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# Torus fibrations on symplectic four-manifolds

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## 1. Introduction

This paper comprises two parts: both are concerned with the symplectic geometry of four-manifolds fibred by tori. The first part concerns the topological constraints placed on a closed four-manifold by the existence of an integrable system; the second seeks to understand a particular class of Lefschetz pencils. In the first half, we narrow ourselves to quadratic singularities by virtue of the geometric restrictions we impose, whereas in the second they appear in abundance as in Morse theory; but these remarks aside, the two parts are logically unrelated and can be read entirely independently. At the start of each part, and to establish a wider context, we have collected various motivations for the questions addressed, particularly from mirror symmetry and Seiberg-Witten theory; we should make clear, however, that we say nothing significant about either here.

### Part 1. Essential Lagrangian fibrations

The aim of this part of the paper is to describe those closed symplectic four-manifolds which admit a "controlled" and "tame" fibration by homologically essential Lagrangian tori over a compact base surface (Theorem 2.1). Tameness enters in assuming control on the possible singular fibres. As one might expect, to fibre-preserving diffeomorphism the only examples are the $K3$ surface and torus bundles over tori with $b_1 \geq 3$. Our proof will compare ideas from integrable systems, which we survey at some length, with results from gauge theory. The arguments are straightforward, but do not seem to have been made explicit before: the natural questions in the community of topologists are perhaps less motivated to those working in dynamical systems, and vice versa. Before stating a precise result we provide some background to questions of this type.

## 2. Motivations

**Mirror symmetry:** In recent years, the phenomenon of "mirror symmetry" for Calabi-Yau manifolds has attracted a great deal of attention. Coming initially from physics, there are now various strands to mirror symmetry in mathematics, revolving around (i) dualities between counting curves and variation of Hodge structure, (ii) certain natural

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equivalences of derived categories of sheaves and of Lagrangian cycles, and (iii) the existence of toric fibrations of rather particular kinds, usually in each case on suitable pairs of "mirror" Calabi-Yau manifolds. It is the third flavour that gives the motivation for this paper. Precisely, mirror symmetry conjectures that a closed Calabi-Yau manifold near a "large complex structure limit" point of moduli space should admit a fibration by closed special Lagrangian submanifolds. In particular, a Kähler symplectic manifold with trivial first Chern class should always admit a Lagrangian fibration. Here the term "fibration" is used as in algebraic geometry, and singular fibres are expected. Conjecturally, dualising this fibration appropriately gives a "mirror" Calabi-Yau and sets up equivalences both of topological data and of derived categories as hinted above. The structure sheaf of the Calabi-Yau is mirror to a section of the special Lagrangian fibration, which must therefore have essential fibres. The distinguished Kähler metric on a Calabi-Yau manifold is part of the given data, and is reflected in the final structure: a fibration by minimal submanifolds. Given the notorious difficulties in constructing special Lagrangian submanifolds, it is natural to discard this particular requirement and look first at the topology or symplectic geometry of the manifolds under consideration. In three dimensions this has been addressed by Gross [15] and Ruan [25]. In two complex dimensions, aspect (iii) of mirror symmetry is well understood, since there are so few examples of Kähler manifolds with trivial first Chern class. Any such is hyperkähler and in fact biholomorphic to a $K3$ surface or to an abelian surface. By the "hyperkähler rotation" trick, one can find explicit special Lagrangian fibrations. However, if we drop the requirement that our manifold be Kähler, then in principle we might expect large classes of new examples, and the notion of a Lagrangian fibration if not a special Lagrangian fibration still makes good sense. In fact symplectically there seem to be more manifolds with trivial first Chern class than with essential Lagrangian fibrations, whilst inessential Lagrangian fibrations exist on many manifolds with $c_1 \neq 0$.

**Integrable systems:** Foliations of symplectic manifolds by Lagrangian tori arise from completely integrable systems of equations, and describe a number of classical problems in mechanics [1]. These families are usually non-compact; however, in algebraic geometry *toric varieties* provide a large and often tractable source of manifolds for study. A variety is toric if it contains a dense algebraic torus $(\mathbb{C}^*)^n$ of its complex dimension. In this case the obvious action of the torus on itself extends to give an action of the torus on the total space. In symplectic geometry the equivalent notion is that of a Hamiltonian torus action of *complexity zero*, that is a Hamiltonian action by a torus of half the real dimension of the manifold. When such an action exists, there is a *moment mapping* which defines a fibration of the manifold over a compact polytope in some Euclidean space. The fibres here are homologically trivial, and the singular fibres are of "elliptic type" (cf. Definition 3.2) being tori of smaller dimension. The toric structure constrains the manifold considerably. In particular, the preimages of the different boundary strata of the moment polytope - which represent the strata of the locus of critical values of the fibration - give distinguished cycles representing the Chern classes of the manifold, each cycle having a smooth dense subset which is naturally symplectic. (The analogue of this result for more general Lagrangian fibrations gives strata which may be symplectic with the opposite orientation.) There is an old classification of symplectic four-manifolds admitting Hamiltonian torus actions of complexity zero: the only such are the projective plane and Hirzebruch surfaces, and certain blow-ups of these [2], all of which are toric varieties.

In general, the literature on integrable systems concentrates on trying to understand the topological structure of the *flows* of the Hamiltonian function on the involutive submanifolds, particularly near singular fibres. In four dimensions, considerable work has been done in this direction by Lerman and Umanskiy [19], who classify Poisson actions of the Euclidean plane on a symplectic four-manifold in order to obtain detailed information on possible orbit structures. In another direction, Fomenko and his collaborators [4] have studied the orbit structures for geodesic flows on cotangent bundles of surfaces of small genus. By contrast, we are concerned with the global topology. Moreover, a seemingly innocuous demand that our fibrations have connected and essential fibres will eliminate much of the structure often seen in the foliations arising in dynamical systems and shall lead to the very restricted class of examples.

**Lefschetz fibrations:** Symplectic manifolds which are the total spaces of fibrations with *symplectic fibres* have been of particular prominence following Donaldson's existence theorem for Lefschetz pencils [5]. Moreover their holomorphic analogues provide a classical source of examples, much as integrable systems provide a classical source of Lagrangian fibrations. The only complete classification for (four-dimensional) Lefschetz fibrations is when the fibres have genus one; all such are holomorphic. There are various possible generalisations of this result one might look for: a classification of holomorphic genus two fibrations, of all Lefschetz genus two fibrations, of torus fibrations over higher dimensional base etc. Only the first of these goals has been substantially realised. In another direction, one might hope to classify elliptic fibrations with more general singularities, and in the symplectic context the singularities that arise (at least generically) in a Lagrangian context are very natural.

**Statements of results:** In order to state a theorem, we need to give a definition (and refer to some others which we will defer):

**Definition 2.1.** Let $(X, \omega)$ be a closed symplectic manifold. $X$ is *Lagrangian fibred* if there is a compact manifold $B$ and a smooth map $f : X \to B$ with compact fibres, such that the generic fibre is smooth and $\omega$ restricts trivially to the smooth locus of every fibre. The fibration is *essential* if $[f^{-1}(p)] \neq 0 \in H_2(X; \mathbb{Z})/\langle Torsion\rangle$ and *connected* if all the fibres are connected.

In general, the critical locus of $f$ will have codimension one in the base, and the smooth fibres may not all have the same topological type. It is a standard fact ([1] and see later) that the generic fibre of such a system has closed torus components. Given the requirements of mirror symmetry, and conversely the existence of a satisfactory classification of Hamiltonian $\mathbb{T}^2$-spaces, we will later insist that our fibrations are essential and connected. We will also need to assume that the singularities of the fibration are "controlled" or "tame"(Definition 3.3); this is a non-degeneracy condition, roughly ensuring that the critical values form a stratified symplectic space which is not of excess dimension in any fixed fibre. The topological restrictions on the fibres and on the singularities play off against one another, and lead to the following (which should be well known to experts):

**Theorem 2.1.** *The manifold $(X^4, \omega)$ admits a tame essential connected Lagrangian fibration if and only if $X$ is diffeomorphic to a $K3$ surface or to a smooth torus bundle over a torus with $b_1 \geq 3$.*

In the next section, we discuss the roles of the various assumptions. It will follow from the proof that the diffeomorphism preserves fibres. Up to arbitrary diffeomorphisms, these torus bundles are all of the form $\mathbb{S}^1 \times Y_n$ where $Y_n$ is the mapping torus of the $n$-th power of a Dehn twist about a meridian of the torus, for some $n \in \mathbb{Z}$. For $n = 0$ we obtain the four-torus, whilst $n = 1$ corresponds to the Thurston manifold [32]. Whenever $n \neq 0$ the manifold admits two essentially different torus fibrations; each admits Lagrange forms, but only one fibration admits sections. We remark that *not all* symplectic four-manifolds with $c_1 = 0$ admit Lagrangian fibrations, even allowing inessential fibres. We will give examples, presumably well-known, to illustrate this later in the paper. Remark also that the classification of symplectic four-manifolds with smooth Lagrangian fibrations is relatively simple, and was accomplished (even to symplectomorphism) by Mishachov in [22]. Under our numerous restrictions, the proof of (2.1) will reduce to a more general result: no relation in the mapping class group composed of both positive *and* negative Dehn twists can give rise to a symplectic manifold with a smooth fibre a symplectic submanifold. As a statement on achiral Lefschetz fibrations (4.1), this may be of independent interest.

## 3. Singularities and Chern classes

This section surveys some of the classical theory of integrable systems, with a view towards the geometry of relevance for the global topology of a closed four-dimensional phase space. In particular, we aim to understand the first Chern class of the total space. None of the material of the section is original. A critical error in an earlier version was pointed out to the author by N.T. Zung. Our treatment is a blend of those of Duistermat [7] and Gross [15]; the classic work of Arnol'd [1] also covers the relevant material.

Suppose $(X, \omega)$ is a symplectic manifold and $f_1, \ldots, f_n$ are smooth functions on (some patch of) $X$ which are in involution: the Poisson brackets $\{f_i, f_j\}$ all vanish. The differentials $df_i$ are dual, via $\omega$, to vector fields $X_i$ whose flows preserve the fibres of $F = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$. Integrating these flows, we define an action of the vector space $\mathbb{R}^n$ on the fibre $F^{-1}(t)$; since the group and the orbit have the same dimension, the set of vector fields acting trivially must be discrete. It follows that the orbits of the action - components of the fibres of $F$ - are products $\mathbb{T}^a \times \mathbb{R}^{n-a}$, and any compact connected fibres must be closed *affine tori.* Note that, by associating to a vector field with periodic flow a solution curve to that flow, we can identify the lattice $L_t \subset \mathbb{R}^n$ of vector fields which stabilise a point of the fibre with the first homology $H_1(F^{-1}(t);\mathbb{Z})$.

Now let $f : X \to B$ be a smooth Lagrangian fibration. If we choose a co-ordinate chart $\chi : U \to \mathbb{R}^n$ on a patch $U \subset B$ with components $\chi_i$, then one easily checks that the functions $\chi_i \circ f$ are in involution on $X$. We obtain an exact sequence

$$0 \to TF_x \to TX_x \to f^*TB_x \to 0;$$

the dual of the last map gives a natural map $T^*B_{f(b)} \to (\nu^*_{F/X})_x$ from the cotangent bundle of the base to the conormal bundle of the fibre (that is, the covectors on the total space vanishing on the tangent space to the fibre). The symplectic form on $X$ gives a natural identification between the conormal bundle of the fibre and the tangent bundle of the fibre; hence a covector on the base defines a vector field on the fibre. One can exponentiate this to define a transitive fibre-preserving action of $T^*B$ on $X$, which is just the invariant description of the action above. In the case where the fibres are compact, the lattices in each fibre form a submanifold $L \subset T^*B$ which is Lagrangian with respect to the natural symplectic form on the cotangent bundle. Moreover locally, $X$ can be identified with the quotient $T^*B/L$, and the projection of $L$ to $B$ is a topological covering map. The duals of the lattices $L_b \subset T^*B_b$ give a smoothly varying family of lattices in the fibres of $TB$, which is just a *flat affine connexion* in the base. Indeed the connexion is "integral affine"; if we identify $(L_b)^*$ with $H_1(f^{-1}(b);\mathbb{Z})$ as above then the monodromy of the connexion is determined by a representation of the fundamental group

$$\rho : \pi_1(B;b) \;\to\; \mathrm{Aut}(L_b) \cong Sp_{2n}(\mathbb{Z})$$

which defines the covering $L \to B$ induced from the projection $L \subset T^*B \to B$. Hence the affine connexion has monodromy inside $GL_{2n}(\mathbb{Z}) \subset GL_{2n}(\mathbb{R})$ in a suitable basis. (The only closed surfaces with flat affine structures have Euler characteristic zero by a theorem of Milnor [23]; this was used by Mishachov [22] to classify to symplectomorphism the closed symplectic four-manifolds admitting smooth Lagrangian fibrations.)

**Lemma 3.1.** *If $f : X \to B$ is a smooth Lagrangian fibration, then the Chern classes of $X$ all vanish.*

*Proof.* From the exact sequence above, we identify the tangent bundle $TX_x$ with the sum $f^*TB_{f(x)} \oplus Tf^{-1}(f(x))_x$, and the second term is canonically identified by $\omega$ with $f^*T^*B_{f(x)}$. Hence $TX = f^*(TB \oplus T^*B)$ and the total Chern class $c(X)$ of $X$ is just the pullback $f^*p(B)$ of the total Pontrjagin class of the base. But the base has a flat affine structure, and hence has trivial Pontrjagin classes. □

Hence, for instance, there is no "hyperkähler rotation" trick which makes the trivial four-torus fibration of $K3 \times \mathbb{T}^4 \to K3$ a Lagrangian fibration.

**Remark 3.1.** If $f$ has a global section then $X$ is isomorphic to $T^*B/L$. More generally, the choice of a local section provides such an identification locally, and gives distinguished local (*action-angle*) co-ordinates on a patch of $X$. Different choices of local section define a cocyle in $H^1(B; \Lambda(T^*B/L))$ where $\Lambda(\cdot)$ denotes the sheaf of Lagrangian sections. From the coboundary map in a long exact sequence induced by

$$0 \to L \to \Lambda(T^*B) \to \Lambda(T^*B/L) \to 0$$

we obtain a "Chern class" in $H^2(B; L)$ - a twisted sum of $n$ copies of $H^2(B; \mathbb{Z})$ - which measures the obstruction to the existence of a global smooth section; such a section is homotopic to a Lagrangian section whenever the symplectic form is exact on its image. This is the starting point for the "classical" investigation of Lagrangian fibrations and integrable systems, as for instance in the papers of Nguyen Tien Zung [35].

We now move to the situation where the fibration has singular fibres. By construction, even where the map $df$ is not surjective, but on the smooth parts of the singular fibres, the above considerations still define an action of $T^*B$, which is trivial on a subspace defined by the kernel of $df^*$. Gross [15] shows that $X^0 = \{x \in X \mid df_x \text{ is onto}\}$ is locally a fibre space of groups, and for any open set $U \subset B$ there is an action of the smooth one-forms over $U$ on $f^{-1}(U)$. Given any point $x \in X$ the orbit of the group action applied to $x$ is given by $\mathbb{T}^a \times \mathbb{R}^b$ with $a + b = rk(df_x)$, for the orbit is a discrete quotient of the vector space $T^*B_{f(x)} / \ker(df^*)_{f(x)}$. We make a provisional:

**Definition 3.1.** Suppose $(X^4, \omega)$ is Lagrangian fibred. The fibration has *controlled singularities* if the critical points of $f$ form a finite union of closed submanifolds of dimension at most two, whose intersection with any fibre is of dimension at most one.

Note that we are excluding multiple fibres. It is reasonable to expect that "many" Lagrangian fibrations can be perturbed to have controlled singularities, but we shall not attempt to prove such a statement. Rather, we work by analogy: not every holomorphic singularity admits a Morsification, but nonetheless Lefschetz fibrations play a central role in various aspects of complex geometry. With this in mind, in the Lagrange setting we shall discuss local models for the possible singularities of the map $f : X \to B$. We give the following definition with the four-dimensional case in mind, but the concept is clearly more general.

**Definition 3.2.** Let $P$ be a critical point of a Lagrangian fibration $f : X \to D$ and choose real co-ordinates $(p, q, r, s)$ at $P$ such that the symplectic form is given by $dp \wedge dq + dr \wedge ds$. The point is of

- *elliptic type* if $f$ has the model $(p, q, r, s) \mapsto (p^2 + q^2, r)$;
- *hyperbolic type* if $f$ has the model $(p, q, r, s) \mapsto (p^2 - q^2, r)$;
- *focus-focus type* if $f$ has the model $(p, q, r, s) \mapsto (ps - qr, pr + qs)$

with respect to some such set of real Darboux co-ordinates. Singularities modelled on (products of) the above will be called *non-degenerate* singularities.

We claim that, in a precise sense, these singularities represent the only generic local degenerations of a four-dimensional integrable system. Let $P$ be a singular point of a

Lagrangian fibration, that is a point where $df$ is not surjective. Suppose locally the map $f$ is given near $P$ by a pair of real-valued functions $f = (F_1, F_2)$. As before, the differentials $dF_i$ define commuting vector fields $X_i$ near $P$; the quadratic parts $(d^2F_i)_P$ define an abelian subalgebra $\mathcal{A}$ of the Lie algebra of quadratic forms on the tangent space to the manifold at $P$. In the presence of the symplectic form, this Lie algebra structure is defined by setting $[d^2g_P, d^2h_P] = d^2\{g, h\}_P$. It is naturally equivalent to the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$.

The dimension $c$ of the algebra $\mathcal{A}$ is exactly the corank of the differential, that is $c = 2 - rk(df_P)$. There is a symplectic quotient space $Q = \ker(df)_P/\langle X_1, X_2\rangle$ and $\mathcal{A}$ is of maximal rank amongst abelian subalgebras of the algebra of quadratic forms on $Q$, which can be identified with $\mathfrak{sp}_{2c}(\mathbb{R})$. Now such a maximal rank abelian subalgebra is *generically* a Cartan subalgebra; it contains some element with distinct eigenvalues, equivalently some diagonalisable element. Indeed the space of abelian non-Cartan subalgebras of rank $k$ in $\mathfrak{sp}_{2k}$ is a real subvariety and hence of positive real codimension in the space of all such abelian subalgebras. A theorem of Williamson [33] (which is also treated in [1]) shows that every Cartan subalgebra of a real symplectic algebra $\mathfrak{sp}_{2k}(\mathbb{R})$ has a basis consisting of functions or function-pairs of elliptic, hyperbolic or focus-focus type. Accordingly, these singularities are *non-degenerate*, where we borrow the terminology of Eliasson [8] and other authors. The analogous statements in higher dimensions are all valid.

Remark that for systems with two degrees of freedom (that is, integrable systems on two-manifolds), non-degenerate singularities are indeed exhaustive after perturbation; that is, they are dense in a suitable function space. This is precisely the content of the Morse lemma for functions on surfaces.

**Definition 3.3.** A Lagrangian fibration $f : X^4 \to B$ is *tame* if it has controlled singularities and every isolated fixed point is non-degenerate.

Equivalently, we could assume that above all the smooth points of the locus of critical values $f$ is given by a product of elliptic, hyperbolic or focus-focus singularities. For if $P$ is a critical but not fixed point, the rank of $df$ at $P$ is one and there is a non-trivial circle action on the fibre. Using this, Eliasson and other authors [8] have shown that one can "split off" this non-degenerate factor and reduce to an integrable system on a smaller phase space: locally the Lagrangian fibres are products of circles and one-dimensional graphs. But the only singularities in the lower-dimensional case which persist after a perturbation are the non-degenerate singularities, by Morse theory. A controlled Lagrangian fibration has as locus of critical values some one-dimensional CW-complex. The smooth subsets of the edges have non-degenerate singularities, whilst the singularities of the complex do not arise from isolated critical points: tameness amounts to fixing the fibres over isolated vertices in the critical complex to be of focus-focus type. (These are fixed points of the Poisson action of vector fields on the fibre.) After perturbation, four-dimensional Lagrangian fibrations with controlled singularities are expected to obey the tame condition, since the critical locus in the base is compact one-dimensional whilst the locus of non-Cartan algebras has real codimension one.

**Remark 3.2.** There is a misleading but conventional gloss here. "Non-degenerate" singularities of integrable systems are traditionally those corresponding to Cartan subalgebras, and they are generic in the sense that Cartan subalgebras are generic amongst abelian subalgebras. Nonetheless, we do not assert (and the author knows of no theorem in the literature which asserts) that a Lagrangian fibration can be deformed, in the space of families of Poisson-commuting functions, to a family for which generic singular fibres are of Cartan type.

It may be helpful to note possible topological types of generic singular fibres:

- an elliptic fibre is a circle $\mathbb{S}^1$ given by collapsing one of the circle factors in $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ to a point;
- a hyperbolic fibre is the product of a circle and a figure-eight $\mathbb{S}^1 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$ given by collapsing two points of one $\mathbb{S}^1$ factor of $\mathbb{T}^2$ together;
- a focus-focus fibre is a nodal (semistable) elliptic curve, given by collapsing a single meridian circle $\mathbb{S}^1 \subset \mathbb{T}^2$ to a point.

We can illustrate these three behaviours with examples of manifolds displaying such singular foliations (the second example was pointed out to the author by N.T. Zung):

- $\mathbb{CP}^2$ admits a Lagrangian fibration over a closed triangle, with elliptic singular fibres over the edges and product elliptic singular fibres - that is, points - over the vertices; this is just its moment map fibration as a toric variety. (Products of elliptic singularities are the only ones appearing in the theory of moments maps and for Hamiltonian systems.)
- A product of Riemann surfaces $\Sigma_g \times \Sigma_h$ admits a smooth map to the torus with Lagrangian fibres; take the obvious map $\Sigma_g \to \mathbb{S}^1$ with $2g - 2$ hyperbolic singular points, arising from a circle-valued Morse function. Note that not all fibres of the map are connected.
- The $K3$ surface admits a Lagrangian torus fibration with precisely 24 focus-focus singular fibres and no others.

Lerman and Umanskiy [19] show that for non-degenerate fibres, the "obvious" cell structure on the fibre does indeed correspond to the stratification of the fibre in terms of orbits of any local Hamiltonian flow. In the examples, focus-focus singularities occur in real codimension two in the base, whilst elliptic or hyperbolic singularities occur generically over one-dimensional loci. This is always the case:

**Lemma 3.2.** *Let $p$ be a point on a singular fibre which is an elliptic or hyperbolic singularity. Then the locus $C$ of critical points of $f$ near $p$ is smooth and two-dimensional, and the symplectic form $\omega$ restricts as a non-degenerate form to $C$.*

This is a direct consequence of the local algebras of $f$ being Cartan. It leads to the following important:

**Proposition 3.3.** *Let $f : X^4 \to B$ be a Lagrangian fibration with tame singularities. The first Chern class is supported on the locus of critical values of $f$, and satisfies $c_1(X) \cdot [\omega] =$*

[Ell] − [Hyp]*; here the terms* $[\cdot]$ *denote the integral of the symplectic form over the strata of elliptic (resp. hyperbolic) singularities.*

*Proof.* Recall the following construction (cf. [24], p.171) of the Chern classes of a vector bundle $E \to M$. Write $V_k(E)$ for the complex bundle with fibre at $m$ the Stiefel manifold of $k$-frames in $E_m$. The projection $p_k : V_k(E) \to M$ induces

- an isomorphism $H^i(M;\mathbb{Z}) \to H^i(V_k(E);\mathbb{Z})$ for each $0 \le i \le 2n-2k$, since $V_k(\mathbb{C}^n)$ is $(2n-2k)$-connected;
- a pullback bundle $p_k^* E \to V_k(E)$ which splits canonically as the sum of a trivial rank $k$ bundle and an orthogonal complement $F_{(k)}$ of rank $n-k$.

Then we can set $c_{n-k}(E)$ to be the Euler class of $F_{(k)}$; when $k=0$ this is the Euler class of $E$. Thus the $k$-th Chern class $c_k(E)$ is exactly the primary obstruction to the existence of a cross-section of the bundle with fibre $V_{n-k+1}(E_m)$. In particular, if a section of this bundle exists over a set $U \subset M$ then $c_j(E)$ belongs to the image of the natural map $H^{2j}(M,U;\mathbb{Z}) \to H^{2j}(M;\mathbb{Z})$ for each $j \ge n-k+1$, or by excision some geometric representative for $c_j(E)$ will be supported on $M\backslash U$. It follows from the proof of (3.1) that the Chern classes are all supported on the critical fibres of a Lagrangian fibration $f : X \to B$. Explicitly, at each point $x \in X^0$ of the smooth locus of $f$ we have a natural half-dimensional subspace of the tangent bundle, namely $\ker(df)_x$, the vertical tangent space. We can lift the section defined by the vertical tangent spaces from the Grassmann manifold $Gr_n(TX)$ to $V_n(TX)$, and hence $c_1(X)$ is supported on the critical locus.

If $f$ has tame singularities, then there are finitely many points $b \in B$ which are critical values of $f$ over which the singular fibres are not non-degenerate. In each of these, by Definition (3.1), the critical set is one-dimensional, so we can remove these loci from $\mathrm{Crit}(f)$ without affecting the value of $c_1(X)\cdot[\omega]$. We are now in a situation where all the singular fibres of $f$ are of elliptic, hyperbolic or focus-focus types, or are products thereof. As we have noted, the fact that these singularities give rise to Cartan abelian subalgebras says that the critical strata

$$\{x \in X \mid rk(df_x) = i\} \;=\; \Delta_i \qquad \text{for } 0 \le i \le n-1$$

with $n$ the real dimension of the base $B$ ($n=2$ in our case), are locally submanifolds to which the symplectic form restricts as a non-degenerate form. In a neighbourhood of such a non-degenerate singular fibre, the topology of the model and of the map $f$ is completely determined. This uniqueness is treated by Eliasson [8] and in the first paper by N.T. Zung [35]. Using this model, one can compute the contribution of the Chern class $c_1(X)$ made by each stratum in the locus $\mathrm{Crit}(f)$. There are two relevant local models: for elliptic singularities, as with toric varieties, the symplectic form is positively oriented on the strata and these contribute positively to $c_1$. For hyperbolic singularities, the sign of the symplectic form is opposite: as above, a model is given by the fibration of a product of Riemann surfaces over a torus. The result follows. □

One can also use the local models to study the higher Chern classes. We draw attention to the easiest case, which we shall need later: a focus-focus singularity which occurs as an isolated point, by comparison to an elliptic Lefschetz fibration, will[1] contribute +1 to $c_2(X)$. More generally, one can proceed with a local computation, akin to the $K$-theory presented by Gross in ([16], Theorem 2.17), to find the contribution of any given cell.

We should say a few words about neighbourhoods of singular circles. If $p$ is an elliptic or hyperbolic point, and $f$ has rank 1 at $p$, then there is a circle action (from the non-degenerate component of $df$) on the fibre through $p$. Flowing along this circle gives rise to a circle of elliptic or hyperbolic singular points. If we take a small arc in the base $B$ transverse to the (smooth) point of the locus of critical values over which $p$ lies, then we have a fibration over an interval with a single singular fibre. A neighbourhood of the critical circle in this three-manifold has one of three types:

- For an elliptic singular circle, the neighbourhood is a solid torus and the circle action defines the trivial foliation given by $\mathbb{S}^1 \times \mathbb{S}_t$ for $t$ a radial co-ordinate in the transverse disc.
- For an *oriented hyperbolic* singular fibre, after flowing around the critical circle we re-glue the disc to itself by a map isotopic to the identity. The critical circle appears as the intersection of two annuli given by sweeping out a cross neighbourhood of the hyperbolic point in the set of orbits in the critical fibre. The foliation of the solid tube has one ($\mathbb{S}^1 \times$ Cross) fibre and the others are unions of ($\mathbb{S}^1 \times$ Two arcs).
- For an *unoriented hyperbolic* singular fibre, after flowing around the critical circle we re-glue the disc by a map isotopic to the rotation by $\pi$. The critical circle now appears as the intersection of two Möbius bands in the solid tube, and the other leaves of the local foliation are connected.

The oriented hyperbolic singularities occur for the fibration of $\Sigma_g \times \Sigma_h \to \mathbb{S}^1 \times \mathbb{S}^1$, but if we start with an unoriented two-dimensional surface then we obtain unoriented hyperbolic singularities. For instance, the Klein bottle $\mathcal{K}$ maps to the circle $\mathbb{S}^1$ with a single hyperbolic singularity, and the neighbourhood of the resulting circle in $\mathcal{K} \times \mathcal{K} \to \mathbb{S}^1 \times \mathbb{S}^1$ is unoriented. Now we add the hypothesis that the fibres are essential and connected. As our local models suggest, this has a striking effect:

**Proposition 3.4.** *If $X$ is Lagrangian fibred and all the smooth fibres are connected and essential, then $c_1(X) = 0$.*

*Proof.* We can eliminate elliptic singular fibres, which are of a smaller dimension; by Poincaré duality and general position arguments, these cannot occur in essential fibrations. For hyperbolic singularities, there are two local models. Let $P \in B$ be a smooth point of the one-dimensional locus of critical values of $f$ and let $S$ be the circle of hyperbolic singularities lying over $P$. Since the rank of $df$ is one at $P$, if we take a tranverse arc $I$ to Crit$(f)$ in $B$ then the three-manifold $f^{-1}(I)$ has a global $\mathbb{S}^1$ action; we can split this off as a product, locally, according to the normal form results of Eliasson, Zung

[1] N.B. independent of its orientation (chirality)

and others. (In fact we only use this as a topological statement, which is fairly easy.) In the simplest case when the fibre contains no other critical circles, there are now two models: either the resulting two-manifold is modelled on a neighbourhood of a hyperbolic singularity in $\Sigma_g \to \mathbb{S}^1$ or it is modelled on a neighbourhood of a hyperbolic singularity in $\mathcal{K} \to \mathbb{S}^1$. In the first instance, on one side of the critical point $P \in I$ the fibres are disconnected, and in the second instance the fibres over points of $I$ are homologically trivial. Since the singular fibre over $P$ is compact and connected, we know it is a product of $\mathbb{S}^1$ with some "generalised figure-eight graph" (i.e. an immersed circle in the plane). The generalisation to the local model here is straightforward.

It follows that the fibration can have no elliptic or hyperbolic singularities. But then if the one-dimensional locus of $\mathrm{Crit}(f)$ is nowhere smooth it is empty, and the proposition (3.3) then implies that $c_1(X) = 0$. □

Our observations on Chern classes tie back into familiar results from other areas. For instance, a result due to Liu [20], following from work of Taubes, asserts that all symplectic four-manifolds with $c_1 \cdot [\omega] > 0$ have $b_+ = 1$. Now an easy lemma on intersection forms on manifolds with $b_+ = 1$ shows that given cohomology classes $\alpha$ and $\beta$ with $\alpha^2 > 0$ and $\alpha \cdot \beta = \beta^2 = 0$, we must have $\beta = 0$. We can apply this with $\alpha$ the symplectic form and $\beta$ the essential Lagrangian fibre of a hypothetical fibration to see that any Lagrangian surface of square zero must be inessential. But of course if we have no hyperbolic singularities, we have seen that $c_1(X) \cdot [\omega] = [\mathrm{Ell}]$ is non-negative, and positive only if indeed the fibres are inessential and elliptic singularities are present. Of course one can achieve the classification of Hamiltonian spaces using moment map techniques without recourse to the Seiberg-Witten theory. Note also that even if we restrict to fibrations with connected fibres over a closed base, the need for essential fibres above is highlighted by the Enriques surface, which admits inessential Lagrangian fibrations over $\mathbb{RP}^2$.

## 4. Achiral genus one fibrations

Returning to the realm of mirror symmetry, we start with a symplectic four-manifold $X$ with an essential connected Lagrangian fibration $f$ and with trivial first Chern class. Going back to the models for our possible controlled and tame singular fibres, it follows that the critical locus has complex codimension two in $X$; that is, $\mathrm{Crit}(f)$ comprises finitely many points. Each of these must be a focus-focus singularity, which is just an *achiral Lefschetz singularity* in the usual terminology of [13]. Thus we are led to consider achiral genus one Lefschetz fibrations. One can complete the proof of (2.1) directly, but we will digress in order to offer a result valid in greater generality. Recall that positive and negative Lefschetz singularities correspond to the respective local models $(z_1, z_2) \mapsto z_1^2 + z_2^2$ and $(z_1, z_2) \mapsto \overline{z}_1^2 + z_2^2$, and an "achiral" fibration is one in which *a priori* both models could occur.

**Proposition 4.1.** *Let $(X, f)$ be an achiral Lefschetz fibration and suppose $X$ admits a symplectic structure for which at least one fibre is a symplectic submanifold. Then the singularities of $f$ all have the positive orientation.*

*Proof.* Suppose for contradiction that there are singular fibres of both orientations. By the neighbourhood theorems for symplectic submanifolds, we may perturb the symplectic form on $X$ near a symplectic fibre to make the fibres over a small disc of regular values of $f$ symplectic submanifolds. (If we do not assume that any fibres of $f$ are symplectic, it may not be possible to perform this perturbation without losing non-degeneracy elsewhere on $X$.) Fix a base-point $u \in X$ inside such a small disc, and choose a set of paths to the critical values of $f$ to yield a sequence of vanishing cycles in the fixed smooth fibre $F_u$. Suppose $\delta$ and $\sigma$ are vanishing cycles corresponding to critical points of opposite sense. Assume both are either non-separating or both are separating in the fibre $F$; then there is some diffeomorphism $\phi : F \to F$ which takes the circle $\delta$ to $\sigma$. (If $\delta$ separates and $\sigma$ does not, fibre sum with a symplectic Lefschetz fibration which contains both separating and non-separating positively oriented singular fibres, and then continue as before.)

Now form a fibre sum of two copies of $X$ along the (symplectic regular) fibre $F_u$, twisting by the diffeomorphism $\phi$. The resulting manifold $Z = X \sharp_F X$ is symplectic by Gompf's theorem [12], and is the total space of an achiral Lefschetz fibration which has two singular fibres with isotopic vanishing cycles but occuring with opposite orientations. Fibre sum $Z$ with itself three times by the identity, after perturbing the symplectic form to give a symplectic structure near a regular fibre as before. Call the new fibration $\mathcal{Z}$. We are now in the following situation:

- We have an achiral Lefschetz fibration $\mathcal{Z}$ for which the monodromy word has the shape $u \cdot (\delta\delta^{-1}) \cdot (\delta\delta^{-1}) = \prod [a_i, b_i]$. Here $u$ is some word in positive and negative twists in a mapping class group, giving a fibration over a disc $\Delta \supset \{\text{Crit}\}$, and the commutators arise from the fundamental group of the base $\pi_1(B \backslash \Delta)$.
- $\delta$ represents the positive Dehn twist along a curve $C$ which is homotopically trivial in the fibration over the disc $\Delta$ defined by the word $u$.

Inserting a trivial relation $\langle \delta\delta^{-1} = 1 \rangle$ into a mapping class group relation amounts to performing a surgery on the manifold along the curve $C = \text{support}(\delta)$. Then an elementary exercise in Kirby calculus ([13], Example 8.4.6) shows that when $C$ is nullhomotopic in the complement of this inserted relation, the surgery connect sums with a sphere bundle over a sphere. It follows that $\mathcal{Z}$ is described as

$$\mathcal{Z} \;\cong\; W \sharp (\mathbb{S}^2 \tilde{\times} \mathbb{S}^2) \sharp (\mathbb{S}^2 \tilde{\times} \mathbb{S}^2)$$

for some manifold $W$. Here the $\tilde{\cdot}$ denotes either the product or non-trivial sphere bundle over the sphere. But this contradicts Seiberg-Witten theory: each of the pieces $W \sharp (\mathbb{S}^2 \tilde{\times} \mathbb{S}^2)$ and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ have positive $b_+$, and hence the connected sum has trivial Seiberg-Witten invariants [34], whereas all symplectic manifolds have non-trivial SW invariants [31]. □

Regress to the main argument. Suppose again that $X$ is an essential Lagrangian fibration. We know that $c_1(X) = 0$, hence $X$ is an achiral genus one fibration. By Gompf's observation we can perturb one Lagrangian fibre to be symplectic: then by the above $X$ is in fact the total space of a chiral symplectic Lefschetz fibration over some base surface $B$. It follows by a standard argument also due to Gompf [13] that there is another symplectic structure on $X$ which is symplectic on *all* the (smooth parts of the) fibres of the fibration. By deforming the given Lagrange form by adding a small multiple of this new symplectic form, we can see that there are symplectic forms on $X$ symplectic on the fibres and deformation equivalent to the given form; in particular, the first Chern class of the deformed symplectic structure is the same as that of the Lagrange structure.

**Proposition 4.2.** *Let $f : X \to B$ be a genus one symplectic Lefschetz fibration with $c_1(X) = 0$. Then $X$ is either an elliptic $K3$ surface or a torus bundle over a torus.*

*Proof.* The Euler class of $X$ is non-negative, given by the number of singular fibres of the fibration. By Rokhlin's theorem we know that the signature of the four-manifold is divisible by sixteen, and on the other hand we know that $2e(X) + 3\sigma(X) = c_1^2(X) = 0$. We need one final ingredient, also due to Matsumoto: a symplectic genus one Lefschetz fibration over a base $B$, with at least one singular fibre, is given by fibre summing $\mathbb{T}^2 \times B$ with a symplectic Lefschetz fibration over $\mathbb{P}^1$ [9]. All the latter are given by fibre sums of the rational elliptic surface $E(n) = E(1)\sharp \cdots \sharp E(1)$. If $B$ has genus $g$ then the first Chern class of the standard symplectic structure on $(\mathbb{T}^2 \times B)\sharp E(n)$ is given by $(2 - 2g - n)[F]$ where $F$ is a torus fibre. This can vanish only if $(g, n) \in \{(1, 0), (0, 2)\}$. Then (for instance by considering Seiberg-Witten basic classes) one can check that on these manifolds if the standard symplectic structure does not have $c_1 = 0$ then nor does any other structure. More simply, if the manifold has a symplectic section, and if the base has genus $g \geq 2$ then one can derive a contradiction to the adjunction formula using $c_1 = 0$. For twisted torus bundles over surfaces of higher genus, sections need not exist. But we can choose an almost complex structure such that the fibration is pseudoholomorphic and see that $c_1 \neq 0$ by taking Chern classes in the exact sequence induced by the differential of the projection. □

It only remains to discuss the geometry of different torus bundles over tori. These have been classified in a manner very suitable for our purposes by Geiges [11] following on from work of Ue. We simply summarise the results for the reader, and note that the main theorem follows.

1. Let $X$ be the total space of a torus bundle over a torus. The bundle structure is determined by the diffeomorphism type unless $b_1 = 3$; in this case there are essentially two distinct torus fibration structures, and the total space is diffeomorphic to $\mathbb{S}^1 \times Y_n$ for $Y_n$ the mapping torus of the $n$-th power of a Dehn twist.
2. If $b_1(X) = 3$ then there is a unique torus fibration with homologically essential fibres, and this admits Lagrange forms.

3. If there is a symplectic structure which is trivial on the fibres, then we are in one of three situations: $X$ is the four-torus, $b_1(X) = 3$ or $X$ is a nilmanifold - a principal $\mathbb{S}^1$-bundle over a principal $\mathbb{S}^1$-bundle over $\mathbb{T}^2$ - with inessential fibres.

For definiteness, here is a typical construction. The manifold $\mathbb{S}^1 \times Y_n$ is the quotient of $\mathbb{R}^4$ by a lattice generated by the unit translations along the $y, z, t$ axes and by the map

$$(x, y, z, t) \ \mapsto \ (x + 1, y, z + ny, t).$$

The projection takes $(x, y, z, t) \mapsto (x, t)$. It is easy to check that $H^2(X)$ is generated by

$$dx.dt, \ dy.dz, \ dy.dt, \ dx.dz - nx\, dx.dy$$

and hence for instance the form

$$\omega \ = \ dy.dt - (dx.dz - nx\, dx.dy)$$

is symplectic but trivial on the fibres; and if we perturb it by adding a small multiple of $dy.dz$ then we have a deformation equivalent form which is symplectic on the fibres.

**Example 4.1.** Let $X$ be a torus bundle over the torus of the form $\mathbb{S}^1 \times Y$ where $Y$ is the mapping torus of an element $A \in SL_2(\mathbb{Z})$. If $A$ has two eigenvalues equal to 1 then $X \cong \mathbb{T}^4$ has $c_1(X) = 0$ for every symplectic structure [31]. If $A$ has one eigenvalue equal to 1 then $X$ admits a Lagrangian fibration and again there is a symplectic structure with $c_1 = 0$. Lastly, if $A$ is generic in $SL_2(\mathbb{Z})$ and has no eigenvalues equal to 1, then there is a symplectic structure on $A$ with $c_1 = 0$ but $X$ admits *no* Lagrangian fibration. To see that $c_1 = 0$, observe we can find a basis of $H_2(X, \mathbb{Z})$ comprising the fibre and a section of $X \to \mathbb{T}^2$, which can both be made symplectic by the Thurston construction [32], and which both have square zero. The result follows by adjunction. The non-existence of any essential Lagrange fibration follows from our earlier comments on intersection forms for manifolds with $b_+ = 1$. But according to [26] the torus fibration of the manifold is unique up to fibre-preserving diffeomorphisms, so $X$ admits no inessential torus fibration either.

Thus in four dimensions, admitting a Lagrangian fibration is a quite distinct requirement on the symplectic topology than demanding that the first Chern class vanishes, a result which runs orthogonal to the general flavour of mirror symmetry.

## Part 2. Lefschetz pencils on torus bundles

In this part of the paper we shall provide (partial) details on an infinite family of genus three Lefschetz pencils; the total spaces of these pencils will be smooth torus bundles over tori with essential fibres. The constructions were first suggested by Donaldson (in private conversations in 1996). The existence of the Lefschetz pencils can be interpreted as a non-trivial theorem on the structure of the genus three mapping class group. It is worth noting that this establishes a purely algebraic theorem via four-manifold geometry. Again, we begin by establishing a wider context.

## 5. Motivations

**Lefschetz pencils:** One of the most striking achievements of symplectic topology in the last years has been the classification, in essentially algebraic terms, of a dense set of symplectic structures on any symplectic manifold. In particular, the diffeomorphism classification of symplectic four-manifolds has in principle been reduced to hard algebra, much as surgery theory reduces diffeomorphism problems for higher dimensional smooth manifolds to hard algebra (of a rather different sort). The central piece of this classification is Donaldson's existence theorem for Lefschetz pencils [5]. On the other hand, despite universal existence theorems for Lefschetz pencils, it's surprisingly hard to find them in practice. In the next section of the paper we present an example which provides some insight into a Lefschetz pencil of genus three curves on the four-torus. The proof relies on reversing the Kummer construction of a $K3$ surface. The fibred nature of the original manifold is crucial here. The example, which is the basis for other constructions later in the paper, is intended to illustrate the gulf between the elementary "existence theorem" for pencils in algebraic geometry and the more demanding task of "seeing" where even the simplest pencils come from.

**Seiberg-Witten invariants:** Gauge theory invariants, particularly arising from the Seiberg-Witten equations, have led to huge advances in our understanding of symplectic four-manifolds. One cornerstone of these applications is the theorem of Taubes [31] equating Seiberg-Witten invariants and Gromov invariants (counting pseudoholomorphic curves) on a large class of symplectic four-manifolds; this gives a direct geometric interpretation of the invariants. Nonetheless, from this perspective, it seems rather mysterious that the Gromov invariants are (almost) independent of the particular symplectic structure. Towards, though by no means reaching, an explanation of this, the author suggests the following

**Question 5.1.** Are the Seiberg-Witten invariants of a symplectic four-manifold $X$ with $b_+ > 1$ completely determined by the homological monodromy representation of a (sufficiently high degree) Lefschetz pencil on $X$?

That Lefschetz pencils exist in generality is a striking theorem of Donaldson [5]. For four-manifolds of the form $X = \mathbb{S}^1 \times Y$, where $Y$ is a fibred 3-manifold $Y \to \mathbb{S}^1$ defined by a surface diffeomorphism $f : \Sigma_g \to \Sigma_g$, it is known that the Seiberg-Witten invariants[2] of $X$ are entirely determined by the action of $f$ on $H_1(\Sigma_g)$. To prove this, one uses a "stretching the neck" argument to reduce to studying three-dimensional SW-equations on $Y$, and then it is a consequence of a theorem of [21]. From our remarks on Lefschetz pencils on torus bundles, which can be extended to describe Lefschetz pencils on general surface bundles, we will see that the action of $f$ on homology is captured by the homological monodromy of the simplest Lefschetz pencil on $X$. This enables us to answer the question in a special case (Corollary 7.5).

[2]For simplicity, in this paper we never mention chamber structures even when $b_+ = 1$.

**Positive relations:** A Lefschetz fibration determines a positive relation in a mapping class group $\Gamma_g$: that is, a relation of the shape "product of positive Dehn twists equals the identity". This combinatorial description of Lefschetz fibrations on four-manifolds was first given by Kas in [17]. He noted that there is a routine way to produce positive relations: string given positive relations together. A relation is called "irreducible" if it describes a four-manifold which is not given by fibre summing two non-trivial Lefschetz fibrations. Equivalently, the positive relation is not Hurwitz equivalent to any product of two strictly smaller non-empty positive relations; for the relevant notions see [13]. Kas raised the following question: *are there only finitely many equivalence classes of irreducible positive relation in* $\Gamma_g$? At genus one this holds [13], and at genus two a weak version was proven in [28]: there are only finitely many positive relations in non-separating Dehn twists which come from Lefschetz pencils (Lefschetz fibrations which admit sections of square $-1$). Our treatment of Lefschetz pencils on torus bundles will provide a negative answer to Kas' question at the next possible stage; $\Gamma_3$ contains infinitely many irreducible positive relations. Incidentally, these contain only non-separating twists and do come from pencils.

Another combinatorial question addressed in the literature concerns the *minimal length* of any positive relation in a mapping class group $\Gamma_g$ (cf. [30]). For every genus $g \neq 3$ the known and conjectured words of minimal length arise from pencils of curves on ruled surfaces. Moreover, there is strong evidence to suggest that these words are the unique ones of minimal length. At genus three, by contrast, the words we obtain from torus bundles are all shorter than those which can arise from any pencils on ruled surfaces [30]. It is curious that the situation at genus three should be qualitatively and quantitatively different.

**Statements of results:** The rest of the paper shall be devoted to the following:

**Theorem 5.1.** *If $X^4$ is the total space of a torus bundle over a torus which admits a section, then $X$ admits a Lefschetz pencil of genus three curves with four base-points.*

The existence statement is trivial for the four-torus; the pencil is holomorphic for a suitable abelian surface (which is not isogenous to a product of elliptic curves). Nonetheless, we shall spend some time providing a rather precise description of this pencil. We argue that, at least in principle, this gives an explicit handle on the pencils on other torus bundles by a degeneration argument. From this we will give a negative answer to a question of Kas from [17].

**Corollary 5.2.** *The genus three mapping class group $\Gamma_3$ admits infinitely many irreducible and inequivalent positive relations.*

From similar arguments one can give a positive answer to Question (5.1) for certain classes of symplectic four-manifold: for instance for products $\mathbb{S}^1 \times Y$ where $Y$ is a three-manifold which fibres over the circle. However, this may be artificial evidence for the general case; knowing both the Seiberg-Witten invariants and the monodromy we shall see that the latter contains the information of the former, but we will not provide any general method for extracting that information.

## 6. A Lefschetz pencil on the four-torus

One drawback of the theory of Lefschetz pencils is that it is rather hard to construct examples. Moreover the general theory provides Lefschetz pencils of an arbitrarily high unspecified degree, presumably of arbitrarily confounding complexity. The positive relations known explicitly to date are essentially all derived from a number of "universal" mapping class group relations (with analogues at every genus $g$) by taking fibre sums or inserting non-trivial relations in positive twists. These combinatorial tricks can give manifolds with interesting properties (cf. for instance [28]), but those manifolds are usually unfamiliar. For algebraic surfaces, one can hope to find pencils - with precise existence theorems and at least heuristic combinatorial descriptions - using the machinery of holomorphic geometry. Moreover the local models provided by such pencils can provide constructions on non-algebraic four-manifolds. We will demonstrate this circle of ideas by constructing Lefschetz pencils on the class of manifolds arising in the first half of the paper; since at least one pencil on the $K3$ surface is well-known, we concentrate on torus bundles with trivial "Euler class" (i.e. which admit a section), defined by a pair of monodromy matrices in $SL_2(\mathbb{Z})$. To diffeomorphism, the only projective such is the four-torus, which provides a starting point for the investigation. Algebraic geometry provides the following:

**Lemma 6.1.** *There are abelian surfaces which admit holomorphic Lefschetz pencils of genus three curves with four base-points (and no abelian surface admits a Lefschetz pencil of curves of lower genus).*

*Proof.* Let $X$ be an abelian surface with a $(1,2)$-polarisation $\mathcal{L}$. It is well-known that the projective space $|\mathcal{L}|$ is a copy of $\mathbb{P}^1$. Moreover, according for instance to ([18], Ch. 10), every curve in $|\mathcal{L}|$ has one of four topological types: (a) a smooth genus three curve; (b) an irreducible genus two curve with one node; (c) two elliptic curves meeting at two nodes; (d) a union of three elliptic curves, two of which are disjoint and each meet the third component transversely once. Type (d) occurs iff $X$ is isogenous to a product of elliptic curves $E_1 \times E_2$ and $\mathcal{L} \cong \pi_1^* L_1 \otimes \pi_2^* L_2$, with $L_i$ of degree $i$ on $E_i$; in this case, the pencil of sections of $\mathcal{L}$ arises from a degree two pencil on $E_2$ - defining a map $X \to E_2 \to \mathbb{P}^1$ - together with a fixed component which is a section of $X \to E_2$. Now the moduli space $\mathcal{H}$ of abelian surfaces is three complex-dimensional, and the locus of surfaces isogenous to $E_1 \times E_2$ has positive codimension, so we can choose $X$ such that $\mathcal{L}$ has no sections of type (d). By Bertini's theorem, it is easy to see that for generic $X$ the pencil will contain smooth members, and since all possible singularities are nodal, another perturbation of $X \in \mathcal{H}$ will yield a Lefschetz pencil. The other statements are trivial. □

This existence statement is not especially illuminating, so we will now give a more detailed description of the pencil. The idea is to utilise the *Kummer construction* which allows us to reduce the problem to understanding a certain $K3$ surface. Recall that if we resolve the singularities given by the involution $x \mapsto -x$ we see

$$\big(\mathbb{T}^4 \# 16\overline{\mathbb{CP}}^2\big)/\mathbb{Z}_2 \;\cong\; K3 \qquad \text{the Kummer surface.}$$

As explained below, (5.1) will follow from the following

**Proposition 6.2.** *There is a $K3$ surface elliptically fibred over $\mathbb{S}^2$, containing 16 disjoint $(-2)$-spheres and such that*

- *four are sections of the fibration;*
- *twelve lie in fibres of the fibration;*
- *the union of the spheres is an even divisor.*

*This $K3$ has precisely twelve (reducible, non-Lefschetz) singular fibres.*

Given this, consider the double cover of $K3$ branched over the union of all sixteen $(-2)$-spheres. On the generic elliptic fibre, we take the double cover over four points to obtain a genus three curve. Moreover, since the $(-2)$-spheres are ramified, they lift upstairs to $(-1)$-curves, which may then be blown down. (Recall that for a cyclic branched cover $\pi$ of order $n$, and divisors $D_1, D_2$ in the branch locus downstairs, $\pi^* D_1 \cdot \pi^* D_2 = \frac{1}{n} D_1 \cdot D_2$.)

It follows that the total space of the cover has a genus three fibration with four exceptional sections. Blowing down the twelve exceptional spheres in fibres and the exceptional sections gives the four-torus by the inverse of the usual Kummer construction. So to understand this pencil, it is enough to understand the $K3$ with the desired properties; in particular once we have done this we can verify *directly* the property that the singular genus three fibres are indeed Lefschetz.

*Proof.* Let $Q \subset \Gamma(\mathbb{P}^3, \mathcal{O}(2))$ be a singular quadric in $\mathbb{P}^3$, that is, the cone on a smooth quadric in $\mathbb{P}^2$:

$$Q = \{[x : y : z : t] \mid x^2 + y^2 + z^2 = 0\}.$$

Then $Q$ is a $\mathbb{P}^1$-bundle over a conic $\mathbb{S}^2 \subset \mathbb{P}^2$ with a section contracted to the vertex of the cone $[0 : 0 : 0 : 1]$. This vertex is an $A_1$ singularity, and it can be smoothed by replacing it with a $(-2)$-sphere. A hyperplane section $D_i$ of $Q$ is a conic in $\mathbb{P}^2$ and clearly $D_1 \cdot D_2 = \{2 \text{ points}\}$.

Now choose four planes in $\mathbb{P}^3$ giving on $Q$ a system of four conics each meeting each of the others transversally in two points. Define $Z$ to be the double cover of the quadric $Q$ over the union of the four conics $\bigcup_{i=1}^4 C_i$. $Z$ has two singular points from the vertex of $Q$; each of the twelve nodes in the branch locus also lifts to an $A_1$ singularity. We can smooth these fourteen singular points by adding $(-2)$-spheres to obtain a manifold $Z^{\mathrm{sm}}$, where "sm" denotes the smoothed space. We claim that $Z^{\mathrm{sm}}$ is the required $K3$ surface. Certainly we have found two $(-2)$-sections and 12 $(-2)$-spheres in fibres; here the fibration by tori comes from pulling back the $\mathbb{P}^1$-bundle structure of $Q$ - the family of $\mathbb{P}^1$'s through the vertex - under the branch map over the conics $\cup C_i$. Note that for generic $C_i$ no $\mathbb{P}^1$ fibre of $Q$ will meet more than one node of $\cup C_i$. It is easy to check that $\pi_1(Z^{\mathrm{sm}}) = 0$ and $K_{Z^{\mathrm{sm}}} = 0$. However, we need to find two more $(-2)$-sections.

To do this, we need a piece of classical algebraic geometric intersection theory - we give the (entertaining if unenlightening!) proof as an Appendix.

**Lemma 6.3.** *Let $C_i, 1 \leq i \leq 4$ be four conic sections of a quadric in $\mathbb{P}^3$ meeting generically. Then one can choose (at least) two more conic sections which are everywhere tangent to the locus $\cup C_i$.*

The tangency condition on these two conics means that under the double cover $Z \to Q \supset \cup C_i$ they lift to reducible curves formed from two two-spheres (that is, the two sheets of the cover form distinct irreducible components). One can check these are $(-2)$-spheres, by construction sections. Precisely in the case that we pick two of these additional tangent sections, *we can choose sheets from the lifts which are disjoint*, giving the final two disjoint sections upstairs. It only remains to see that the union of the sixteen spheres is an even divisor. The sixteen spheres occur as a pair $N_1, N_2$ from resolving the node, a pair $T_1, T_2$ from the tangent conics and twelve spheres from the pairs $S^a_{i,j}, S^b_{i,j}$ of intersections of the conics $C_i$ and $C_j$, with $1 \leq i < j \leq 4$. Then it is easy to see that $N_1 + T_1 + \sum C^a_{i,j} = N_2 + T_2 + \sum C^b_{i,j}$ in homology (although none of the individual spheres are homologous to one another; they all intersect). □

**Remark 6.1.** The reader may find it helpful to draw some pictures. Two conics in the plane intersecting transversely in two points obviously allow two (disjoint) conics tangent to both; one inscribed inside both and the other outside both. On the surface of a cone we can push one tangential conic upwards and the other downwards to give an initial pair of angled rings meeting at two points neatly sandwiched between horizontal slices. The postponed lemma allows us to find these tangent conics given four initial slices though this translates less well in a real picture!

In fact we can find more than two of these tangent conics; but we can lift at most two to give disjoint spheres upstairs. This is because (real pictures notwithstanding) any pair of such tangent conics *themselves* intersect, and so the best we can do upstairs is find two lifts $C_1 \cup C_2$ and $C_1' \cup C_2'$ such that $C_1$ meets only $C_2'$ and $C_2$ meets only $C_1'$; then $C_1$ and $C_1'$ will be disjoint spheres in the cover.

Now consider the form of the singularities in the genus three fibration. Rather than smoothing the nodes and then blowing down, we consider a model where we simply branch over the $A_1$ singularities arising from the double cover over the nodal locus of conics. An $A_1$ singularity has local model the origin in $X$ defined by

$$X = \{(x, y, w) \mid x^2 + y^2 + w^2 = 0\}.$$

It occurs as the $\mathbb{Z}_2$ diagonal quotient of a smooth $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $\mathbb{C}^2$ defined by the equivalence relation $(z_1, z_2) \sim (\pm z_1, \pm z_2)$. The relevant fibration of $X$ for modelling our $K3$ comes from the projection map to $w$. Pulling this fibration back to $\mathbb{C}^2$ is easiest using the algebras of functions: the invariant functions on $\mathbb{C}^2/\{\pm I\}$ are $z_1^2, z_2^2, z_1 z_2$ and so the ring of functions on the local model $X$ is just $\mathbb{C}[u,v,w]/\langle uv = w^2\rangle$. The projection map to $\mathcal{O}(\mathbb{C}) = \mathbb{C}[\alpha]$ takes $w \mapsto \alpha$ and so if we pull back to the smooth $\mathbb{C}^2$ covering $X$ - our model for lifting to the torus - the fibration from the final co-ordinate induces on algebras

$$\mathbb{C}[\alpha] \to \mathbb{C}[z_1, z_2] \ : \ \alpha \mapsto z_1 z_2.$$

This just arises from the usual map $\mathbb{C}^2 \to \mathbb{C}$ taking $(z_1, z_2) \mapsto z_1 z_2$, which has a Lefschetz singularity over the origin.

It follows that the involution on the genus three curves locally preserves the two half-cones (copies of $\mathbb{C}^*$) around the node; again the reader may enjoy drawing a topological model. You need to find an involution on a genus three curve with four fixed points and quotient a torus, and for which there is an embedded circle disjoint from the fixed points and preserved by the involution - this is the vanishing cycle, which collapses to an extra branch point of the double cover in the nodal fibres.

For a $(d_1, d_2)$-polarisation on any abelian surface, there is a pencil of curves of genus $1 + d_1 d_2$ with $2d_1 d_2$ base-points. The pencil we have displayed is "minimal" in the sense that it comes from a line bundle with precisely two sections, and any other Lefschetz pencil on $\mathbb{T}^4$ will be by curves of higher genus. A $(1,3)$-polarisation gives a branched covering map of $\mathbb{T}^4$ over $\mathbb{P}^2$ of degree 6 with branch locus a curve of degree 18, whilst the $(1,5)$-polarisation embeds the four-torus in $\mathbb{P}^4$ as the zero-set of a section of the famous Horrocks-Mumford bundle (essentially the only known stable indecomposable rank two bundle on $\mathbb{P}^4$).

## 7. Monodromy and positive relations

In this section we shall outline a proof of Theorem 5.1. A distinct but related construction is presented in great detail in the treatment of stabilisation of Lefschetz pencils by Auroux and Katzarkov [3]. In each case one perturbs a degenerate family of sections of the form $\{[\text{Fixed component}] \cdot \phi_\lambda\}$, where $\{\phi_\lambda\}$ is a given pencil, to a new family which is Lefschetz. For stabilisation, the given pencil $\{\phi_\lambda\}$ is a Lefschetz pencil and one adds in another section in the same homology class (for instance coming from the existence theorem for nets of sections due to Auroux). Here, we shall start with a degenerate family $\{\phi_\lambda\}$, add a component and perturb to yield a Lefschetz pencil. Recall the statement we are after:

**Theorem 7.1.** *Let $X \to \mathbb{T}^2$ be the total space of a torus bundle over the torus which admits a section. Then $X$ admits a Lefschetz pencil of genus three curves.*

*Sketch.* It follows from [11] that $X$ has a symplectic structure which is symplectic on the fibres, so the assertion is consistent with general theory. We need the following topological perspective on the Lefschetz pencil we have already constructed on $\mathbb{T}^4$; let $Z$ be some abelian surface. Compose the projection $\pi : Z \to \mathbb{T}^2$ with the double cover $\mathbb{T}^2 \to \mathbb{S}^2$ branched over four points. As in the previous section, the homology class

$2[\mathrm{Fibre}(\pi)] + [\mathrm{Section}(\pi)]$ is Poincaré dual to $c_1(\mathcal{L})$ with $\mathcal{L}$ a holomorphic line bundle with $h^0(\mathcal{L}) = 2$.

**Lemma 7.2.** *If $Z$ is isogenous to a product of elliptic curves, the pencil of divisors defined by sections of $\mathcal{L}$ contains a fixed section $s$ and is parametrised by a degree two pencil on $\mathbb{T}^2$. The holomorphic Lefschetz pencil of the previous section is a deformation of this arising from deforming the complex structure on the abelian surface away from the product structure.*

From this perspective, the twelve singular fibres of the Lefschetz pencil arise as four sets of three Dehn twists. Using our explicit description of the pencil from the Kummer construction, we can see the relevant degeneration. Allow the four generic conic sections $\cup C_i$ to degenerate so that there are precisely four singular points in the union, at each of which exactly three of the conics meet mutually tangentially. Then the double cover carries a pencil of tori in which four of the fibres are double covers of spheres branched over just two points; that is, there are four spherical fibres. Working backwards from the torus, given the degenerate pencil on a split abelian surface induced from a degree two pencil on $\mathbb{T}^2$ and a fixed torus section $T$, we can divide by $\mathbb{Z}_2$ to obtain a family of connected tori varying over a fixed two-sphere section $T/\mathbb{Z}_2$, with four degenerate fibres $\mathbb{T}^2/\mathbb{Z}_2 \cong \mathbb{S}^2$. Since $T$ has square zero, the spherical section downstairs (after dividing by the involution, with four fixed points on the torus) has square $-2$, as we expect. (To complete the picture, we also need to degenerate the branch locus for the second covering given by the tangent slices above.) The deformation we have described can be effected by a suitable path in the connected moduli space $\mathcal{H}'$ of abelian surfaces equipped with a particular polarisation (the moduli scheme is not fine but for generic choice of path we have a family of abelian surfaces, as the manipulation of the conics suggests). In any case, we have arranged our twelve singular fibres into four triples, where each triple arises from the deformation of a single double fibre in the pencil of curves pulled back from $\mathbb{T}^2 \to \mathbb{S}^2$.

Topologically, away from neighbourhoods of the four branch points in $\mathbb{S}^2$ there is an obvious perturbation of the family of nodal curves given by two fibres and a section meeting transversely to a family of smooth genus three curves: take the connect sum. Thus the monodromy of the final Lefschetz fibration can be viewed as coming from these four branch points. Now the torus bundle $\pi : X \to \mathbb{T}^2$ can also be composed with the projection $p : \mathbb{T}^2 \to \mathbb{S}^2$; the base of the second map is the parameter space for a pencil of real subvarieties (with locally holomorphic singularities), where we add in a fixed section to $\pi$. Thus the generic member of this family, over $t \in \mathbb{S}^2$ say, is given by the union of the fixed section $s$ of $\pi$ with the two disjoint tori defined by $(p \circ \pi)^{-1}(t)$. Describe this pencil of real surfaces by choosing two generating sections $\{s \cdot \phi_1 + s \cdot (\lambda \phi_2)\}_{\lambda \in \mathbb{P}^1}$, where the $\phi_i$ define the degree two pencil on $\mathbb{T}^2$ which yields the map $p$ and $s$ denotes the fixed section. The connect sum deformation away from the double fibres makes good sense, and can be effected by generic perturbations modelled on

$$s \cdot \phi_1 + \lambda s \cdot \phi_2 \;=\; 0 \implies (s \cdot \phi_1 + \varepsilon \psi_1) + \lambda(s \cdot \phi_2 + \varepsilon \psi_2) \;=\; 0$$

for small $\varepsilon$ and smooth sections $\psi_i$ of the bundle with first Chern class $2[\text{Fibre}]+[\text{Section}]$. Observe that the nodes we are smoothing trace out copies of square zero surfaces as we vary over the pencil, and so we can smooth all the nodes in a single deformation without changing the monodromy around the boundaries of the $\varepsilon$-discs. We can complete this family of smooth curves, defined in the $\lambda$-plane away from discs $|\lambda| \leq \varepsilon$ centred on the branch points of $\mathbb{T}^2 \to \mathbb{S}^2$, to a genus three Lefschetz pencil by gluing in the model from the pencil on the four-torus. This is valid since the family of submanifolds for small $\lambda$, up to conjugation by a diffeomorphism, does not see the global monodromy of the fibration. □

We can be more explicit at the level of homology, tying back into the Question (5.1). For any torus bundle over the torus, the homology classes coming from a section are invariant under the monodromy ($b_1 \geq 2$ for these manifolds). We can construct the word in Dehn twists inside $Sp_6(\mathbb{Z})$ (and not $\Gamma_3$) arising from the construction above; reduce to $Sp_4$ by throwing out the section. We are therefore working with the homology of the pencil of disconnected submanifolds given just by $X \to \mathbb{T}^2 \xrightarrow{2:1} \mathbb{S}^2$. Around a branch point the local action on the four-dimensional space given by the homology of the fibres is the map

$$\phi_* = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

It is possible to write down an appropriate triple of symplectic matrices with this product; take the involution $\phi : \Sigma_2 \to \Sigma_2$ which rotates about an axis with two fixed points and expand in Dehn twists (one of which is in the Torelli group). Thus the local monodromies can be captured homologically by four triples of symplectic matrices, one for each branch point. We would like to incorporate the monodromies $\gamma_i \in SL_2(\mathbb{Z})$ into the picture. Fix a loop in $\mathbb{S}^2$, based in a disc with a fixed homology identification of the fibre at one branch point, encircling another branch point. The monodromy about this loop must be conjugate to the local monodromy $\phi_*$ but incorporate the fact that as we move away from the base-point we twist the torus fibre by some monodromy matrix $A$. This idea translates easily into matrices:

**Lemma 7.3.** *We have the identity in $Sp_4$ (writing matrices in block form)*

$$\begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix};$$

The torus fibration defined by monodromy matrices $\gamma_{1*} = A, \gamma_{2*} = B$ is thus homologically compatible with the quadruple of four sets of three twists each, which in block form read

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & BA^{-1} \\ B^{-1}A & 0 \end{pmatrix}.$$

Note that the final matrix is of the correct block form, i.e. the bottom left entry is the inverse of the top right; for $(BA^{-1})^{-1} = B^{-1}A \Leftrightarrow AB = BA$ which is valid since our bundle is defined by a monodromy representation $\pi_1(\mathbb{T}^2) \to SL_2(\mathbb{Z})$ and the fundamental group is abelian. We will leave further details on the construction to another occasion.

**Lemma 7.4.** *All the Lefschetz pencils on torus bundles described above are irreducible; they cannot be written as non-trivial fibre sums.*

This follows from a theorem due to Stipsicz [29], with a simpler proof in [27]: any Lefschetz fibration which contains a sphere of square $-1$ is irreducible. Since our fibrations arise from blowing up base-points of pencils of curves, exceptional sections are necessarily present. From this and the sketch above, the Corollary (5.2) is an immediate consequence. Moreover we see that the homological monodromy of a torus bundle is determined by the homological mondromy of its Lefschetz pencil. Adapting the constructions to general genus fibres, one can prove:

**Corollary 7.5.** *The question (5.1) has a positive answer for four-manifolds which are products of a circle and the mapping torus of a diffeomorphism.*

Indeed, modulo suitable small print on wall-crossing, the restriction to $b_+ > 1$ seems unnecessary for this class of manifolds. The corollary comes from combining the homological description of the Lefschetz pencils with a theorem of Meng and Taubes [21] on Seiberg-Witten invariants in three dimensions. This identifies the solutions of the Seiberg-Witten equations on a mapping cylinder with periodic orbits of the obvious flow; these are (roughly) fixed points for the monodromy action on the symmetric products of the fibre, and an algebraic count of the numbers of fixed points is a homological invariant by the Lefschetz fixed point formula. A pretty exposition of the result is given in Donaldson's paper [6]. That solutions to the Seiberg-Witten equations on $\mathbb{S}^1 \times Y$ are invariant in the circle direction (and hence determined by certain solutions on $Y$) is a standard argument. Roughly, a Floer-type description of the equations on $Y \times \mathbb{R}$ identifies them with gradient flow equations for a Chern-Simons functional. Then one expects solutions to be decreasing in the $\mathbb{R}$ direction, contradicting the existence of non-trivial periodic solutions which could descend to $\mathbb{S}^1 \times Y$.

## 8. Appendix

In this section we prove the Lemma (6.3). The proof is a piece of classical intersection theory in the style of "The problem of five conics" ([14], p.749). We recall the statement.

**Lemma 8.1.** *Let $C_i, 1 \leq i \leq 4$ be four conic sections of a quadric in $\mathbb{P}^3$ meeting generically. Then one can choose (at least) two more conic sections which are everywhere tangent to the locus $\cup C_i$ and disjoint from one another.*

*Proof.* We have $Q \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(2))$ the singular quadric, and the conics $C_i$ are elements of $W = H^0(Q, \mathcal{O}(1)) = \mathbb{P}^4$. Since this is a four-dimensional space, when we impose four conditions - the tangency conditions for four distinct conics $C_i$ - we expect a finite answer; we must check that this answer is at least two.

For $C \in W$ let $V_C$ be the space of conics in $W$ tangent to $C$.

**Lemma 8.2.** *This is a degree 2 hypersurface in $W$.*

To see this, consider a generic line $\mathbb{P}^1$ in $W$. Each element $\eta$ of this $\mathbb{P}^1$ represents a conic curve, and this meets $C$ in two points unless $\eta \in \mathbb{P}^1 \cap V_C$. Thus we define a double cover $C \to \mathbb{P}^1$ branched at $\deg(V_C)$ points, and the result follows. We might then hope that the answer we require is $2^4 = 16$ conics exist tangent to four given conics; but unfortunately the intersection of the different hyperplanes $V_{C_i}$ is not transverse.

**Lemma 8.3.** *The line bundle $\mathcal{O}(1)|_Q$ is divisible in the Picard group.*

Recall that we can smooth the node of $Q$ by inserting a $(-2)$-section; thus we can view $Q$ as given by contracting the section at infinity in the second rational ruled surface $\mathbb{F}_2$. Now the Kähler form for $\mathbb{P}^3$ restricted to $Q$ looks like the zero-section $s_0 = 2F + s_\infty$ in $\mathbb{F}_2$ under our identifications[3], for $F$ a fibre of $\mathbb{F}_2 \to \mathbb{P}^1$. In particular, killing $s_\infty$, we see that $\omega_{\mathbb{P}^3}|_Q$ is divisible by two. Let $\mathcal{O}(1)^{\frac{1}{2}}$ denote a square root.

- Now choose a section of the bundle of the form $2a$ for $a \in \Gamma(\mathcal{O}(1)^{\frac{1}{2}})$, so $a \cdot H = 1$ for a hyperplane $H$ in $\mathbb{P}^3$. Then $2a$ represents a conic which intersects each $C_i$ at a double point (depending on $i$); in particular, having a unique point of intersection multiplicity two, all the conics $2a$ of this form lie in the multiple intersection $\cap_i V_{C_i}$. We can see these singular conics very clearly in a diagram; we are just looking at the conics 2[Line] for the lines $L$ through the vertex $Q$ of the cone. Thus the space of such double lines $Z$ forms a copy of $\mathbb{P}^1$, and we must compute the contribution of this locus to the intersection number 16 we computed above from the degree of the hypersurfaces. In complicated cases this requires the use of excess intersection theory [10] but in this instance it is easy enough to proceed directly.

- First we require the multiplicity of this $Z$ in the intersection of the hyperplanes $V_{C_i}$. Fix a conic $C_0$. For a line in $W$ through some fixed point $2[L]$ of $Z$ consider the pencil of conics and the double branched cover of $C_0$ they define; since there are two branch points, this line $\{C_\lambda\}$ meets $Z$ in one point other than $2[L]$. Thus the two double-lines in the pencil are distinct (distinct branch points) and hence $Z$ must have multiplicity one in $V_{C_0}$.

Blow up $W = \mathbb{P}^4$ (the space of all conics) along $\mathbb{P}^1 = \{\text{Double lines}\}$. If $E$ is the exceptional divisor for this blow-up, then the corrected value for the number of *smooth* conics tangent to the given $C_1, C_2, C_3, C_4$ is given by

$$\widetilde{V_C}^4 \equiv N = (2\tilde{H} - E)^4$$

where $\widetilde{V_C}$ is the proper transform of a generic hypersurface $V_C$ under the blow-up map $\widetilde{W} \xrightarrow{\pi} W$. $H$ represents a hyperplane in $W$ (and hence $2H$ a degree two hypersurface); then $\tilde{H}$ is the pullback of this to $\widetilde{W}$. We correct by $-E$ since the points of $Z$ have multiplicity one; thus $\widetilde{V_C} = 2\tilde{H} - E$ in cohomology.

[3]Here $\mathbb{F}_2$ denotes the ruled surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$

• Write $TW|_Z = TZ \oplus \nu_{Z/W}$ for the normal bundle $\nu$. Taking total Chern classes (recalling $W = \mathbb{P}^4$ and $Z = \mathbb{P}^1$) gives

$$c(TW)|_Z \;=\; (1 + 5\omega + 6\omega^2)|_Z \;=\; (1 + 5h)$$

and hence since $c(TZ) = 1 + 2h$, we see $c(\nu_{Z/W}) = 1 + 3h$. Here $\omega$ denotes the Kähler class of $W$ and $h$ a hyperplane in $Z$. Thus $\omega|_Z = 1h$.

Now $E$ is a $\mathbb{P}^2$-bundle over $Z$ with cohomology ring generated over the cohomology of the base by an element $\xi \in H^2(E, \mathbb{Z})$; here $\xi = c_1(\text{Taut})$ is the first Chern class of the tautological bundle. This cohomology ring is subject to the relation

$$\xi^3 - 3\tilde{h} \cdot \xi^2 = 0$$

where $\tilde{h}$ is the pullback $\pi^* h$. This relation comes from the usual formula for the cohomology of a projective bundle of rank $r$; $H^*(\mathbb{P}(E \to Y))$ is generated over $H^*(Y)$ by $\xi$ subject to

$$\xi^r - c_1(E)\xi^{r-1} + \cdots + (-1)^r c_r(E) = 0.$$

We also have another relation: for on each fibre $\mathbb{P}$ of $E \to Z$ the tautological bundle restricts to $\mathcal{O}(-1)$ over the projective space fibre, and hence $c_1(\text{Taut}|_{\mathbb{P}})^2 = 1$. This gives also

$$\xi^2 \cdot \tilde{h} = 1.$$

Combining these relations, recalling that $[E]|_E = \xi$ and expanding $(2\tilde{H} - E)^4 = (2\tilde{\omega} - [E])^4$ we obtain the final answer. Precisely,

$$\tilde{\omega}^4 = 1; \quad \tilde{\omega}^3 \cdot [E] = (\tilde{\omega}|_E)^3 = \tilde{h}^3 = 0;$$

$$\tilde{\omega}^2 \cdot [E]^2 = \tilde{h}^2(\cdot) = 0; \quad \tilde{\omega} \cdot [E]^3 = (\tilde{h}\xi^2)|_E = 1;$$

$$[E]^4 = 3\tilde{h} \cdot \xi^2 = 3$$

and putting in the constants the final answer is eleven, which is certainly at least as large as the required answer two. (Thus the contribution of the locus $Z$ to the naive intersection number 16 is precisely 5.) □

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