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Sections of Lefschetz fibrations and Stein fillings

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1. Introduction

In this note we give a new proof of the following

Theorem 1.1. *If $f\colon X^4 \to S^2$ is a nontrivial Lefschetz fibration on the 4–manifold X^4 with fiber genus $g > 0$ admitting a section $\sigma\colon S^2 \to X^4$ then $[\sigma(S^2)]^2 < 0$.*

(For definitions and basic notions regarding Lefschetz fibrations see [4, 5, 8]. Unless otherwise stated, we always assume that X is closed, hence the generic fiber of the fibration is a closed Riemann surface.) Theorem 1.1 follows from work of McDuff [6] where symplectic 4-manifolds containing symplectic spheres of nonnegative selfintersection are studied. Theorem 1.1 has been already proved in [9] using Seiberg-Witten theory; an alternative proof was given by I. Smith [7] in which hyperbolic arguments are applied. These proofs will be sketched in Section 4. In Section 3 we describe a proof based on Stein fillings of contact 3-manifolds. We hope that the diversity of these proofs demonstrate the close interplay between the notion of Lefschetz fibrations and various other branches of low dimensional topology.

2. Preliminaries

For sake of brevity we will omit the definition of notions like contact structures, Stein manifolds and Stein fillings. The interested reader is advised to turn to [2, 4, 5]. We will also deliberately use constructions and notions from the theory of Lefschetz fibrations without explicitely describing them — all these ideas are discussed, for example, in [4]. (The word "Lefschetz fibration" is used in the sense of [4], which coincides with "positive Lefschetz fibration" used in [5].) Our new proof of Theorem 1.1 rests on the following results:

Theorem 2.1 (Eliashberg, [2, 3])**.** *The standard 3-sphere S^3 admits a unique Stein fillable contact structure ξ_{st} (called the standard contact structure). The contact 3-manifold (S^3, ξ_{st}) admits a unique Stein filling, which is diffeomorphic to the 4–dimensional disk D^4.* □

Theorem 2.2 (Loi-Piergallini, [5])**.** *Suppose that $f\colon W \to D^2$ is a Lefschetz fibration over the 2-disk D^2 with regular fiber F having nonempty connected boundary ∂F. If the Lefschetz fibration admits only nonseparating vanishing cycles then W is a Stein surface, hence it is a Stein filling of some contact structure of its boundary ∂W.* □

A Lefschetz fibration over the disk D^2 admits a description through the monodromies of its singular fibers. Assuming that the map $f: X \to D^2$ is injective on its critical set (which property can be achieved by a slight perturbation of the map) the monodromy of a singular fiber is a right-handed Dehn twist along a simple closed curve in the generic fiber. (This curve is called the vanishing cycle corresponding to the singularity.) Therefore the fibration can be given by a word in the mapping class group of the generic fiber composed only from right-handed Dehn twists. A fibration over S^2 corresponds to such a word representing 1 in the mapping class group of the closed surface. A section gives a factorization of the unit element in the mapping class group of a once punctured surface, while a section of square n provides a factorization of δ^{-n} into the product of right-handed Dehn twists in the mapping class group of a surface with a unique boundary component. Here δ denotes the right-handed Dehn twist along a simple closed curve parallel to the boundary. (For more about relations between mapping classes and Lefschetz fibrations, see [1, 4, 7].)

3. The proof

Theorem 1.1 turns out to be a direct consequence of the following result.

Proposition 3.1. *The relatively minimal genus-g Lefschetz fibration (with $g > 0$) admits a section $\sigma: S^2 \to X$ with $[\sigma(S^2)]^2 = 0$ if and only if X is the trivial fibration $S^2 \times \Sigma_g \to S^2$.*

Remark 3.2. *From the point of view of monodromies Proposition 3.1 simply states that 1 in the mapping class group $\Gamma(F, \partial F)$ of a surface F with a unique boundary component cannot be written as a product of right-handed Dehn twists.*

Proof. The trivial fibration $S^2 \times \Sigma_g \to S^2$ (where Σ_g stands for the genus-g surface) admits a section with square 0: take $S^2 \times \{\text{pt.}\}$, for example. Conversely, suppose that $X \to S^2$ is a relatively minimal, nontrivial fibration with a section $\sigma: S^2 \to X$ of square 0. Once $g > 0$, by modifying X we can assume that the vanishing cycles of the fibration are all nonseparating: this follows from the fact that a right-handed Dehn twist along a separating curve can be written as a product of right-handed Dehn twists along nonseparating curves, and this equality holds in the mapping class group of the surface F with one boundary component (for more details see the proof of Lemma 3.3 in [5]). Let $f^{-1}(t)$ be a regular fiber of $X \to S^2$ and consider $W = X - \nu(f^{-1}(t) \cup \sigma(S^2))$ equipped with the Lefschetz fibration $f|_W: W \to D^2$. According to Theorem 2.2, the 4-manifold W provides a Stein filling for ∂W. Consider the Lefschetz fibration $V \to D^2$ specified by the $2g$ vanishing cycles $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ where $\{\alpha_i, \beta_i\}_{i=1}^g$ is the standard basis of $H_1(F; \mathbb{Z})$ (and F is the genus-g surface with one boundary component). Easy handlebody argument shows that $V = D^4$, because the 2–handles provided by the vanishing cycles of the singular fibers of the Lefschetz fibration $V \to D^2$ cancel the $2g$ 1–handles of $D^2 \times F$. Notice that (because of the assumption on the self-intersection of the section of $X \to S^2$) we have that $\partial W \cong \partial(D^2 \times F)$. Consider now the fiber sum $W \#_f V$. The diffeomorphism $\partial W \cong \partial(D^2 \times F)$ implies that $\partial(W \#_f V) \cong \partial V \cong S^3$, since in both fibrations the

2-handles corresponding to the singular fibers of V are attached to $\partial(D^2 \times F)$. Therefore (according to Theorem 2.2) $W\#_f V$ provides a Stein filling of some contact structure on S^3. Since $\mathrm{rk} H_2(W\#_f V;\mathbb{Z})$ is positive, this filling contradicts Eliashberg's Theorem 2.1. Therefore W, and so X with the given properties cannot exist. □

Remark 3.3. *Here is the same story from a different angle: Let M_φ be an open book with monodromy $\varphi\colon F \to F$. (For notations see* [5]*; here F denotes the genus-g surface with one boundary component.) According to* [5] *a factorization of φ as $\delta_1 \dots \delta_k$ in $\Gamma(F, \partial F)$ gives a Stein filling of M_φ. (Here each δ_i denotes a right-handed Dehn twist along some nonseparating simple closed curve.) Now if $1 \in \Gamma(F, \partial F)$ admits a factorization as $1 = \eta_1 \dots \eta_l$ with η_i right-handed Dehn twists (which can be chosen to be along nonseparating simple closed curves), then $(\eta_1 \dots \eta_l)^n \delta_1 \dots \delta_k$ still gives φ, hence provides a Stein filling for some contact structure on M_φ. In this way we get infinitely many Stein fillings of M_φ with growing Euler characteristics, which (by taking, e.g., $M_\varphi = S^3$) contradicts Theorem 2.1.*

Proof of Theorem 1.1. Suppose that $X \to S^2$ is a nontrivial fibration with a section of nonnegative square. By blowing down spheres in fibers we can assume that X is relatively minimal. Since for any fiber-genus g there exists a relatively minimal Lefschetz fibration $X_g \to S^2$ with a section of square (-1) (for such examples see [10]), fiber summing X with the appropriate number of X_g (and sewing the sections together) we get $Z = X\#_f X_g \#_f \dots \#_f X_g \to S^2$ with a section of square 0. Since X is nontrivial, Z is nontrivial as well, contradicting the conclusion of Proposition 3.1. □

4. Appendix

In this Appendix we sketch four futher proofs of Proposition 3.1; as it was shown above, this proposition implies Theorem 1.1.

Proof (using Seiberg-Witten theory). Suppose that $X \to S^2$ is nontrivial, relatively minimal and admits a section of square 0. Then, according to a theorem of Gompf (which equips any Lefschetz fibration of fiber genus $g > 1$ with a symplectic structure) $X\#_f X$ carries a symplectic structure ω. It follows from [8] that $b_2^+(X\#_f X) > 2$ (because a nontrivial Lefschetz fibration over S^2 contains a nonseparating vanishing cycle). Sewing the two copies of the section of square 0 together we find a homologically essential sphere with square 0 — contradicting the adjunction inequality for the Seiberg-Witten basic class $c_1(X, \omega)$. □

Proof. Alternatively, using Seiberg-Witten theory it can be shown [8] that the number of singular fibers in a genus-g Lefschetz fibration over S^2 grows linearly with g. Now along a section of square 0 we could Gompf sum the trivial fibration $\Sigma_h \times S^2 \to S^2$ (along a section of it); in this way the fiber genus increases while the number of singular fibers remains unchanged, providing a contradiction to the above mentioned result of [8]. □

Proof (using a theorem of McDuff). Suppose again that $X \to S^2$ is nontrivial, relatively minimal and admits a section of square 0. According to the theorem of Gompf cited above, X admits a symplectic structure, which can be chosen to make a preassigned section symplectic. According to [6] the existence of a symplectic sphere of square 0 implies that the 4-manifold X is ruled, i.e., X is diffeomorphic to the blow-up of some 4–manifold of the form $S^2 \times \Sigma_g$. With a little more care one can also show that the fibration itself is the trivial fibration. □

Remark 4.1. *By taking $X \#_f X$ and using $b_2^+(X\#_f X) > 1$ again, for a nontrivial fibration X we get the contradiction without any further effort: since the section of square 0 in X provides a similar section in $X\#_f X$, according to the above it is the blow-up of a ruled surface. This is, however, a contradiction because the b_2^+ invariant of a ruled surface is 1, while $b_2^+(X\#_f X) > 1$.*

Proof (due to I. Smith, [7]*).* Using the section of the fibration, each Dehn twist admits a lift to the universal cover of the typical fiber. Since we can assume that the genus of the typical fiber is ≥ 2, this universal cover can be identified with the hyperbolic disk. These lifts induce maps $S^1_\infty \to S^1_\infty$ of the boundary of the disk, and it has been proved in [7] that when starting from right-handed Dehn twists these induced maps on the boundary circle "rotate" in the same direction (and not all points are fixed). Now the existence of a Lefschetz fibration with a section of square 0 gives rise to a collection of "rotations" with product equal to $\mathrm{id}_{S^1_\infty}$ — leading us to a contradiction again. (For more details see [7].) □

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