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The canonical class of a symplectic 4-manifold

Ronald Fintushel and Ronald Stern

1. Introduction

In this article we present examples of simply connected symplectic 4-manifolds X whose canonical classes are represented by complicated disjoint unions of symplectic submanifolds of X:

Theorem 1.1. *Given finite collections $\{g_i\}, \{m_i\}$, $i = 1, \dots, n$, of positive integers, there is a minimal symplectic simply connected 4-manifold X whose canonical class is represented by a disjoint union of embedded symplectic surfaces*

$$K \sim \Sigma_{g_1,1} \cup \cdots \cup \Sigma_{g_1,m_1} \cup \cdots \cup \Sigma_{g_n,1} \cup \cdots \cup \Sigma_{g_n,m_n}$$

where $\Sigma_{g_i,j}$ is a surface of genus g_i. Furthermore, $c_1^2(X) = \chi_h(X) - (2+c)$ where $c = \sum_{i=1}^{n} m_i$ is the total number of connected components of this symplectic representative of the canonical class and $\chi_h(X)$ denotes one-quarter the sum of the Euler characteristic and signature of X.

There are several interesting questions which arise:

1. Let X be a symplectic 4-manifold whose canonical class is represented by a disjoint collection of symplectic surfaces of genera and multiplicities $\{g_i, m_i\}$ as above. Suppose also that each $g_i > 1$. Are these numbers a symplectic invariant of X?
2. Let $C_{\mathfrak{K}}(X)$ be the number of components of a symplectic representative $\mathfrak{K}$ of the canonical class of a symplectic 4-manifold X, and let $C(X)$ be the maximum of $C_{\mathfrak{K}}(X)$ over all $\mathfrak{K}$. Is it true that

$$c_1^2(X) \geq \chi_h(X) - (2 + C(X))?$$

This question should be compared with the conjecture of [9] which states that

$$c_1^2(X) \geq \chi_h(X) - 2b - 1$$

where b represents the number of Seiberg-Witten basic classes up to sign. For the manifolds in Theorem 1.1 this would read

$$c_1^2(X) \geq \chi_h(X) - 2^{c-1} - 1$$

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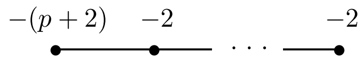
and this is far from optimal, whereas they are likely optimal for the conjecture of (2) above.

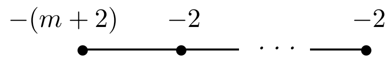
The techniques which are used to prove Theorem 1.1 are standard, namely, 'symplectic sum' [7] and 'rational blowdown' [2]. For the convenience of the reader, we review the notation introduced in [2]. Let C_p denote the simply connected smooth 4-manifold with boundary obtained by plumbing $(p-1)$ disk bundles over the 2-sphere according to the linear diagram:



Here, each node denotes a disk bundle over S^2 with Euler class indicated by the label; an interval indicates that the endpoint disk bundles are plumbed, i.e. identified fiber to base over a hemisphere of each S^2. Label the homology classes represented by the spheres in C_p by $u_1, \ldots, u_{p-1}$ so that the self-intersections are $u_{p-1}^2 = -(p+2)$ and, for $j = 1, \ldots, p-2$, $u_j^2 = -2$. Further, orient the spheres so that $u_j \cdot u_{j+1} = +1$. Then C_p is a 4-manifold with negative definite intersection form and with boundary the lens space $L(p^2, 1-p)$. The lens space $L(p^2, 1-p) = \partial C_p$ bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and a surjective inclusion-induced homomorphism $\pi_1(L(p^2, 1-p) = \mathbf{Z}_{p^2} \to \pi_1(B_p)$. If X is a smooth 4-manifold containing an embedded copy of C_p, its 'rational blowdown' is the result of replacing C_p by the rational ball B_p.

2. The Construction

Lemma 2.1. *The elliptic surface $E(m+2)$ admits a symplectic structure with respect to which it contains a pair of disjoint configurations of smooth, symplectically embedded surfaces. The first is a K3-nucleus and the second is a linear plumbing of $4m-1$ spheres:*



where the sphere of self-intersection $-(m+2)$ is met transversely, in a single positive point, by a symplectically embedded torus F of self-intersection 0.

A 'nucleus' of an elliptic surface $E(n)$, is a regular neighborhood of the union of a cusp fiber and a section C of the fibration. (See [6].) The section C is an embedded sphere of self-intersection $-n$.

Proof. In $\mathbf{S}^2 \times \mathbf{S}^2$ let $S_A = \mathbf{S}^2 \times \{pt\}$ and $S_B = \{pt\} \times \mathbf{S}^2$. Then $E(m)$ may be constructed as the double branched cover of $\mathbf{S}^2 \times \mathbf{S}^2$ branched over 4 copies of S_A and $2m$ copies of S_B. (Either one takes the double branched cover branched over the singular curve as described and then resolves the singularities, or equivalently, smooths the double points of the branch set and then takes the double branched cover. See *e.g.* [13].) In this way, one sees two fibrations on $E(m)$ — the elliptic fibration whose fibers are the double branched covers of the spheres in $\mathbf{S}^2 \times \mathbf{S}^2$ which are parallel to S_A and a genus $(m-1)$ fibration whose fibers are the double branched covers of the spheres in $\mathbf{S}^2 \times \mathbf{S}^2$ which are parallel to S_B. This construction also imbues $E(m)$ with the structure of a Kahler surface.

Take the 4 copies of S_A in the branch set to be $\mathbf{S}^2 \times \{x_i\}$ and $\mathbf{S}^2 \times \{y_i\}$, $i = 1, 2$ where the x_i lie in the northern hemisphere of $\mathbf{S}^2$ and the y_i lie in the southern hemisphere. The double branched cover of $\mathbf{S}^2 \times \mathbf{S}^2$ branched over 2 copies of S_A and $2m$ copies of S_B is a complex surface $R(m)$ which also admits a pair of fibrations, an 'A'-fibration by 2-spheres and a 'B'-fibration by surfaces of genus $m-1$.

It follows that $E(m)$ may be obtained as the fiber sum

$$E(m) \cong R(m) \#_{\Sigma_{m-1}} R(m)$$

of the 'B'-fibrations. It is not difficult to identify $R(m)$. The 'A'-fibration has $2m$ singular fibers as in Figure 1.

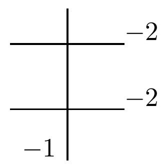


FIGURE 1

The vertical sphere of self-intersection -1 has multiplicity 2. If we blow down this exceptional curve, we obtain a configuration which may be blown down again to obtain a smooth 2-sphere of self-intersection 0. Since there are $2m$ singular fibers, after blowing down $4m$ times, we obtain an S^2-bundle over S^2. It follows that $R(m)$ is the rational surface

$$R(m) = \mathbf{CP}^2 \# (4m+1) \overline{\mathbf{CP}}^2.$$

We next need a second description of $R(m)$. Let $W(m)$ denote the canonical resolution of the $(2, 2m-1, 4m-3)$ Brieskorn singularity. It is given by the plumbing manifold

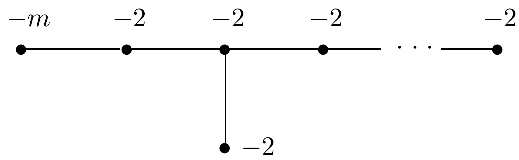


FIGURE 2

where the right-hand 'branch' has $4(m-1)$ nodes. Let $N(m)$ denote the manifold obtained by adding a pair of 2-handles to the 4-ball, the first with framing 0 along the torus knot $T(2, 2m-1)$, the second with framing -1 attached along a meridian to $T(2, 2m-1)$.

For example, $N(2)$ is a neighborhood of a cusp fiber and a section in $E(1)$ (an $E(1)$-nucleus). The boundaries of $W(m)$ and $N(m)$ are easily seen to be orientation-reversing diffeomorphic.

We claim that $W(m) \cup_\partial N(m)$ is a rational surface isomorphic with $R(m)$. To see this, start with a configuration of 4 lines in $\mathbf{CP}^2$, three of them passing through a point x_0, together with a line at infinity. If we blow up $\mathbf{CP}^2$ at x_0 we obtain the configuration of rational curves in $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$ shown in Figure 3.

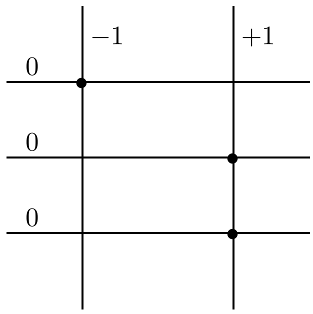


FIGURE 3

Now blow up at each of the three points indicated in Figure 3 to obtain the configuration of rational curves in $\mathbf{CP}^2 \# 4\,\overline{\mathbf{CP}}^2$ as shown in Figure 4.

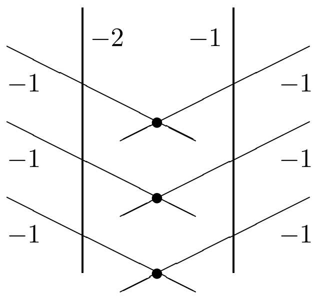


FIGURE 4

Three more blowups at the points indicated in Figure 4 give the configuration of Figure 5 in $\mathbf{CP}^2 \# 7\,\overline{\mathbf{CP}}^2$.

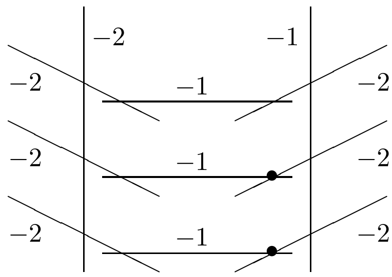


FIGURE 5

Continue blowing up at the indicated points to obtain Figure 6.

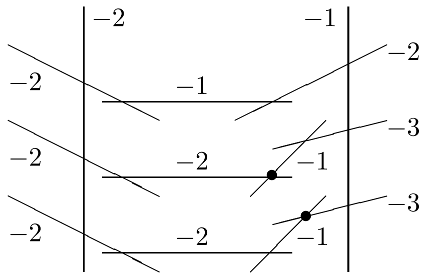


FIGURE 6

Continue blowing up, at both indicated points if $m \geq 3$, otherwise just at the bottom point. After a total of $5m+1$ blowups one achieves a configuration in $\mathbf{CP}^2 \# (5m+1)\overline{\mathbf{CP}}^2$ which consists of two subconfigurations separated by three -1-curves. On the left is the canonical resolution for the $(2, 2m-1, 4m-3)$ Brieskorn singularity and on the right is the configuration whose dual graph (which has $m+2$ nodes) is shown in Figure 7.

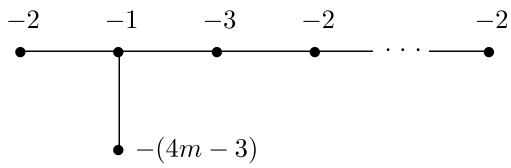


FIGURE 7

Blow up one more time at a point on the curve corresponding to the node labelled $-(4m-3)$. This gives a disjoint pair of configurations in $\mathbf{CP}^2 \# (5m+2)\overline{\mathbf{CP}}^2$. One is the canonical resolution for the $(2, 2m-1, 4m-3)$ Brieskorn singularity and the other is the configuration whose dual graph (which has $m+3$ nodes) is shown in Figure 8.

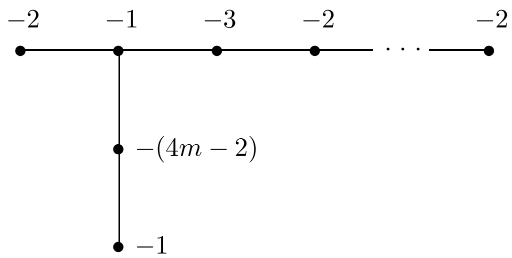


FIGURE 8

Now start blowing down the configuration of Figure 8 starting with the (-1) node of order 3. After $m+1$ blowdowns (along the top row of Figure 8) one obtains the manifold $N(m)$. (See, *e.g.* [1].) We thus decompose $\mathbf{CP}^2 \# (4m+1)\overline{\mathbf{CP}}^2$ as $W(m) \cup_\partial N(m)$. Hence we may identify $R(m)$ with $W(m) \cup_\partial N(m)$.

This identifies an A-fiber of $R(m)$ with $H - E_1$, one of the horizontal lines in Figure 3. Here H denotes the homology class of a line in $\mathbf{CP}^2$ and E_1 is the class of the exceptional curve arising from the first blowup. Looking at Figure 2 we see a linear configuration of $4m-1$ rational curves in $R(m)$ (along the top row). The sphere S labelled '$-m$' represents the sum of m exceptional curves and intersects $H - E_1$ once. (For example, see Figure 6.) In the fiber sum $E(m) \cong R(m)\#_{\Sigma_{m-1}} R(m)$, the A-fibers of the two copies of $R(m)$ glue together to form an elliptic fiber of $E(m)$, thus S is a section of the elliptic fibration.

We consider $E(m+2)$ as the result of a fiber sum $E(m)\#_F E(2)$. The K3-surface $E(2)$ contains three disjoint nuclei. To get an appropriate symplectic structure on $E(2)$, we first fix a hyperkahler metric g. This gives a Kahler form ω_0 on $E(2)$ so that in one nucleus, N_1, the fiber T_1 and section C_1 are both symplectic submanifolds, but for the other two nuclei, N_2, N_3, the fibers F_i and sections C_i are Lagrangian. There are self-diffeomorphisms f_i, $i = 2, 3$, of $E(2)$ which take (N_i, T_i, C_i) to (N_1, T_1, C_1). For small enough $t > 0$, $\omega = \omega_0 + t(f_2^*\omega + f_3^*\omega)$ is a symplectic form on $E(2)$ for which T_i, C_i are symplectic for $i = 1, 2, 3$. Choose one nucleus, say N_1, in order to make the fiber sum $E(m)\#_{F=T_1} E(2)$. (The fact that $E(2)$ has a big diffeomorphism group implies that the result is actually $E(m+2)$.) In this construction, the sections add, and we obtain the plumbing claimed in the statement of the lemma. □

Call a K3-nucleus in which the fiber T and section C are both symplectic a *symplectic nucleus.*

Lemma 2.2. *Suppose that we are given a positive integer g and a symplectic simply connected 4-manifold X whose canonical class K_X is represented by an embedded symplectic surface of m disjoint components of genus $g_1, \ldots, g_m$. Suppose also that X contains a symplectic K3-nucleus N disjoint from this surface. Then there is a symplectic simply* *connected 4-manifold Y whose canonical class K_Y is represented by $m+1$ disjoint embedded surfaces of genus $g, g_1, \dots, g_m$. Furthermore, Y contains a symplectic K3-nucleus disjoint from these surfaces, and*

$$c_1^2(Y) = c_1^2(X) + g - 1, \quad \chi_h(Y) = \chi_h(X) + g.$$

Proof. First suppose that $g \geq 2$. Let $E(g)$ carry the symplectic structure given by Lemma 2.1, and form the symplectic sum of $E(g)$ and X by identifying the torus F given in Lemma 2.1 with a symplectic c-embedded torus T in the symplectically embedded nucleus of X. The existence of the sphere C which is transverse to T in the nucleus implies that the fiber sum $E(g)\#_{F=T}X$ is simply connected. It follows from [12] (see *e.g.* [3]) that the basic classes of $E(g)\#_{F=T}X$ all have the form $k + r[F]$ where k is a basic class of X, $|r| \leq g$, and $r \equiv g \pmod 2$. Since the class $r[F]$ is represented by r symplectic tori, it follows from [16] that the canonical class is given by $K_{E(g)\#_{F=T}X} = K_X + g[F]$.

By Lemma 2.1, $E(g)\#_{F=T}N \subset E(g)\#_{F=T}X$ contains a symplectic configuration C_g and a disjoint symplectic nucleus. (In case, $g = 2$, this follows directly without the use of the lemma.) The configuration C_g is taut in the sense of [2]. Let Y be the result of rationally blowing down C_g. Then Y is simply connected and minimal. According to Symington [14], Y inherits a symplectic structure, and by [2], the basic classes of Y have the form $\bar{k} \pm g[\bar{F}]$, where $\bar{k}$ and $[\bar{F}]$ denote the unique images of these classes in Y. Thus, $K_Y = \bar{K}_X + g[\bar{F}]$ (see [16]). The claims concerning c_1^2 and χ_h follow easily.

The canonical class of $E(g)\#_{F=T}X$ is represented by a disjoint union of symplectic surfaces

$$K_{E(g)\#_{F=T}X} \sim F_1 \cup \cdots \cup F_g \cup \Sigma_{g_1} \cup \cdots \cup \Sigma_{g_m}$$

where $K_X \sim \Sigma_{g_1} \cup \cdots \cup \Sigma_{g_m}$ is the symplectic surface in $X \setminus N$ given in the hypothesis, and $F_1, \dots, F_g$ are symplectic tori. Let the spheres in the configuration C_g be denoted by $U_0, U_1 \dots, U_{g-2}$ where U_0 is the sphere of self-intersection $-(g+2)$. Then for each j, we have $F_j \cdot U_0 = 1$ and $F_j \cdot U_k = 0,\ k > 0$. Also, $\Sigma_i \cdot U_k = 0$ for all i, k. Techniques of [11] show that we may assume that all the intersections $F_j \cap U_0$ are orthogonal with respect to the symplectic structure. The configuration C_g is blown down by replacing a regular neighborhood with a rational ball B_g with $\pi_1 = \mathbf{Z}_g$. The symplectic structure on B_g is induced from an embedding in a ruled surface $\mathbf{F}_{g-1}$: Let S_+ and S_- be the positive and negative sections and f a fiber of $\mathbf{F}_{g-1}$. Then $S_+ + f$ and S_- are represented by rational curves, and the complement of the configuration consisting of these two curves is diffeomorphic to B_g. According to [10], the symplectic structure on $\mathbf{F}_{g-1}$ is determined by the areas of S_+ and S_-. In [14], Symington shows that for an appropriate choice of these areas, we obtain a symplectic structure on B_g which gives the symplectic structure on the rational blowdown.

Note that $S_+ \cdot (S_+ + f + S_-) = g$; so $S_+ \cap B_g$ is a sphere with g holes. Again, we may assume that these intersections are orthogonal with respect to the symplectic form on $\mathbf{F}_{g-1}$. Thus $(F_1 \cup \cdots \cup F_g) \setminus C_g$ glues together with $S_+ \cap B_g$ to form a genus g symplectic surface in Y which represents $g[\bar{F}]$, and this completes the proof. □

We may now prove our theorem:

Proof of Theorem 1.1. First take $X_1 = E(3)$, and, for $g > 1$, let X_g be the result of rationally blowing down the configuration C_g in $E(g+2)$. These minimal symplectic simply connected 4-manifolds are discussed in [2]. The fact that the canonical class of X_g is represented by an embedded symplectic surface of genus g is proved exactly as in Lemma 2.2. Note that $c_1^2(X_g) = g-1$ and $\chi_h(X_g) = g+2$. Now using Lemma 2.2, construct the required manifold X inductively, starting with X_{g_1}. Then

$$c_1^2(X) = \sum_{i=1}^n m_i(g_i - 1), \quad \chi_h(X) = 2 + \sum_{i=1}^n m_i\, g_i$$

To see that X is minimal, compute its Seiberg-Witten invariant: It is clear that (up to sign) X_{g_1} has a single basic class, of self-intersection $g_1 - 1$. Since $g_1 \geq 1$, X_{g_1} is minimal. As in the proof of the above lemma, we see that X_{g_1,g_2} has basic classes of the form $\pm k_1 \pm k_2$ where $k_1^2 = g_1 - 1$, $k_2^2 = g_2 - 1$, and $k_1 \cdot k_2 = 0$. Continuing, one sees that the basic classes of the manifold X constructed in this theorem have the form

$$\sum_{i=1}^n \sum_{j=1}^{m_i} (\pm k_{i,j})$$

where $k_{i,j}^2 = g_i - 1$ and $k_{i,j} \cdot k_{i',j'} = 0$ unless $(i,j) = (i',j')$. It follows easily from the blowup formula [5] that, if X is not minimal, there are two basic classes of the form $A+E$, $A-E$ where E has square -1 and $A \cdot E = 0$. So $((A+E)-(A-E))^2 = -4$. But our formula for the basic classes of X shows that the square of the difference of any two basic classes has the form

$$(\sum_{i=1}^n \sum_{j=1}^{m_i} (\pm \eta_{i,j} k_{i,j}))^2 = \sum_{i=1}^n \sum_{j=1}^{m_i} \eta_{i,j}^2 (g_i - 1) \geq 0$$

where $\eta_{i,j} = 0$ or 2 (and $g_i \geq 1$). Thus X is minimal. □

References

[1] R. Fintushel and R. Stern, *Constructing lens space by surgery on knots*, Math. Z. **175** (1980), 33–51.

[2] R. Fintushel and R. Stern, *Rational blowdowns of smooth* 4*-manifolds*, Jour. Diff. Geom. **46** (1997), 181–235. alg-geom/9505018.

[3] R. Fintushel and R. Stern, *Knots, links, and* 4*-manifolds*, Invent. Math. **134** (1998), 363–400. dg-ga/9612014

[4] R. Fintushel and R. Stern, *Surfaces in 4-manifolds*, Math. Res. Letters **4** (1997), 907–914. dg-ga/9705009.

[5] R. Fintushel and R. Stern, *Immersed* 2*-spheres in* 4*-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995), 27–39.

[6] R. Gompf, *Nuclei of elliptic surfaces*, Topology **30** (1991), 479–511.

[7] R. Gompf, *A new construction of symplectic manifolds*, Ann. Math. **142** (1995), 527–595.

[8] E. Ionel and T. Parker, *Gromov-Witten invariants of symplectic sums*, preprint, math.SG/9806013.

[9] M. Marino, G. Moore, and G. Peradze, *Four-manifold geography and superconformal symmetry*, preprint. math.DG/9812042.

[10] D. McDuff, *The structure of rational and ruled symplectic 4-manifolds*, J. Amer. Math. Soc. **3** (1990), 679–712.

[11] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994), 405–429.

[12] J. Morgan, T. Mrowka and Z. Szabo, *Product formulas along T^3 for Seiberg-Witten invariants*, in preparation.

[13] U. Persson, *An introduction to the geography of surfaces of general type*, in 'Algebraic Geometry Bowdoin 1985', Proc. Symp. Pure Math. Vol. 46, 1987, pp.195–220.

[14] M. Symington, *Symplectic rational blowdowns*, Jour. Diff. Geom. **50** (1998), 505–518.

[15] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1** (1994), 809–822.

[16] C. Taubes, *More constraints on symplectic manifolds from Seiberg-Witten invariants*, Math. Res. Letters **2** (1995), 9–14.

[17] E. Witten, *Monopoles and four-manifolds*, Math. Res. Letters **1** (1994), 769–796.

E. Lansing, MI, USA
E-mail address: `ronfint@math.msu.edu`

Irvine, CA, USA
E-mail address: `rstern@math.uci.edu`