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A partial order on the group of contactomorphisms of $\mathbb{R}^{2n+1}$ via generating functions

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1. Introduction

In an attempt to find an analogue of the Hofer metric in the contact case, Eliashberg and Polterovich [4] found that, for certain classes of contact manifolds, the universal cover of the space of contactomorphisms carried a natural partial order. They proposed that it should be possible to prove the same result for a large class of contact manifolds using the idea of contact homology. In this note, we show that, for $\mathbb{R}^{2n+1}$ with the standard contact structure, the identity component of the group of compactly supported contactomorphisms, itself, carries a natural partial order. We note that this is the first time such an existence result for a contactomorphism group has appeared in the literature. An outline of our construction follows.

Consider $\mathbb{R}^{2n+1}$ endowed with its standard contact structure ξ given as the kernel of the 1-form $\alpha = dz - \sum_i y_i dx_i$. Let $\psi\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ be a compactly supported contactomorphism and denote by $g\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ the logarithm of the factor by which ψ scales α, that is, $\psi^*\alpha = e^g\alpha$. Here ψ is said to be compactly supported if it agrees with the identity outside of a compact set. We associate with ψ the Legendrian embedding $\iota_\psi\colon \mathbb{R}^{2n+1} \to (\mathbb{R}^{2(2n+1)+1}, \Xi)$ given by

$$\iota_\psi(p) = (p, \psi(p), g(p)),$$

where, in coordinates $(x, y, z, X, Y, Z, \theta)$ on $\mathbb{R}^{2(2n+1)+1}$, the contact structure Ξ is given as the kernel of the 1-form

$$A = e^\theta\Big(dz - \sum_i y_i dx_i\Big) - \Big(dZ - \sum_i Y_i dX_i\Big).$$

Next, let $J^1\mathbb{R}^{2n+1} = T^*\mathbb{R}^{2n+1} \times \mathbb{R}$ denote the 1-jet bundle of $\mathbb{R}^{2n+1}$ and equip it with its standard contact structure ξ given as the kernel of the 1-form $\alpha_0 = dw - \lambda_{\mathrm{can}}$, where w denotes the coordinate on $\mathbb{R}$ and λ_{can} denotes the canonical 1-form on $T^*\mathbb{R}^{2n+1}$. Consider the contact embedding $\Phi\colon (\mathbb{R}^{2(2n+1)+1}, \Xi) \to J^1\mathbb{R}^{2n+1}$ given by

$$\Phi(x, y, z, X, Y, Z, \theta) = (x, Y, z, e^\theta y - Y, X - x, 1 - e^\theta, \langle Y, X - x\rangle - Z + z)$$

and denote by $\Gamma_\psi \subset J^1\mathbb{R}^{2n+1}$ the image of $\Phi \circ \iota_\psi$.[1] Note that Γ_ψ agrees with the zero-section outside of a compact set, thus we may uniquely extend it to a Legendrian submanifold, which we denote $\widetilde{\Gamma}_\psi$, of J^1S^{2n+1}. Here we think of S^{2n+1} as the one-point compactification of $\mathbb{R}^{2n+1}$. We identify $S^{2n+1} \setminus \{q_s\}$, where q_s denotes the "south pole", with $\mathbb{R}^{2n+1}$ via stereographic projection.

Let, next, $\mathrm{Cont}^0 = \mathrm{Cont}^0(\mathbb{R}^{2n+1})$ denote the connected component of the identity in the group of compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$. For $\psi \in \mathrm{Cont}^0$, we define $c(\psi) = c(1, \widetilde{\Gamma}_\psi)$, where $c(1, \widetilde{\Gamma}_\psi)$ is as defined by Viterbo in [6] (see also Section 2). Note that, according to Viterbo [6, Proposition 4.2], $c(\psi) \leq 0$ for every $\psi \in \mathrm{Cont}^0$. Proceeding as in the symplectic case considered by Viterbo [6], we define a relation $\succ$ on Cont^0 as follows.

Definition 1.1. $\psi \succ \phi$ if $c(\psi\phi^{-1}) = 0$.

Theorem 1.1. *Let ψ, ϕ be contactomorphisms in* Cont^0*. The relation $\succ$ has the following properties:*

(i) $$\psi \succ \mathrm{id} \quad \Longrightarrow \quad \phi\psi\phi^{-1} \succ \mathrm{id}\,.$$

(ii) $$\psi \succ \mathrm{id} \ \textit{and}\ \psi^{-1} \succ \mathrm{id} \quad \Longleftrightarrow \quad \psi = id.$$

(iii) $$\psi \succ \mathrm{id} \ \textit{and}\ \phi \succ \mathrm{id} \quad \Longrightarrow \quad \psi\phi \succ \mathrm{id}\,.$$

Corollary 1.2. *The relation $\succ$ defines a partial order on $Cont^0$.*

Suppose, next, that (M, ξ) is an arbitrary contact manifold and let $\alpha \in \Omega^1(M)$ be a contact 1-form. Given a contact isotopy $\psi_t\colon M \to M$ (with $\psi_0 = \mathrm{id}$), recall that the associated contact vector field $X_t \in \mathrm{Vect}(M)$ is given by

$$\frac{\partial}{\partial t}\psi_t = X_t \circ \psi_t,$$

and the associated contact Hamiltonian $H_t\colon M \to \mathbb{R}$ is given by

$$H_t = -\iota(X_t)\alpha.$$

In [4], Eliashberg and Polterovich call a contact isotopy ψ_t of a contact manifold (M, ξ) *nonnegative* if the associated contact Hamiltonian H_t satisfies $H_t \geq 0$ for every t. Given an element f in the universal cover of $\mathrm{Cont}^0(M, \xi)$, they write $f \geq \mathrm{id}$ if f can be represented by a nonnegative contact isotopy ψ_t. They show that, for certain classes of closed contact manifolds, this relation gives rise to a genuine partial order on the universal cover of the space of contactomorphisms of (M, ξ). The following proposition, inspired by [6, Proposition 4.6], shows that, in case (M, ξ) is $\mathbb{R}^{2n+1}$ with its standard contact structure, if ψ_t is nonnegative, then $\psi_1 \succ \mathrm{id}$.

Proposition 1.3. *Let ψ_t be a contact isotopy of* $\mathbb{R}^{2n+1}$ *and $H\colon [0,1] \times \mathbb{R}^{2n+1} \to \mathbb{R}$ be the associated contact Hamiltonian. Suppose that $H(t,x) \geq 0$ for all (t,x). Then $\psi_1^H \succ$* id.

[1] We remark that, in the terminology of [1], when ψ is C^1-close to the identity, the Legendrian submanifold Γ_ψ is precisely the 1-graph of V_ψ where V_ψ is the generating function of type V of ψ.

2. Generating functions

In this section we give a brief introduction to the theory of generating functions and state the results that will be needed in later sections. For the purposes of this note it will only be necessary to consider generating functions $S\colon E \to \mathbb{R}$ defined on trivial bundles $E = B \times \mathbb{R}^k$. The definition in this case is as follows.

Assume that 0 is a regular value of the fibre derivative $\partial S/\partial\xi\colon B \times \mathbb{R}^k \to \mathbb{R}^k$ so that the set $\mathcal{C}_S = (\partial S/\partial\xi)^{-1}(0)$ is a submanifold of E. Now note that at a point $(x,\xi) \in \mathcal{C}_S$ the covector $dS(x,\xi) \in T^*_{(x,\xi)}E$ can be identified with a unique covector $v^* \in T^*_x B$. It can now easily be checked that the map

$$\mathcal{C}_S \xrightarrow{\iota_S} J^1B\colon (x,\xi) \mapsto (x, v^*, S(x,\xi))$$

is a Legendrian immersion. In this situation the function $S\colon E \to \mathbb{R}$ is called a *generating function for the Legendrian immersion* $\iota_S\colon \mathcal{C}_S \to J^1B$. More generally, given a Legendrian immersion $f\colon L \to J^1B$, $S\colon E \to \mathbb{R}$ is called a *generating function for f* if there exists a diffeomorphism $h\colon L \to \mathcal{C}_S$ such that $f = \iota_S \circ h$.

A function $S\colon E \to \mathbb{R}$ defined on a vector bundle $\pi\colon E \to B$ is called *quadratic at infinity* if $S(x,\xi) = q_\infty(x,\xi)$, for $|\xi|$ sufficiently large, where $q_\infty(x,\cdot)$ is a nondegenerate quadratic form for each $x \in B$.

Assume now that B is a closed (that is, compact and boundaryless) manifold and let $\mathrm{Leg} = \mathrm{Leg}(B)$ denote the set of Legendrian submanifolds $L \subset J^1B$ which are isotopic through Legendrian submanifolds to the zero-section L_0. The following theorem is due to Chekanov [3] (see also [5] and [1]).

Theorem 2.1. *Every Legendrian submanifold $L \in \mathrm{Leg}$ admits a generating function $S\colon E \to \mathbb{R}$ which is quadratic at infinity. Moreover, every family of Legendrian submanifolds $L_t \in \mathrm{Leg}$ admits a family of generating functions $S_t\colon E \to \mathbb{R}$, each member of which is a quadratic at infinity.*

Since we shall only be considering generating functions which are quadratic at infinity, from now on we omit the phrase "which is quadratic at infinity".

Suppose now that $f\colon L \to J^1B$ is a Legendrian immersion and let $S\colon E \to \mathbb{R}$ be a generating function for f. Then given any fibre preserving diffeomorphism $\Phi\colon E \to E$, $S \circ \Phi$ will also be a generating function for f. Two generating functions $S, S'\colon E \to \mathbb{R}$ are called *equivalent* if they are related in this way. Also, if $q\colon E' \to \mathbb{R}$ is a nondegenerate quadratic form on the fibres of $E' \to B$, then the function $\tilde{S}\colon E \oplus E' \to \mathbb{R}$ given by

$$\tilde{S}(x,\xi,\xi') = S(x,\xi) + q(x,\xi')$$

will be a generating function for f as well. This operation is called *stabilisation.* The uniqueness theorem of Viterbo–Theret [6, 5] now states:

Theorem 2.2. *If S_1 and S_2 are two generating functions for a Legendrian submanifold $L \in \mathrm{Leg}$, then after stabilisation they are equivalent.*

Given a function $S\colon E \to \mathbb{R}$ which is quadratic at infinity, set

$$E^\lambda = \{c \in E \mid S(c) \leq \lambda\}$$

for $\lambda \in \mathbb{R}$ and let $D(E^-)$ and $S(E^-)$ denote the disc and sphere bundles respectively of the negative bundle E^- associated to q_∞. The Thom isomorphism now gives an isomorphism between the cohomology of B and the cohomology of the pair $(D(E^-), S(E^-))$, shifting the grading by $k = \operatorname{rank} E^-$. By homotopy, the cohomology of the latter is the same as the cohomology of the pair $(E^\mu, E^{-\mu})$ for μ sufficiently large. We thus obtain an isomorphism

$$T\colon H^*(B) \underset{\simeq}{\longrightarrow} H^*(E^\mu, E^{-\mu}),$$

for μ sufficiently large, shifting the grading by k.

Given a cohomology class $u \in H^*(B)$, the invariant $c(u, S) \in \mathbb{R}$ is now defined by

$$c(u, S) = \inf\{\lambda \mid Tu \text{ is nonzero in } H^*(E^\lambda, E^{-\mu})\}$$

for μ sufficiently large. By Lusternik–Schnirelman theory, $c(u, S)$ is a critical value of S. By the uniqueness theorem, given a Legendrian submanifold $L \in \mathrm{Leg}$, $c(u, L)$ may be defined to be $c(u, S)$ for any S which is a generating function for L.

We now quote some further results from Viterbo's paper [6], which we state in the language of Legendrian submanifolds and which will be needed in later sections.

Given a vector bundle $\pi\colon E \to B$, denote by $\mathcal{Q}(E)$ the space of quadratic-at-infinity functions $S\colon E \to \mathbb{R}$.

Proposition 2.3 ([6, Proposition 1.2])**.** *For every $u \in H^*(B)$, the restriction of $c(u, \cdot)$ to $\mathcal{Q}(E)$ is continuous with respect to the C^0-topology on $\mathcal{Q}(E)$.*

Let $\mu \in H^n(B)$ denote the orientation class of B.

Proposition 2.4 ([6, Corollary 2.3])**.** *Let L be a Legendrian submanifold in* Leg*. Then $c(1, L) = c(\mu, L)$ if and only if $L = L_0$.*

Given a Legendrian submanifold $L \subset J^1B$, denote by $\overline{L}$ the image of L under the map $(x, y, z) \mapsto (x, -y, -z)$. Of course, $\overline{L}$ is again Legendrian. Moreover

Proposition 2.5 ([6, Corollary 2.8])**.** *If L is a Legendrian submanifold in* Leg*, then $c(\mu, \overline{L}) = -c(1, L)$.*

Legendrian submanifolds may be summed in the following manner.

Let $f_i = (l_i, S_i)\colon L_i \to J^1B = T^*B \times \mathbb{R}$, $i = 1, 2$ be two Legendrian immersions and assume that $l_1 \times l_2\colon L_1 \times L_2 \to T^*(B \times B)$ is transverse to the submanifold $T^*(B \times B)|_\Delta$, where Δ denotes the diagonal in $B \times B$. As $T^*(B \times B)|_\Delta$ is coisotropic it admits a reduction, which may be identified with T^*B. The corresponding reduction of $l_1 \times l_2$ is denoted $l_1 \sharp l_2\colon L_1 \sharp L_2 \to T^*B$, where $L_1 \sharp L_2 = (l_1 \times l_2)^{-1} T^*(B \times B)|_\Delta$. The lift $f = (l_1 \sharp l_2, S)\colon L_1 \sharp L_2 \to J^1B$, where S is given by composing $S_1 \times S_2|_{L_1 \sharp L_2}$ with the summation map $(z_1, z_2) \mapsto z_1 + z_2$, is denoted $f_1 \sharp f_2$ and is called the *Legendrian sum.*

Note that when L_1, L_2 are submanifolds of J^1B the Legendrian sum is given by

$$L_1\sharp L_2 = \{(x,y,z) \in J^1B \mid y = y_1 + y_2,\, z = z_1 + z_2, (x,y_1,z_1) \in L_1,\, (x,y_2,z_2) \in L_2\}.$$

In view of this the Legendrian sum $L_1\sharp L_2$ is also denoted L_1+L_2. By convention $L_1-L_2 = L_1+\overline{L_2}$. Also, note that if $S_i\colon E_i \to \mathbb{R}$ is a generating function for L_i, $i = 1,2$, then $S_1\sharp S_2\colon E_1 \oplus E_2 \to \mathbb{R}$, given by

$$S_1\sharp S_2(x,\xi_1,\xi_2) = S_1(x,\xi_1) + S_2(x,\xi_2),$$

is a generating function for $L_1\sharp L_2$.

Proposition 2.6 ([6, Proposition 3.3])**.** *For every $u, v \in H^*(B)$,*

$$c(u\cup v, S_1\sharp S_2) \geq c(u,S_1) + c(v,S_2).$$

3. Proof of Theorem 1.1

We make use of the following lemma in the proof of Theorem 1.1.

Lemma 3.1. *Let ψ be a contactomorphism in* Cont^0 *with $\psi^*\alpha = e^g\alpha$ and $S\colon E \to \mathbb{R}$ be a generating function for Γ_ψ. Then if $c \in E$ is a critical point of S, then $\psi(x,y,z) = (x,y,z-S(c))$ and $g(x,y,z) = 0$, where $(x,y,z) = \pi(c)$. In particular, if c is a critical point of S with critical value zero, then $\pi(c)$ is a fixed point of ψ and $g(\pi(c)) = 0$.*

Furthermore, suppose that c is a critical point of S with critical value zero, then the following are equivalent:

(i) *c is a nondegenerate critical point of S;*
(ii) *Γ_ψ intersects the zero-wall, $Z_{\mathbb{R}^{2n+1}}$, that is, the product of the zero-section of the $T^*\mathbb{R}^{2n+1}$ with the real line, transversely at $\iota_S(c)$;*
(iii) *the linearised equations $d\psi(\pi(c))X = X$, $dg(\pi(c))(X) = 0$ admit no common non-trivial solution $X \in \mathbb{R}^{2n+1}$.*

Proof. Let c be a critical point of ψ. Then, by definition, we have $\iota_S(c) = (\pi(c), 0, S(c)) \in \Gamma_\psi$. Letting $\pi(c) = (x,y,z)$ and $\psi(x,y,z) = (X,Y,Z)$ and recalling that $\Gamma_\psi = \Phi\circ\iota_\psi$ we find that

$$e^\theta y - Y = 0, \quad X - x = 0, \quad 1 - e^\theta = 0, \quad \langle Y, X-x\rangle - Z + z = S(c)$$

or, equivalently,

$$\theta = 0, \quad X = x, \quad Y = y, \quad -Z + z = S(c).$$

It follows that $\psi(x,y,z) = (x,y,z-S(c))$, $g(x,y,z) = 0$ proving the first part of the lemma. We now prove the second part of the lemma.

(i) is equivalent to (ii). We use the following elementary fact. Let (V,ω) be a symplectic vector space and $N \subset V$ be a coisotropic subspace. Suppose that Λ_0 and Λ_1 are two Lagrangian subspaces of V satisfying

$$\Lambda_0 \subset N, \qquad \Lambda_1 \cap N^\omega = \{0\},$$

where N^ω denotes the symplectic complement of N. Then Λ_0 is transverse to Λ_1 if and only if, in the quotient, the reduced spaces $\bar{\Lambda}_0$ and $\bar{\Lambda}_1$ are transversal. Now set $V = T_a(T^*E)$, $N = T_aN_E$, $\Lambda_0 = T_aE_0$ and $\Lambda_1 = T_a(\mathrm{graph}(dS))$, where $a = (c, dS(c))$,

$$N_E = \left\{(c, \eta) \in T^*E \,\middle|\, \eta \in (\ker d\pi(c))^\perp\right\}$$

and E_0 denotes the zero section of T^*E.

(ii) is equivalent to (iii). Γ_ψ intersects $Z_{\mathbb{R}^{2n+1}}$ nontransversally at $(x, y, z) = \iota_S(c)$ if and only if there exists a nonzero vector $(\xi, \eta, \zeta) \in \mathbb{R}^{2n+1}$ such that

$$e^\theta y\Theta + e^\theta\eta - \eta' = 0, \quad \xi' - \xi = 0, \quad e^\theta\Theta = 0,$$

where $\theta = g(x, y, z)$, ξ' and η' are defined by $(\xi', \eta', \zeta') = d\psi(x, y, z)(\xi, \eta, \zeta)$ and $\Theta = dg(x, y, z)(\xi, \eta, \zeta)$. Since $g(x, y, z) = 0$, this is equivalent to

$$\xi' = \xi, \quad \eta' = \eta, \quad \Theta = 0. \tag{1}$$

Also, using the fact that ψ is a contactomorphism it follows that

$$\zeta' - \langle Y, \xi'\rangle = e^\theta(\zeta - \langle y, \xi\rangle).$$

It thus follows that if the equalities in (1) hold, then $\zeta' = \zeta$ holds automatically, that is, (ξ, η, ζ) is a fixed point of $d\psi(x, y, z)$ and $dg(x, y, z)(\xi, \eta, \zeta) = 0$. □

Proof of Theorem 1.1. (i) Fix a smooth family ϕ_t of compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$ satisfying $\phi_0 = \mathrm{id}$, $\phi_1 = \phi$. For each t, define $\psi_t = \phi_t\psi\phi_t^{-1}$ and suppose that $\psi_t^*\alpha = e^{g_t}\alpha$. Also, let $S_t\colon E \to \mathbb{R}$ be a smooth family of generating functions for the associated family $\widetilde{\Gamma}_{\psi_t}$ of Legendrian submanifolds of J^1S^{2n+1}. By definition, outside of some compact set $C \subset E$, which we can assume is away from the fibre over the south pole $q_s \in S^{2n+1}$, all of the functions S_t are of the form $S_t(q, \xi) = q_{t,\infty}(q, \xi)$, where $q_{t,\infty}(q, \cdot)$ is a nondegenerate quadratic form for each $q \in S^{2n+1}$. We now prove (i) assuming, to begin with, that all critical points $c \in \mathrm{int}\, C$ of $S = S_0$, with critical value zero, are nondegenerate.

By Lemma 3.1, away from the fibre over q_s, every critical point $c \in E$ of S, with critical value zero, corresponds to a point $p \in \mathbb{R}^{2n+1}$ satisfying $\psi(p) = p$, $g(p) = 0$, where $g = g_0$. Also, a point $p \in \mathbb{R}^{2n+1}$ satisfies $\psi(p) = p$, $g(p) = 0$ if and only if $p_t = \phi_t(p)$ satisfies $\psi_t(p_t) = p_t$, $g_t(p_t) = 0$ for every t. Now, if p, and hence also $p_t = \phi_t(p)$ for every t, satisfies the preceding, then, by the nondegeneracy assumption, the linearised equations $d\psi(p)X = X$, $dg(p)(X) = 0$, and hence also $d\psi_t(p_t)X = X$, $dg_t(p_t)(X) = 0$ for every t, admit no common nontrivial solution $X \in \mathbb{R}^{2n+1}$. It follows easily that to every nondegenerate critical point $c \in E$ of S with critical value 0 there corresponds a smooth family c_t, $0 \le t \le 1$, such that, for each t, c_t is a nondegenerate critical point of S_t with critical value zero. Under the nondegeneracy assumption, (i) now follows from Proposition 2.3. In general we approximate ψ as follows.

Let $U \subset E$ be a sufficiently large open set, with compact closure and away from the fibre over q_s, such that outside U S is of the form $S(q, \xi) = q_\infty(q, \xi)$. Perturb S in U so that all critical points in the interior of U become nondegenerate. By adding a small bump function, now, if necessary, which takes the value ε in U, for some small constant

$\varepsilon > 0$, and is identically zero outside of some neighbourhood of U, we may assume that the resulting function S' satisfies $c(1, S') = 0$. Indeed, let ρ be a bump function which takes the value 1 in U and which is identically zero outside of some neighbourhood of U and let S^ε denote the function obtained by first perturbing S and then adding $\varepsilon\rho$. The critical values of this function S^ε are given by

$$\operatorname{crit} S^\varepsilon = (\operatorname{crit} \hat{S} + \varepsilon) \cup \{0\},$$

where $\hat{S}$ denotes the perturbed function. Since $\operatorname{crit} \hat{S}$ is a totally disconnected set, it is easy to see that if $c(1, \hat{S}) < 0$, then $c(1, S^\varepsilon) = 0$ for some (small) $\varepsilon > 0$.

It follows that, for a sufficiently small perturbation, S' will be a generating function for $\widetilde{\Gamma}_{\psi'}$ for some contactomorphism ψ' C^1-close to ψ. A continuity argument now completes the proof of (i).

For the proof of (ii) and (iii) we require the following result.

Lemma 3.2. *For every $u \in H^*(S^{2n+1})$ and $\psi \in \operatorname{Cont}^0(\mathbb{R}^{2n+1})$,*

$$c(u, \widetilde{\Gamma}_\psi) = 0 \quad \Longrightarrow \quad c(u, \overline{\widetilde{\Gamma}_{\psi^{-1}}}) = 0.$$

Before proceeding to prove this lemma we introduce some notation.

Let ψ be a contactomorphism of $\mathbb{R}^{2n+1}$ with $\psi^*\alpha = e^g\alpha$. Then the map

$$\widetilde{\psi} \colon \mathbb{R}^{2(2n+1)+1} \to \mathbb{R}^{2(2n+1)+1}$$
$$(p, P, \theta) \mapsto (p, \psi(P), g(P) + \theta)$$

is a contactomorphism of $(\mathbb{R}^{2(2n+1)+1}, \Xi)$. Denote by Ψ_ψ the composition $\Phi \circ \widetilde{\psi} \circ \Phi^{-1}$ and extend this to a contactomorphism $\widetilde{\Psi}_\psi$ of a subset of J^1S^{2n+1} containing the zero section, L_0, in the obvious way. Note that $\widetilde{\Psi}_\psi$ satisfies

$$\widetilde{\Psi}_\psi \widetilde{\Gamma}_\phi = \widetilde{\Gamma}_{\psi\phi}, \qquad \widetilde{\Psi}_\psi^{-1} = \widetilde{\Psi}_{\psi^{-1}}$$

for any contactomorphism ϕ of $\mathbb{R}^{2n+1}$. Lemma 3.2 is now an immediate consequence of the following.

Lemma 3.3. *Assume that U is an open subset of J^1S^{2n+1} containing L_0. Let $\Psi \colon U \to U$ be a contactomorphism which is isotopic to the identity and $L \subset U$ be a Legendrian submanifold which is Legendrian isotopic to L_0, then*

$$c(u, \Psi(L)) = 0 \quad \Longrightarrow \quad c(u, L - \Psi^{-1}(L_0)) = 0$$

for any $u \in H^*(S^{2n+1})$.

Proof. Choose a smooth family of contactomorphims $\Psi_t \colon U \to U$ such that $\Psi_0 = \mathrm{id}$, $\Psi_1 = \Psi$ and, for $t \in [0,1]$, when defined, set $\Lambda_t = \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(L_0)$. Then

$$\Lambda_0 = \Psi(L) - L_0 = \Psi(L), \qquad \Lambda_1 = L - \Psi^{-1}(L_0).$$

Now let $S_t \colon E \to \mathbb{R}$ be a smooth family of generating functions for the Legendrian submanifolds Λ_t obtained by considering generating functions individually for the families of Legendrian submanifolds $\Psi_t^{-1}\Psi(L)$ and $\Psi_t^{-1}(L_0)$. Note that, with this definition, S_t is defined for every $t \in [0,1]$, even when Λ_t is not defined. The idea of the proof now is to study the bifurcation diagram of the critical points of S_t and to show that if $c(u,\Lambda_0) = c(u,S_0) = 0$, then $c(u,S_t) = 0$ for every t and, in particular, that $c(u,\Lambda_1) = c(u,S_1) = 0$.

First note that critical points $c \in E$ of S_t with critical value zero correspond, in a straightforward way, to points

$$b \in \Psi_t^{-1}\Psi(L) \cap \Psi_t^{-1}(L_0)$$

which in turn correspond to points

$$b' = \Psi_t(b) \in \Psi(L) \cap L_0.$$

It follows that every critical point $c \in E$ of S_t with critical value zero belongs to a continuous family of points c_t, $0 \le t \le 1$, such that $dS_t(c_t) = 0$, $S_t(c_t) = 0$ for each t. This does not, in itself, prove the lemma as one of these families may bifurcate.

Now note that every critical point $c \in E$ of S_t corresponds to a point

$$a \in \Pi\Psi_t^{-1}\Psi(L) \cap \Pi\Psi_t^{-1}(L_0),$$

where $\Pi\colon J^1S^{2n+1} \to T^*S^{2n+1}$ denotes the natural projection. Also, if $\Pi\Psi(L)$ intersects ΠL_0 transversely at $\Pi(b')$, where $b' \in \Psi(L) \cap L_0$, then $\Pi\Psi_t^{-1}\Psi(L)$ intersects $\Pi\Psi_t^{-1}(L_0)$ transversely at $\Pi\Psi_t^{-1}(b')$. It follows that if $\Psi(L)$ intersects $Z_{S^{2n+1}}$ transversely and $c(u,\Lambda_0) = c(u,S_0) = 0$, then $c(u,S_t) = 0$ for every $t \in [0,1]$. The proof is now completed by approximating L by Legendrian submanifolds L_ε such that $\Psi(L_\varepsilon)$ is transverse to $Z_{S^{2n+1}}$ and $c(u,\Psi(L_\varepsilon)) = 0$, arguing as in the proof of Theorem 1.1 (i) □

Proof of Theorem 1.1 continued. (ii) This follows immediately from Lemma 3.2, Proposition 2.4 and Proposition 2.5.

(iii) Set $\Psi = \widetilde{\Psi}_{\phi^{-1}\psi^{-1}}$, $L = \widetilde{\Gamma}_\psi$. Then

$$0 = c(1,\widetilde{\Gamma}_\phi) = c(\mu,\overline{\widetilde{\Gamma}_\phi}) = c(\mu,\widetilde{\Gamma}_{\phi^{-1}}) = c(\mu,\Psi(L)),$$

where the second equality follows from Proposition 2.5 and the third equality from Lemma 3.2. Thus, by Lemma 3.2 and Proposition 2.6,

$$0 = c(\mu, L - \Psi^{-1}(L_0)) \ge c(1,\widetilde{\Gamma}_\psi) + c(\mu,\overline{\Psi^{-1}(L_0)}) = -c(1,\Psi^{-1}(L_0)) = -c(1,\widetilde{\Gamma}_{\psi\phi}),$$

that is, $c(\psi\phi) \ge 0$. This completes the proof of (iii) and the theorem. □

4. Proof of Proposition 1.3

We first prove the following auxiliary lemma.

Lemma 4.1. *Let ψ_t be a contact isotopy of $\mathbb{R}^{2n+1}$ with $\psi_1^*\alpha = e^g\alpha$ and $H\colon [0,1] \times \mathbb{R}^{2n+1} \to \mathbb{R}$ be the associated contact Hamiltonian. Then the vector field $X_t \in \mathrm{Vect}(\mathbb{R}^{2n+1})$ associated with ψ_t is given by*

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i} - y_i\frac{\partial H}{\partial z}, \quad \dot{z} = \langle y,\dot{x}\rangle - H. \tag{2}$$

Furthermore,

$$g(p) = \int_0^1 -\partial_z H(t, \psi_t(p))\, dt$$

for any $p \in \mathbb{R}^{2n+1}$.

Proof. The first part of the lemma is easy. For the second part, fix a point $p = (x, y, z) \in \mathbb{R}^{2n+1}$ and consider a curve $s \mapsto (x_s, y_s, z_s)$ passing through p at $s = 0$. Also, set

$$(x_s(t), y_s(t), z_s(t)) = \psi_t(x_s, y_s, z_s),$$

abbreviate $(x(t), y(t), z(t)) = (x_0(t), y_0(t), z_0(t))$ and denote

$$(\xi(t), \eta(t), \zeta(t)) = \left.\frac{\partial}{\partial s}\right|_{s=0} (x_s(t), y_s(t), z_s(t)).$$

Now consider the expression

$$\dot{z}_s = \langle y_s, \dot{x}_s \rangle - H.$$

Differentiate this with respect to s and set $s = 0$ to obtain

$$\begin{aligned}
\dot{\zeta} &= \langle \eta, \dot{x} \rangle + \langle y, \dot{\xi} \rangle - \partial_x H \xi - \partial_y H \eta - \partial_z H \zeta \\
&= \langle \eta, \dot{x} - \partial_y H \rangle - \langle \dot{y} + \partial_x H + y \partial_z H, \xi \rangle + \langle y \partial_z H, \xi \rangle - \partial_z H \zeta + \partial_t \langle y, \xi \rangle.
\end{aligned}$$

Thus, using the first two equations in (2), obtain

$$\frac{d}{dt}\big(\zeta - \langle y, \xi \rangle\big) = -\partial_z H \big(\zeta - \langle y, \xi \rangle\big).$$

It follows that

$$\zeta(1) - \langle y(1), \xi(1) \rangle = e^{\int_0^1 -\partial_z H(t, p(t))\, dt} \big(\zeta(0) - \langle y(0), \xi(0) \rangle\big), \tag{3}$$

where $p(t) = \psi_t(p)$. The lemma now follows by comparing (3) with the defining equation $\psi_1^* \alpha = e^g \alpha$. $\square$

Proof of Proposition 1.3. For $0 \le \lambda \le 1$, let ψ_t^λ be the contact isotopy associated to the contact Hamiltonian $H_\lambda = \lambda H$. Also, let $S_\lambda \colon E \to \mathbb{R}$ be a family of generating functions for the $\widetilde{\Gamma}_{\psi_1^\lambda}$. We claim that for every critical point $c \in E$ of S_λ

$$\frac{\partial S_\lambda}{\partial \lambda}(c) \ge 0.$$

This claim will be proved at the end of the proposition.

We now prove the proposition assuming the validity of the claim. To begin with, we make the stronger assumption that $\partial_\lambda S_\lambda(c) > 0$ for every point $c \in E$ satisfying $dS_\lambda(c) = 0$. We also assume that the family S_λ is generic, that is, for every λ in the complement of a finite set, $\lambda_0 < \lambda_1 < \cdots < \lambda_k$, S_λ is a Morse function with isolated critical values. Now, for $\lambda \in [\lambda_{i-1}, \lambda_i]$, $i = 1, \ldots, k$, let c_λ be the unique critical point of S_λ such that $S_\lambda(c_\lambda) = c(1, S_\lambda)$ and abbreviate $c(S_\lambda) = c(1, S_\lambda)$. Then

$$\frac{d}{d\lambda} c(S_\lambda) = \frac{d}{d\lambda} S_\lambda(c_\lambda) = \frac{\partial S_\lambda}{\partial \lambda}(c_\lambda) > 0,$$

where the second equality holds as c_λ is a critical point of S_λ. Thus $c(S_\lambda)$ is strictly increasing on each segment $[\lambda_{i-1}, \lambda_i]$ and, as it is also continuous, it follows that $c(S_1) \geq 0$, that is, $\psi_1 \succ \mathrm{id}$. This proves the proposition under the assumption that $\partial_\lambda S_\lambda(c) > 0$ for every point $c \in E$ satisfying $dS_\lambda(c) = 0$ and that the family S_λ is generic. These assumptions are removed as follows.

The genericity assumption can be removed by approximating S_λ in the C^1-topology. This does not destroy the assumption that $\partial_\lambda S_\lambda(c) > 0$ for every point $c \in E$ satisfying $dS_\lambda(c) = 0$. The latter in turn can be removed by replacing S_λ by $S_\lambda + \varepsilon\lambda\rho$ for ε sufficiently small, where ρ is a suitable cut-off function. All that remains now is to prove the claim.

Assume now that c_λ, $\lambda \in I$, for some interval $I \subset [0,1]$, is a family of points in E such that $dS_\lambda(c_\lambda) = 0$ for each λ. By Lemma 3.1, to the family of points $c_\lambda \in E$ there corresponds a family of points $p_\lambda = (x_\lambda, y_\lambda, z_\lambda) \in \mathbb{R}^{2n+1}$ such that

$$\psi_1^\lambda(x_\lambda, y_\lambda, z_\lambda) = (x_\lambda, y_\lambda, z_\lambda - S_\lambda(c_\lambda)), \qquad g_\lambda(x_\lambda, y_\lambda, z_\lambda) = 0$$

for each λ, where g_λ is defined by ${\psi_1^\lambda}^*\alpha = e^{g_\lambda}\alpha$.

Set, now,

$$p_\lambda(t) = (x_\lambda(t), y_\lambda(t), z_\lambda(t)) = \psi_t^\lambda(x_\lambda, y_\lambda, z_\lambda)$$

and denote

$$(\xi_\lambda(t), \eta_\lambda(t), \zeta_\lambda(t)) = \frac{\partial}{\partial\lambda}(x_\lambda(t), y_\lambda(t), z_\lambda(t)).$$

In this notation the critical values of the S_λ are given by

$$S_\lambda(c_\lambda) = -z_\lambda(1) + z_\lambda(0).$$

Also

$$\frac{\partial S_\lambda}{\partial\lambda}(c_\lambda) = \frac{d}{d\lambda}S_\lambda(c_\lambda) = -\zeta_\lambda(1) + \zeta_\lambda(0). \tag{4}$$

Consider now the expression

$$\dot z_\lambda = \langle y_\lambda, \dot x_\lambda\rangle - H_\lambda.$$

By the same calculation as in the proof of Lemma 4.1, with varying $H_\lambda = \lambda H$, we have

$$\frac{d}{dt}\big(\zeta - \langle y, \xi\rangle\big) = -\partial_z H_\lambda\big(\zeta - \langle y, \xi\rangle\big) - H.$$

Here the argument λ has been suppressed, when this leads to no confusion. It follows that

$$\begin{aligned}\zeta(1) - \langle y(1), \xi(1)\rangle &= e^{\int_0^1 -\partial_z H_\lambda(t,p(t))\,dt}\big(\zeta(0) - \langle y(0), \xi(0)\rangle\big)\\ &\quad - \int_0^1 e^{\int_s^1 -\partial_z H_\lambda(t,p(t))\,dt} H(s, p(s))\,ds.\end{aligned}$$

Now from Lemma 4.1

$$0 = g(p(0)) = \int_0^1 -\partial_z H_\lambda(t, p(t))\,dt.$$

Also, $y(0) = y(1)$ and $\xi(0) = \xi(1)$. It follows, using (4), that

$$\frac{\partial S_\lambda}{\partial \lambda}(c_\lambda) = -\zeta(1) + \zeta(0) = \int_0^1 e^{\int_s^1 -\partial_z H_\lambda(t,p(t))\,dt} H(s, p(s))\, ds \geq 0$$

for every $\lambda \in I$. This proves the claim and the proposition. □

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