientation of such a graph is an equivalence class of cyclic orderings of the edges at each trivalent vertex, with two such being equivalent if they differ at an even number of vertices. Note that more than one leg may be labelled by the same integer, and we do not need to use all labels. The graphs may also be disconnected, and may contain connected components with no legs.

We will consider the space of rational linear combinations of marked unitrivalent graphs modulo the AS and IHX relations. This space is graded by half the number of vertices (univalent and trivalent) of a unitrivalent graph D, and we denote the graded completion by $\mathcal{B}_g$.

Let I be the interval, and let

$$\chi : \mathcal{B}_g \longrightarrow \mathcal{A}(\coprod_{i=1}^{g} I)$$

be the map given by averaging over all ways of joining the legs of a marked unitrivalent graph to the intervals such that the legs labelled by $j \in \{1, \dots, g\}$ are joined to the j^{th} interval.

Proposition 3.1. *The map χ is an isomorphism of $\mathcal{A}(\emptyset)$-modules.*

Proof: This result is well-known among knot-theorists, and is described for $g = 1$ in Bar-Natan [2][6] and for general g in Bar-Natan et al [3] (Definition 2.7). In the former case, the argument relies on using the IHX relations to rewrite an element of $\mathcal{B} := \mathcal{B}_1$ in such a way that its legs are arranged 'symmetrically'. The map χ is then simply given by gluing the legs to the interval and is an isomorphism. Since this is a 'local' operation, it generalizes to arbitrary g. In other words, we can rewrite an element of $\mathcal{B}_g$ in such a way that its legs are arranged in g symmetric collections, and the proposition follows. □

There is a surjective map

$$\mathcal{A}(\coprod_{i=1}^{g} I) \longrightarrow \mathcal{A}(\coprod_{i=1}^{g} S^1)$$

given by closing up the intervals into circles. This is only an isomorphism when $g = 1$ (Theorem 8 in Bar-Natan [2]; see also Lemma 3.17 in Thurston [23] and the discussion there). However, this is not the map we wish to use. Instead we wish to attach a tree to the intervals as shown in Figure 8, resulting in a chain graph Γ_g. This gives us a map

$$\rho : \mathcal{A}(\coprod_{i=1}^{g} I) \longrightarrow \mathcal{A}(\Gamma_g).$$

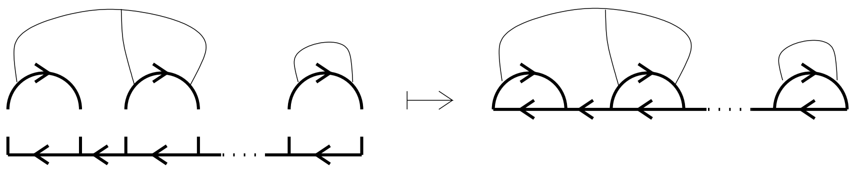

FIGURE 8. Attaching a tree to a collection of intervals.

Proposition 3.2. *The map ρ is an isomorphism of $\mathcal{A}(\emptyset)$-modules.*

Proof: We postpone the proof to an appendix. □

Composing χ and ρ gives us an isomorphism

$$\tau := \rho \circ \chi : \mathcal{B}_g \longrightarrow \mathcal{A}(\Gamma_g).$$

[6] Note that Bar-Natan uses the map given by summing over all ways of joining the legs to the interval, whereas we use the average.

This means that given a chord diagram D on the chain graph Γ_g, we can rewrite it in an equivalent way as a marked unitrivalent graph, namely $\tau^{-1}(D)$.

3.3. The hyperkähler weight system

Recall that $\mathcal{B}_g$ is graded by half the number of vertices. It also has a multi-grading given by the number of legs with label 1, with label 2, etc. A unitrivalent graph with no legs is simply a trivalent graph, and thus $\mathcal{B}_0 = \mathcal{A}(\emptyset)$. Our aim in this subsection is to show the following.

Proposition 3.3. *There is a homomorphism of vector spaces*

$$W_X : \mathcal{B}_g \longrightarrow \mathcal{H}_g := \bigoplus_q \mathrm{H}^q(X, (\Lambda^\bullet T)^{\otimes g}).$$

If a marked unitrivalent graph D has q trivalent vertices and l_i legs labelled by $i \in \{1, \ldots, g\}$, then $W_X(D)$ lies in

$$\mathrm{H}^q(X, \Lambda^{l_1} T \otimes \cdots \otimes \Lambda^{l_g} T).$$

The map

$$W_X|_{\mathcal{B}_0 = \mathcal{A}(\emptyset)} : \mathcal{A}(\emptyset) \longrightarrow \mathcal{H} := \mathcal{H}_0 = \bigoplus_q \mathrm{H}^q(X, \mathcal{O}_X)$$

is a homomorphism of commutative rings, where the product in $\mathcal{H}$ is cup-product. In general the map W_X intertwines the $\mathcal{A}(\emptyset)$-module and $\mathcal{H}$-module structures on $\mathcal{B}_g$ and $\mathcal{H}_g$ respectively.

Proof: We will give a definition of W_X, after which the rest of the statements in the proposition will follow. We begin with $g = 0$. This is the situation of the original Rozansky-Witten invariants [17]. We review instead the approach of Kapranov [8] which leads immediately to cohomology classes.

Suppose D is a trivalent graph with q vertices (in this case q must be even). We place a copy of the Atiyah class α_T at each vertex of D and a copy of the dual of the holomorphic symplectic form $\tilde{\omega}$ at each edge. If α_{ijk} has been placed at some vertex, we label the outgoing edges by i, j, and k. Similarly, if ω^{ij} has been placed on some edge, we label the ends of the edge by i and j[7]. We then contract all indices and take the cup product of the α_Ts. This results in an element

$$W_X(D) \in \mathrm{H}^q(X, \mathcal{O}_X)$$

which is precisely what we wanted. That this construction is compatible with the AS and IHX relations follows (respectively) from careful consideration of the orientation of D and from an identity satisfied by the Atiyah class. See Kapranov [8] or Hitchin and Sawon [7] for details.

[7]These labellings must be done in a way compatible with the orientation of D. This is discussed at length in Kapranov [8], and reproduced in Hitchin and Sawon [7]. We do not wish to repeat the argument for a third time, so the reader is urged to consult those articles regarding questions of orientation.

The general case is much the same. Let D be a marked unitrivalent graph with q trivalent vertices and l_i legs labelled by $i \in \{1, \ldots, g\}$. Since we have the isomorphism

$$\chi : \mathcal{B}_g \longrightarrow \mathcal{A}(\coprod_{i=1}^{g} I),$$

we may assume that the legs of D are already arranged into g symmetric collections. As before, we place a copy of α_T at each trivalent vertex and a copy of $\tilde{\omega}$ at each edge, and label outgoing edges and ends of edges by indices. Contracting indices and taking the cup product of the α_Ts gives us an element of

$$\mathrm{H}^q(X, T^{\otimes(l_1+\ldots+l_g)}),$$

where the copies of T occur because there are uncontracted indices labelling the legs. The fact that the legs are arranged into g symmetric collections means that we actually get an element

$$W_X(D) \in \mathrm{H}^q(X, \Lambda^{l_1} T \otimes \cdots \otimes \Lambda^{l_g} T)^8.$$

Compatibility with the AS and IHX relations follow as before. □

3.4. The hyperkähler TQFT

We wish to define a TQFT by applying the hyperkähler weight system of the previous subsection to the Murakami-Ohtsuki TQFT. Recall that the vector spaces of the latter are modules over the commutative ring $\mathcal{A}(\emptyset)$. Elements of this ring are mapped to the space

$$\mathcal{H} = \bigoplus_q \mathrm{H}^q(X, \mathcal{O}_X)$$

which will be the commutative ring of our hyperkähler TQFT. Note that for irreducible hyperkähler manifolds the cohomology groups $\mathrm{H}^q(X, \mathcal{O}_X)$ vanish for odd q and are one dimensional and generated by

$$[\bar{\omega}^l] \in \mathrm{H}^{0,2l}_{\bar{\partial}}(X) = \mathrm{H}^{2l}(X, \mathcal{O}_X)$$

for even $q = 2l$. Up to scale, there is a unique graph cohomology class of degree one, represented by the graph Θ for example. When X is irreducible

$$W_X(\Theta) = \beta_\Theta[\bar{\omega}] \in \mathrm{H}^2(X, \mathcal{O}_X)$$

for some scalar β_Θ. This scalar is proportional to the $\mathcal{L}^2$-norm of the curvature of X and is therefore non-zero (see Equation 9 in Hitchin and Sawon [7]). Similarly, for the disjoint

[8] The fact that the legs are arranged *symmetrically* but give us a cohomology class with values in an *anti-symmetric* bundle, namely $\Lambda^{l_1} T \otimes \cdots \otimes \Lambda^{l_g} T$, is a manifestation of the general 'reversal of statistics' from Bose-Einstein to Fermi-Dirac which is inherent in Rozansky-Witten theory. The same phenomena occurs with the wheels in Hitchin and Sawon [7], and is the reason why Kapranov considers desuspensions of operads in [8]. If you think this is some kind of skullduggery, then check it directly by considering the orientations!

union of l copies of Θ we have

$$W_X(\Theta^l) = \beta_\Theta^l[\bar{\omega}^l] \in \mathrm{H}^{2l}(X, \mathcal{O}_X).$$

Thus $W_X(\Theta)$ generates the commutative ring $\mathcal{H}$.

In the original paper of Rozansky and Witten [17] there was a further map from $\mathcal{H}$ to the real numbers. This was given by taking trivalent graphs of degree k where the real-dimension of X is $4k$; W_X applied to such a graph gives us an element of $\mathrm{H}^{2k}(X, \mathcal{O}_X)$, and taking the Serre duality pairing with the generator $[\omega^{2k}]$ of $\mathrm{H}^0(X, \Lambda^{2k}T^*)$ gives us a number (which is real after the appropriate normalization). If we begin with a trivalent graph of degree less than k, it is still possible to get a real number by including some *observables*. Basically this involves multiplying by some power of $[\bar{\omega}]$, or taking the disjoint union with some copies of Θ (these are equivalent for X compact and irreducible). We shall say more about these and other observables in the next section. From the TQFT perspective, however, it makes more sense to work over the commutative ring $\mathcal{H}$ in order to preserve more of the original structure.

Recall that the vector spaces of the Murakami-Ohtsuki TQFT are the spaces $\mathcal{A}(\Gamma_g)$. Applying the isomorphism τ^{-1} takes us to $\mathcal{B}_g$, to which we apply the hyperkähler weight system W_X, taking us into

$$\mathcal{H}_g = \bigoplus_q \mathrm{H}^q(X, (\Lambda^\bullet T)^{\otimes g}).$$

We must be careful here. Whereas we can ensure that the map

$$W_X : \mathcal{A}(\emptyset) \longrightarrow \mathcal{H}$$

is onto, this is never likely to be the case for

$$W_X \circ \tau^{-1} : \mathcal{A}(\Gamma_g) \longrightarrow \mathcal{H}_g.$$

This suggests that the vector spaces of our hyperkähler TQFT should be subspaces of $\mathcal{H}_g$, though exactly how to define these subspaces is still unclear.

Putting aside this problem, let us just assume the vector spaces are the $\mathcal{H}_g$s themselves. Then a 3-manifold M with boundary $\partial M \cong \Sigma_g$ gives rise to an element

$$Z(M) \in \mathcal{A}(\Gamma_g)$$

after a choice of longitudes and meridians in ∂M. Applying $W_X \circ \tau^{-1}$ takes us into $\mathcal{H}_g$ and completes the definition of our TQFT.

Note that we assumed M has a single connected boundary component. The application of the weight system W_X in the more general case remains problematical. Presumably we would like axiom (A1)

$$V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$$

to be satisfied. For example

$$V(\Sigma_{g_1} \sqcup \Sigma_{g_2}) = \mathcal{H}_{g_1} \otimes_{\mathcal{H}} \mathcal{H}_{g_2}$$

and every vector space will be a tensor product of $\mathcal{H}_g$s, tensored over $\mathcal{H}$ since these are $\mathcal{H}$-modules. However, the Murakami-Ohtsuki TQFT satisfies the modified axiom (A1)$'$

$$V(\Sigma_1) \otimes V(\Sigma_2) \hookrightarrow V(\Sigma_1 \sqcup \Sigma_2)$$

i.e.

$$\mathcal{A}(\Gamma_1) \otimes_{\mathcal{A}(\emptyset)} \mathcal{A}(\Gamma_2) \hookrightarrow \mathcal{A}(\Gamma_1 \sqcup \Gamma_2).$$

Although the left hand side sits inside the right as a direct summand, projecting onto it results in a significant loss of information. On the other hand, there does not appear to be an isomorphism from $\mathcal{A}(\Gamma_1 \sqcup \Gamma_2)$ to the space of marked unitrivalent graphs (or some similar space) which generalizes τ^{-1}. Thus it is not clear how to extend the hyperkähler weight system W_X in this case.

Next let us consider the pairing of axiom (A2)$'$. Let D_1 and D_2 be chord diagrams on Γ_g and $-\Gamma_g$ respectively. Note that a chord diagram on $-\Gamma_g$ is the same as a chord diagram on Γ_g, up to a sign, so we may as well take $D_2 \in \mathcal{A}(\Gamma_g)$. We described the pairing

$$\langle D_1, D_2 \rangle_{\Sigma_g} \in \mathcal{A}(\emptyset)$$

in Subsection 2.5. Applying the hyperkähler weight system directly

$$\langle W_X \circ \tau^{-1}(D_1), W_X \circ \tau^{-1}(D_2) \rangle_{\mathcal{H}_g} = W_X(\langle D_1, D_2 \rangle_{\Sigma_g}) \in \mathcal{H}$$

gives us an implicit description of the pairing in the hyperkähler TQFT. This is somewhat unsatisfactory. We'd like to have an explicit description of the pairings in the spaces $\mathcal{H}_g$. In the hyperkähler context, the operation of summing over all pairings of legs becomes a combination of wedge product of $\Lambda^\bullet T$s and fiberwise convolution, i.e.

$$\Lambda^{2m} T^* \otimes \Lambda^{2m} T \longrightarrow \mathcal{O}_X,$$

with a power of the holomorphic symplectic form ω, which gives a section of $\Lambda^{2m} T^*$ (an example of such a calculation was carried out in detail in Hitchin and Sawon [7]). Combining this with cup-product of cohomology classes no doubt gives the pairing on $\mathcal{H}_g$, though unfortunately this is rather difficult to see from the implicit description given above. The problem is that in order to reduce the LMO 'circle removing operation' (see Subsection 2.2) to a sum over all pairings of legs, we first need to rewrite the chord diagrams D_1 and D_2 in some canonical way. On the other hand, we need to apply the isomorphism τ^{-1} before acting with the hyperkähler weight system W_X. This is also equivalent to choosing some canonical way of writing D_1 and D_2, and these two canonical representations are unlikely to be compatible.

The third axiom (A3) simply tells us that closed 3-manifolds will gives us $\mathcal{H}$-valued invariants. These are the original Rozansky-Witten invariants of [17], after including observables where necessary and mapping $\mathcal{H}$ to the real numbers, as described earlier in this subsection. The final axiom (A4)$'$ should still be satisfied, though as we have yet to fully extend our hyperkähler TQFT to all 3-manifolds we will postpone our investigations to a future article.

4. Observables

There are three kinds of observables occurring in the Rozansky-Witten theory. The first was introduced by Rozansky and Witten [17] and the second and third by Thompson in [22] and [21] respectively. In this subsection we review these and try to place them within the context of our hyperkähler TQFT.

1. The first observable is one we have already mentioned, namely the inclusion of additional copies of Θ to bring the degree of a trivalent graph in $\mathcal{A}(\emptyset)$ up to k, for a hyperkähler manifold X of real-dimension $4k$. More precisely, suppose $D \in \mathcal{A}(\emptyset)$ has degree l, that is $2l$ vertices, where $l < k$. Then
$$W_X(D) \in \mathrm{H}^{2l}(X, \mathcal{O}_X).$$
In order to integrate this, as in the original Rozansky-Witten invariants, we first need to multiply by
$$[\bar{\omega}^{k-l}] \in \mathrm{H}^{2(k-l)}(X, \mathcal{O}_X)$$
or equivalently
$$W_X(\Theta)^{k-l} \in \mathrm{H}^{2(k-l)}(X, \mathcal{O}_X).$$
More generally, an arbitrary element of $\mathrm{H}^{2(k-l)}(X, \mathcal{O}_X)$ may be included as an observable, though for irreducible X this cohomology space is one-dimensional and hence generated by $W_X(\Theta)^{k-l}$.
2. The second type of observable arises from taking a knot K in our 3-manifold M and a holomorphic vector bundle E over X. We then take the trace in a fibre $\mathbb{E}$ of E of the holonomy around the knot K of the gauge field A, to get
$$\mathcal{O}(K; \mathbb{E}) = \mathrm{Tr}_{\mathbb{E}} \exp(\oint_K A).$$
Note that originally Rozansky and Witten introduced a similar observable with $\mathbb{E}$ a representation of $\mathrm{Sp}(k)$. Since X has holonomy $\mathrm{Sp}(k)$ (or contained in $\mathrm{Sp}(k)$ if X is reducible), the frame bundle of X is a principal $\mathrm{Sp}(k)$-bundle. Thus the representation $\mathbb{E}$ induces a bundle E over X, which is a tensor bundle. Thompson's construction is a generalization to arbitrary holomorphic[9] vector bundles E over X.
3. The third type of observable comes from choosing a $(0, q)$-form λ on X with values in $\Lambda^{l_1} T \otimes \cdots \otimes \Lambda^{l_r} T$, where $b_1(M) = r > 0$. See Thompson [21] for how these are constructed.

In the present context, the first kind of observable is not relevant as we are working over the commutative ring $\mathcal{H}$, and so we do not require our cohomology classes to be of top degree.

[9]When X is hyperkähler (not just holomorphic symplectic) we can obtain a 'good' observable by adding the condition that E be hyper-holomorphic (see [22]). This gives the observable some invariance properties under variation of the compatible complex structure on X. For example, all tensor bundles are hyper-holomorphic.

The second kind of observable comes from including a knot K and associating a holomorphic vector bundle E to it. The inclusion of a knot in our 3-manifold M means that instead of the LMO invariant of M, the Murakami-Ohtsuki TQFT will give us the Kontsevich integral

$$Z(K \subset M) \in \mathcal{A}(S^1)$$

of the knot K in M. If we then apply the hyperkaähler weight system to $Z(K \subset M)$, as in Thompson [22] or the author's article [20], we get the expectation value of the observable, instead of the usual partition function.

Proposition 4.1. *The dependence on the vector bundle E is purely through its Chern character.*

Proof: As with the usual construction of a hyperkähler weight system, when we include a vector bundle E we place its Atiyah class

$$\alpha_E \in \mathrm{H}^1(X, T^* \otimes \mathrm{End}E)$$

at the univalent vertices, which lie on the circle S^1. We then contract indices and complete the construction as before. Thus E enters into the construction through the appearance of the cohomology classes

$$\mathrm{Tr}(\alpha_E^l) \in \mathrm{H}^l(X, \otimes^l T^*)$$

coming from *wheels* (also known as *hedgehogs*) with l spokes, as shown in Figure 9. Note that powers of α_E are obtained by composing elements of End$E$ and taking the cup-product in cohomology. The traces $\mathrm{Tr}(\alpha_E^l)$ are known as the *big Chern classes* of E (see Kapranov [8]).

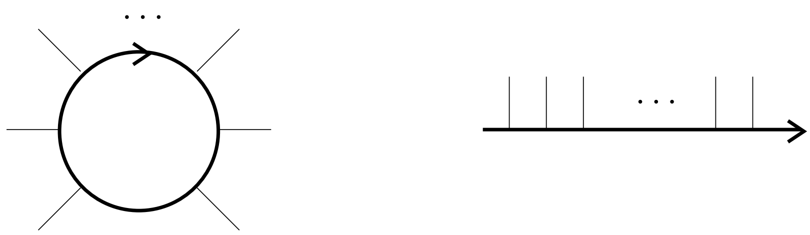

FIGURE 9. A wheel and a comb (or a hedgehog and a flat hedgehog).

Recall that we have isomorphisms

$$\mathcal{B} \xrightarrow{\chi} \mathcal{A}(I) \xrightarrow{\rho} \mathcal{A}(S^1).$$

By virtue of the isomorphism ρ, we can break a wheel to make a *comb*, as shown in Figure 9. By virtue of the isomorphism χ, we may assume that the legs of an element of $\mathcal{A}(I)$ (and hence of an element of $\mathcal{A}(S^1)$) are arranged symmetrically. In the hyperkähler context, symmetry becomes antisymmetry, and in effect we have projected the big Chern class to the exterior product, where it becomes the usual Chern class

$$\mathrm{ch}_l(E) \in \mathrm{H}^l(X, \Lambda^l T^*) = \mathrm{H}^{l,l}_{\bar\partial}(X) \subset \mathrm{H}^{2l}(X).$$

This completes the proof. □

These observables can be generalized to links with $l > 1$ components, where we associate a holomorphic vector bundle to each link component. By generalizing the above argument we can show that the dependence on these vector bundles is once again purely through their Chern classes. The crucial point is the symmetry of the diagrams which ultimately reduces the big Chern class to the usual Chern class. Recall that we have maps

$$\mathcal{B}_g \stackrel{\chi}{\longrightarrow} \mathcal{A}(\coprod_{i=1}^{g} I) \longrightarrow \mathcal{A}(\coprod_{i=1}^{g} S^1).$$

The first of these is an isomorphism, and implies that the legs of an element of $\mathcal{A}(\coprod_{i=1}^{g} I)$ may be arranged in g symmetric collections. The second map, while not an isomorphism, is surjective. Indeed $\mathcal{A}(\coprod_{i=1}^{g} S^1)$ is isomorphic to the quotient of $\mathcal{A}(\coprod_{i=1}^{g} I)$ by the link relations (this is Theorem 3 in Bar-Natan et al [4]). This implies that the legs of an element of $\mathcal{A}(\coprod_{i=1}^{g} S^1)$ may also be arranged symmetrically, and the rest of the argument follows as before.

Let us return to the knot case. In terms of the TQFT we can think of this observable in the following way. A toroidal neighbourhood $N(K)$ of the knot is a solid torus with an embedded S^1. Thought of on its own, the solid torus is a 3-manifold with boundary a genus one Riemann surface, and hence gives us an element

$$Z(N(K)) \in \mathcal{H}_1 = \bigoplus_q \mathrm{H}^q(X, \Lambda^\bullet T).$$

The complement of $N(K)$ in M also gives rise to an element

$$Z(M \backslash N(K)) \in \mathcal{H}_1 = \bigoplus_q \mathrm{H}^q(X, \Lambda^\bullet T),$$

and pairing as in the previous section gives

$$Z(M) = \langle Z(N(K)), Z(M\backslash N(K))\rangle_{\mathcal{H}_1} \in \mathcal{H} = \bigoplus_q \mathrm{H}^q(X, \mathcal{O}_X).$$

Note that the space $\mathcal{H}_1$ is the Hochschild cohomology of X, as on page 67 of Kontsevich [11] where it is denoted

$$\mathrm{HH}^m(X) := \bigoplus_{p+q=m} \mathrm{H}^q(X, \Lambda^p T).$$

The Hodge cohomology

$$\bigoplus_{i,j} \mathrm{H}^j(X, \Lambda^i T^*) = \bigoplus_{i,j} \mathrm{H}^{i,j}_{\bar\partial}(X) \subset \bigoplus_n \mathrm{H}^n(X)$$

of X acts linearly on the Hochschild cohomology $\mathcal{H}_1$, with the action given by taking the cup product in cohomology combined with the convolution operator

$$\Lambda^i T^* \otimes \Lambda^p T \longrightarrow \Lambda^{p-i} T$$

acting fiberwise. Pure Hodge classes, i.e. those lying in

$$\bigoplus_i \mathrm{H}^i(X, \Lambda^i T^*),$$

preserve the $\mathbb{Z}-$grading on $\mathcal{H}_1$ (though not the (p,q) bi-grading).

The observable is added by including the embedded S^1 with its associated vector bundle E. This changes the element $Z(N(K))$ by twisting by some element $\delta(E)$ of Hodge cohomology, using the action described above. By virtue of Proposition 4.1, $\delta(E)$ must be a characteristic class of E, and presumably it is the Chern character. In any case, it will be of pure Hodge type and so the twist preserves the degree of $Z(N(K))$. Pairing the twisted term with $Z(M\backslash N(K))$ as before gives us the expectation value of the observable

$$Z(M, \mathcal{O}(K;E)) = \langle \delta(E).Z(N(K)), Z(M\backslash N(K))\rangle_{\mathcal{H}_1} \in \mathcal{H}.$$

The action of the Hodge cohomology on the Hochschild cohomology is the hyperkähler analogue of a well-known diagrammatic operation and will be discussed in more detail in a forthcoming paper of Roberts and Willerton [18].

It is not clear how this interpretation can be modified to describe the third kind of observable, but let us make the following comments. These observables are defined for 3-manifolds with $b_1(M) = r > 0$, whereas the Murakami-Ohtsuki TQFT is only defined on rational homology spheres. So we need to assume that the hyperkähler TQFT can be extended to arbitrary 3-manifolds. Now involved in the construction is a choice of basis $\{\gamma_i\}$ for $\mathrm{H}_1(M)$ (see Thompson [21]) which could be regarded as a $b_1(M) = r$ component link. If we consider a neighbourhood of the link, as we did for knots earlier, we are led to 3-manifolds with boundaries consisting of r connected components, each component a genus one Riemann surface. Of course we don't yet have a definition of the TQFT for such 3-manifolds when $r > 1$. Nonetheless, the form λ should be holomorphic and hence we have a Dolbeault cohomology class

$$[\lambda] \in \bigoplus_q \mathrm{H}^{0,q}_{\bar{\partial}}(X, \Lambda^{l_1}T \otimes \cdots \otimes \Lambda^{l_r}T) = \bigoplus_q \mathrm{H}^q(X, \Lambda^{l_1}T \otimes \cdots \otimes \Lambda^{l_r}T) \subset \mathcal{H}_r$$

which sits inside the vector space associated to a genus r Riemann surface. This suggests that $\mathcal{H}_r$ must be closely related to the vector space associated to r disjoint genus one Riemann surfaces: the latter is probably a quotient space of the former.

We can be a little more specific when $r = 1$. Then the generator γ_1 of $\mathrm{H}_1(M)$ can be regarded as a knot K in M, and as before we get

$$Z(N(K)) \qquad \text{and} \qquad Z(M\backslash N(K)) \in \mathcal{H}_1.$$

We can discard $Z(N(K))$ and pair $Z(M\backslash N(K))$ with $[\lambda] \in \mathcal{H}_1$ instead, giving

$$\langle [\lambda], Z(M\backslash N(K))\rangle_{\mathcal{H}_1} \in \mathcal{H}$$

which presumably is the expectation value of the observable.

Note that in general given a 3-dimensional TQFT we can construct knot invariants in the following way. By removing a toroidal neighbourhood of the knot we get a 3-manifold with boundary a torus. Applying the TQFT we therefore get a vector in the vector space associated to the torus. To get knot invariants we can take the components of this vector with respect to some choice of basis of the vector space. Equivalently, we can pair this vector with another vector in the vector space. In effect all the observables in this section can be interpreted in this way.

5. Appendix

We now give the proof of Proposition 3.2 which we postponed earlier on. In fact the result we shall prove is slightly more general, and is due to Lescop [14]. The author is grateful to Christine Lescop for devising this proof and explaining it to him.

Proposition 3.2 says that the space of chord diagrams on the collection of g intervals is isomorphic to the space of chord diagrams on the chain graph Γ_g, that is

$$\rho : \mathcal{A}(\coprod_{i=1}^{g} I) \xrightarrow{\cong} \mathcal{A}(\Gamma_g).$$

The map ρ is given by attaching a certain (unitrivalent) tree to the intervals. More generally, let Δ be an arbitrary tree which has precisely $2g$ univalent vertices. By 'arbitrary' we mean that some of the vertices may have valency greater than three. When we attach this to the collection of g intervals (in some way), the resulting graph Γ may also have vertices of valency greater than three. We can consider the space of chord diagrams on Γ modulo the AS, IHX, STU, and branching relations, though we need to add additional branching relations for vertices of higher valency (see Figure 10). We denote the resulting space $\mathcal{A}(\Gamma)$ as before.

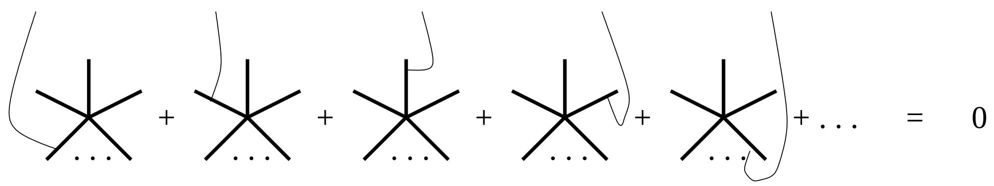

Figure 10. Branching relations for higher valency vertices.

The operation of attaching the tree Δ to the collection of g intervals as shown in Figure 11 results in a map

$$\rho : \mathcal{A}(\coprod_{i=1}^{g} I) \longrightarrow \mathcal{A}(\Gamma).$$

We use the same notation as in Proposition 3.2 as this is a generalization of that map.

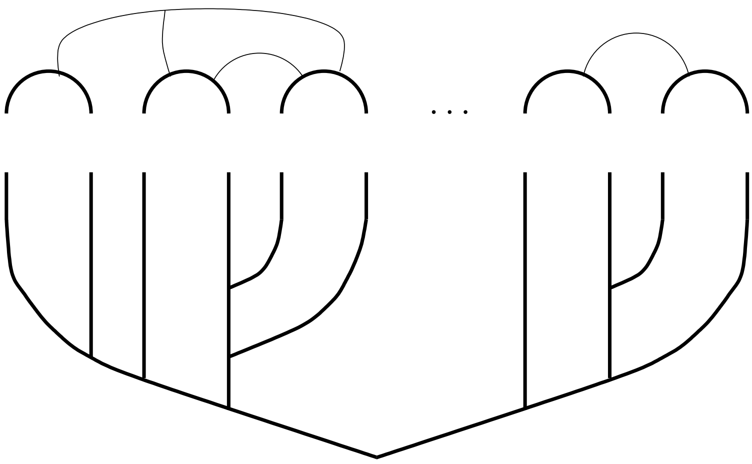

FIGURE 11. Joining a tree to a collection of intervals.

By constructing an explicit inverse to ρ we shall prove the following.

Proposition 5.1. *The map ρ is an isomorphism of $\mathcal{A}(\emptyset)$-modules.*

Proof: Recall that a chord diagram D on Γ is the union of Γ and a unitrivalent graph, the chord graph Q. Given D, we want to 'push' all the legs of the chord graph Q off the 'tree part' of Γ and onto the 'interval part' of Γ. In Figure 12 this corresponds to pushing the legs up past the dotted line. The branching relations allow us to push legs past the vertices of Γ. In order to turn this into a rigorous proof, we need to make this 'pushing' operation more precise. The following idea is due to Lescop [14].

We may assume that we begin with no legs above the dotted line. We label all the legs of Q by natural numbers 1, 2, 3, etc. Next we choose a generic point of Γ which lies below the dotted line, and call it the *root*. The dotted line intersects Γ in $2g$ points. If p is such a point, we trace down along Γ to the root making a note of which legs are encountered along the way; then we add 'labelled tabs' to Γ above p corresponding to the legs encountered and *in the same order*. We do this for all such points p. Figure 12 shows an example of how the resulting picture may look.

We are now in a position to define the inverse to ρ. Given a chord diagram D on Γ, we detach Q from Γ but keep note of the labellings of the legs. Then we take the sum

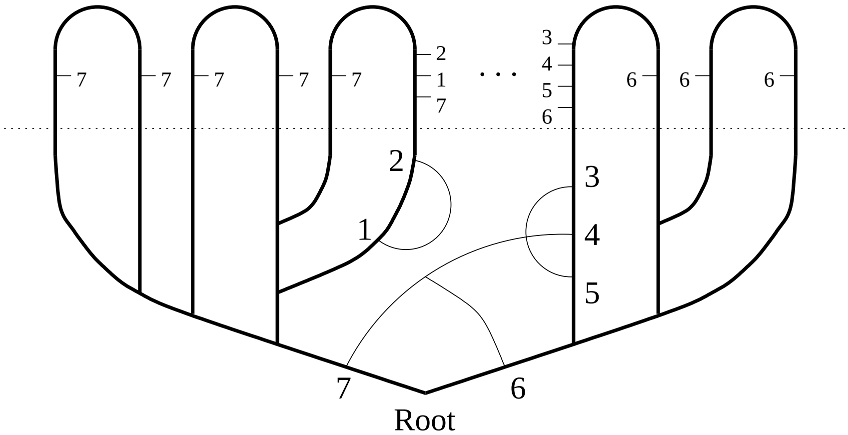

FIGURE 12. Pushing a chord diagram off a tree and onto a collection of intervals.

over all the ways to reattach Q to the labelled tabs above the dotted line. Of course a leg labelled i may only be attached to a tab labelled i. Since there are no longer any legs attached to the part of Γ below the dotted line (that is, the tree part Δ), we may remove this part and are left with a sum of chord diagrams on a collection of g intervals. We call this $\sigma(D)$.

As an element of $\mathcal{A}(\Gamma)$, we only regard D up to the AS, IHX, and STU relations. It is easy to see that if D and D' are equivalent under these relations, then $\sigma(D)$ and $\sigma(D')$ are also equivalent under the AS, IHX, and STU relations (applied to chord diagrams on the collection of g intervals). It is also clear from the way σ is defined that if D and D' are equivalent under the branching relations (*not* including the branching relation at the root) then $\sigma(D) = \sigma(D')$. This means that the only possible ambiguity in the construction concerns the root.

If we slide a leg of D past the root to get D' then the difference between $\sigma(D)$ and $\sigma(D')$ will be a sum of terms as on the left hand side of the equation in Figure 13. Note that we have only drawn the legs which are closest to the endpoints of the intervals, and there may be many more legs which we have not shown.

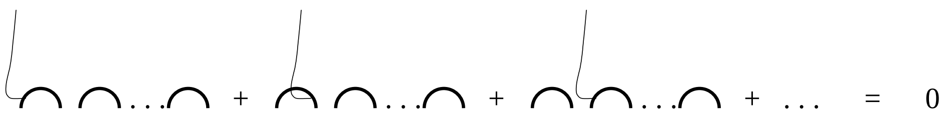

FIGURE 13. Relation in $\mathcal{A}(\coprod_{i=1}^{g} I)$.

However, the vanishing of this sum is a well-known relation in the space $\mathcal{A}(\coprod_{i=1}^{g} I)$. Thus $\sigma(D)$ and $\sigma(D')$ are equivalent chord diagrams. If the root happens to be a vertex of Γ, then 'sliding a leg past the root' would mean applying the branching relation at that vertex, and the conclusion is still valid. Instead of moving legs past the root, we could move the root itself, thus resulting in a new map σ'. An argument analogous to the one above shows that $\sigma(D)$ and $\sigma'(D)$ are equivalent chord diagrams.

Therefore we have a well-defined map

$$\sigma : \mathcal{A}(\Gamma) \longrightarrow \mathcal{A}(\coprod_{i=1}^{g} I).$$

It is now a simple exercise to verify that σ and ρ are inverses, thus proving the proposition. □

References

[1] M. Atiyah, *Complex analytic connections in fibre bundles,* Trans. Am. Math. Soc. **85** (1957), 181–207.

[2] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995), 423–472.

[3] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston, *The Århus integral of rational homology 3-spheres I: A highly non trivial flat connection on S^3,* to appear in Selecta Mathematica, preprint **math.QA/9706004 v6** (1999).

[4] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston, *The Århus integral of rational homology 3-spheres II: Invariance and universality,* to appear in Selecta Mathematica, preprint **math.QA/9801049 v4** (1999).

[5] D. Guan, *Examples of compact holomorphic symplectic manifolds which are not Kählerian II,* Invent. Math. **121** (1995), no. 1, 135–145.

[6] N. Habegger and G. Thompson, *The universal perturbative quantum 3-manifold invariant, Rozansky-Witten invariants, and the generalized Casson invariant,* preprint **math.GT/9911049** (1999).

[7] N. Hitchin and J. Sawon, *Curvature and characteristic numbers of hyperkähler geometry,* to appear in Duke Math. Jour., preprint **math.DG/9908114** (1999).

[8] M. Kapranov, *Rozansky-Witten invariants via Atiyah classes,* Compositio Math. **115** (1999), 71–113.

[9] M. Kontsevich, *Vassiliev's knot invariants,* Advances in Soviet Mathematics **16** (1993), 137–150.

[10] M. Kontsevich, *Rozansky-Witten invariants via formal geometry,* Compositio Math. **115** (1999), 115–127.

[11] M. Kontsevich, *Operads and motives in deformation quantization,* Lett. Math. Phys. **48** (1999), no. 1, 35–72.

[12] T. T. Q. Le, J. Murakami, and T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds,* Topology **37** (1998), no. 3, 539–574.

[13] C. Lescop, *Global surgery formula for the Casson-Walker invariant,* Annals of Mathematical Study **140** (1996), Princeton University Press.

[14] C. Lescop, *private communication,* (2000).

[15] J. Murakami, *Representation of the mapping class groups via the universal perturbative invariant,* preprint (1999).

[16] J. Murakami and T. Ohtsuki, *Topological quantum field theory for the universal quantum invariant,* Commun. Math. Phys. **188** (1997), 501–520.

[17] L. Rozansky and E. Witten, *Hyperkähler geometry and invariants of three-manifolds,* Selecta Math. **3** (1997), 401–458.
[18] J. Roberts and S. Willerton, in preparation.
[19] J. Sawon, *Rozansky-Witten invariants of hyperkähler manifolds,* PhD thesis, University of Cambridge (1999), **www.maths.ox.ac.uk/∼sawon**.
[20] J. Sawon, *A new weight system on chord diagrams via hyperkähler geometry,* preprint **math.DG/0002218** (2000).
[21] G. Thompson, *On the generalized Casson invariant,* Adv. Theor. Math. Phys. **3** (1999), no. 2, 249–280.
[22] G. Thompson, *Holomorphic vector bundles, knots and the Rozansky-Witten invariants,* preprint **hep-th/0002168** (2000).
[23] D. P. Thurston, *Wheeling: A diagrammatic analogue of the Duflo isomorphism,* PhD thesis, Berkeley (2000), preprint **math.QA/0006083**.

Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, United Kingdom
E-mail address: sawon@maths.ox.ac.uk