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Knotting of algebraic curves in complex surfaces

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1. Introduction

Theorem 1.1. *For any $d \geq 5$ there exist infinitely many smooth oriented closed surfaces $F \subset \mathbb{C}\mathrm{P}^2$ representing class $d \in H_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$, having* $\mathrm{genus}(\mathrm{F}) = \frac{1}{2}(\mathrm{d}-1)(\mathrm{d}-2)$ *and* $\pi_1(\mathbb{C}\mathrm{P}^2 \smallsetminus \mathrm{F}) \cong \mathbb{Z}/\mathrm{d}$*, such that the pairs* $(\mathbb{C}\mathrm{P}^2, \mathrm{F})$ *are pairwise smoothly non-equivalent. Moreover, d-fold cyclic coverings over* $\mathbb{C}\mathrm{P}^2$ *branched along F differ by their Seiberg-Witten invariants and thus are non-diffeomorphic.*

This theorem, which answers an old question (cf. [6], Problem 4.110), is proved in [2] for even $d \geq 6$. In this paper the proof for odd d and generalized Theorem 1.1 (see below Theorem 1.6) are added. Sections 2-3 and the Appendix reproduce the content of [2] whereas Section 5 extends the results from there.

Remark 1.1. Note that the surfaces that are constructed are not symplectic. Some speculation referring to Gromov's theorem suggests that any symplectic surface in $\mathbb{C}\mathrm{P}^2$ may be isotopic to an algebraic curve. As far as I know, at the moment it is proved only for degrees $d \leq 4$.

The knotting construction used to obtain surfaces F is a relative of the rim-surgery defined in [5]. An alternative way to achieve Theorem 1.1 is to use the tangle-surgery of Viro introduced in [3]. For technical reasons I prefer to use the rim-surgery in this paper, and give below an idea about the other approach just because it inspired this paper.

1.1. The idea that inspired my construction

Any kind of a surgery on a codimension two submanifold, F, in some fixed n-manifold X gives rise to some n-dimensional surgery on the double covering $Y \to X$ branched along F. Vice versa, considering a surgery on Y, one can try to perform it equivariantly with respect to the covering transformation, which results in some surgery on a pair (X, F). Sometimes X is preserved, and only F as an embedded submanifold is modified by this surgery. Such an ambient surgery on F in X will be called *the folding* of the corresponding surgery on Y.

For example, if Y is a complex surface defined over $\mathbb{R}$, and $X = Y/\mathrm{conj}$ is the quotient by the complex conjugation $\mathrm{conj} : Y \to Y$, then the projection $p : Y \to X$ is a double covering branched along $F = \mathrm{Fix}(\mathrm{conj})$ (the real locus of Y). Algebraic transformations (say, a blow-up, or a logarithmic transform) can be applied to Y in the real category. It turns out (at least in the examples known to the author) that the quotient $X = Y/\mathrm{conj}$ is not changed if a transformation is irreducible over $\mathbb{C}$, .i.e., if it does not contain a pair of conj-symmetric transformations localized outside the real part F.

Say, the folding of a blow-up at a real point of Y is a real blow-up of F, that is an ambient connected sum $(X, F)\#(S^4, \mathbb{R}\mathrm{P}^2)$, because $\mathbb{C}\mathrm{P}^2/\mathrm{conj} \cong \mathrm{S}^4$. Viro observed [3] that the folding of a logarithmic transform is a certain tangle-surgery on F. This yields "exotic knottings" of $F = \#_{10}\mathbb{R}\mathrm{P}^2$ in $S^4 = Y/\mathrm{conj}$, where $Y = E(1) = \mathbb{C}\mathrm{P}^2\#_9\overline{\mathbb{C}\mathrm{P}}^2$ is a rational elliptic surface, being modified by logarithmic transforms (which produce Dolgachev surfaces defined over $\mathbb{R}$).

The same construction applied to a K3 surface, $Y = E(2)$, instead of $E(1)$, gives "exotic knottings" of $F = \mathrm{Fix}(\mathrm{conj})$ in $X = Y/\mathrm{conj}$. For a suitable choice of the real structure in Y, the quotient X is diffeomorphic to $\mathbb{C}\mathrm{P}^2$ and F becomes a sextic in X, so the surgery gives examples for $d = 6$ in Theorem 1.1. Viro's tangle surgery can be applied, in general, along any null-framed annulus membrane on a surface in a four-manifold, which gives in the covering space a logarithmic transform. Suitable membranes on algebraic curves in $\mathbb{C}\mathrm{P}^2$ are described in what follows.

It turned out that the Fintushel-Stern's surgery on Y admits also a folding, i.e., can be made equivariantly, with the quotient X being preserved, provided the knot that we use is a double knot, i.e., $K\#K$. This folding is just what I call below "an annulus rim surgery".

1.2. An annulus rim-surgery

Our surgery, like the Viro tangle surgery, requires a suitable annulus membrane and produces a new surface via knotting an old one along such a membrane. By an annulus membrane for a smooth surface F in 4-manifold X we mean a smoothly embedded surface $M \subset X$, $M \cong S^1 \times I$, with $M \cap F = \partial M$ and such that M comes to F normally along ∂M. Assume that such a membrane has framing 0, or equivalently, admits a diffeomorphism of its regular neighborhood $\phi : U \to S^1 \times D^3$ mapping $U \cap F$ onto $S^1 \times f$, where $f = I \amalg I \subset D^3$ is a disjoint union of two segments, which are unknotted and unlinked in D^3, that is to say that a union of f with a pair of arcs on a sphere ∂D^3 bounds a trivially embedded band, $b \subset D^3$, $b \cong I \times I$, so that $f = I \times (\partial I) \subset b$ (see Figure 1). The annulus M can be viewed as $S^1 \times \{\frac{1}{2}\} \times I$ in $S^1 \times b \subset S^1 \times D^3 \cong U$.

If X and F are oriented, then f inherits an orientation as a transverse intersection, $f = F \pitchfork D^3$, and we may choose a band b so that the orientation of f is induced from some orientation of b. It is convenient to view $f = I \amalg I$ as is shown on Figure 1, so that the segments of f are parallel and oppositely oriented, with b being a thin band between them. Such a presentation is always possible if we allow a modification of ϕ, since one of the segments of f may be turned around by a diffeomorphism of $D^3 \to D^3$ leaving the other segment fixed.

Given a knot $K \subset S^3$, we construct a new smooth surface, $F_{K,\phi}$, obtained from F by tying a pair of segments $I \amalg I$ along K inside D^3, as is shown on Figure 1. More precisely, we consider a band $b_K \subset D^3$ obtained from b by knotting along K and let f_K denote the pair of arcs bounding b_K inside D^3. We assume that the framing of b_K is chosen the same as the framing of b, or equivalently, that the inclusion homomorphisms from $H_1(\partial D^3 \smallsetminus (\partial f)) = H_1(\partial D^3 \smallsetminus (\partial f_K))$ to $H_1(B^3 \smallsetminus f)$ and to $H_1(B^3 \smallsetminus f_K)$ have the same kernel. Then $F_{K,\phi}$ is obtained from F by replacing $S^1 \times f \subset S^1 \times D^3 \cong U$ with $S^1 \times f_K$. It is obvious that $F_{K,\phi}$ is homeomorphic to F and realizes the same homology class in $H_2(X)$.

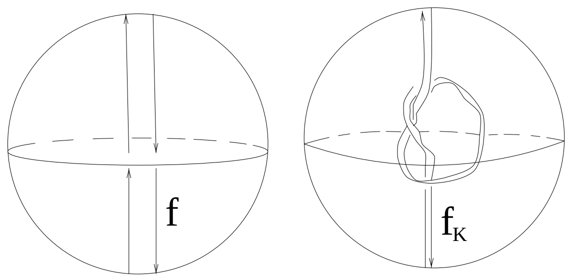

FIGURE 1. Knotting of a band b_K

The above construction is called in what follows *an annulus rim-surgery*, since it looks like the rim-surgery of Fintushel and Stern [5], except that we tie two strands simultaneously, rather then one. Recall that the usual rim-surgery is applied in [5] to surfaces $F \subset X$ which are *primitively embedded*, that is $\pi_1(X \smallsetminus F) = 0$, which is not the case for the algebraic curves in $\mathbb{C}\mathrm{P}^2$ of degree > 1. The primitivity condition is required to preserve the fundamental group of $X \smallsetminus F$ throughout the knotting. An annulus rim-surgery may preserve *a non-trivial* group $\pi_1(X \smallsetminus F)$, if we require commutativity of $\pi_1(X \smallsetminus (F \cup M))$, instead of primitivity of the embedding.

Proposition 1.2. *Assume that X is a simply connected closed 4-manifold, $F \subset X$ is an oriented closed surface with an annulus-membrane M of index 0, $\phi : U \to S^1 \times D^3$ is a trivialization like described above and $K \subset S^3$ is any knot. Assume furthermore that $F \smallsetminus \partial M$ is connected and the group $\pi_1(X \smallsetminus (F \cup M))$ is abelian. Then the group $\pi_1(X \smallsetminus F_{K,\phi})$ is cyclic and isomorphic to $\pi_1(X \smallsetminus F)$.*

1.3. Maximal nest curves

To prove Theorem 1.1, we apply an annulus rim-surgery inside $X = \mathbb{C}\mathrm{P}^2$ letting $F = \mathbb{C}A$ be the complex point set of a suitable non-singular real algebraic curve, containing an annulus, M, among the connected components of $\mathbb{R}\mathrm{P}^2 \smallsetminus \mathbb{R}\mathrm{A}$, where $\mathbb{R}A = \mathbb{C}A \cap \mathbb{R}\mathrm{P}^2$ is the real locus of the curve.

One may take, for instance, a real algebraic curve $\mathbb{C}A$ of degree d, with *a maximal nest real scheme*. Such a curve for $d = 2k$ is constructed by a small real perturbation of a union of k real conics, whose real parts (ellipses) are ordered by inclusion in $\mathbb{R}\mathrm{P}^2$. For

$d = 2k+1$, we add to such conics a real line not intersecting the conics in $\mathbb{R}P^2$ and then perturb the unions. The real part, $\mathbb{R}A$, of our non-singular curve contains k components, $O_1, \dots, O_k$, called ovals (just deformed ellipses). We order the ovals so that O_i lies inside O_{i+1} and denote by R_i the annulus-component of $\mathbb{R}P^2 \setminus \mathbb{R}A$ between O_i and O_{i+1} for $i = 1, \dots, k-1$. R_0 is a topological disk bounded from outside by O_1, and R_k is the component bounded from inside by O_k.

The closures, $\mathrm{Cl}(R_i)$, for $i = 1, \dots, k-1$ are obviously 0-framed annulus-membranes on $\mathbb{C}A$. For simplicity, let us choose $M = \mathrm{Cl}(R_1)$.

Proposition 1.3. *The assumptions of Proposition 1.2 hold if we put $X = \mathbb{C}P^2$, let $F = \mathbb{C}A$ be a maximal nest real algebraic curve of degree $d \geq 5$ and choose $M = \mathrm{Cl}(R_1)$.*

1.4. Proof of Theorem 1.1 for even d

Assuming that the class $[F] \in H_2(X;\mathbb{Z}/2)$ vanishes, one can consider a double covering $p : Y \to X$ branched along F; such a covering is unique if we require in addition that $H_1(X;\mathbb{Z}/2) = 0$. Similarly, we consider the double coverings $Y(K,\phi) \to X$ branched along $F_{K,\phi}$. To prove non-equivalence of pairs $(\mathbb{C}P^2, F_{K,\phi})$ for some family of knots K, it is enough to show that $Y(K,\phi)$ are not pairwise diffeomorphic. It follows from that $Y(K,\phi)$ is diffeomorphic to the 4-manifolds $Y_{K\#K}$ obtained from Y by a surgery introduced in [4] (Let us call it *FS-surgery*).

Proposition 1.4. *The above $Y(K,\phi)$ is diffeomorphic to a 4-manifold obtained from Y by the FS-surgery along the torus $T = p^{-1}(M)$ via the knot $K\#K \subset S^3$.*

To distinguish the diffeomorphism types of $Y_{K\#K}$ one can use the formula of Fintushel and Stern [4] for SW-invariants of a 4-manifold Y after FS-surgery along a torus $T \subset Y$. Recall that this formula can be applied if the SW-invariants of Y are well-defined and a torus T, realizing a non-trivial class $[T] \in H_2(Y)$, is *c-embedded* (the latter means that T lies as a non-singular fiber in a cusp-neighborhood in Y, cf. [4]). Being an algebraic surface of genus ≥ 1, the double plane Y has well-defined SW-invariants. The conditions on T are also satisfied.

Proposition 1.5. *Assume that X, F and M are like in Proposition 1.2, $[F] \in H_2(X;\mathbb{Z}/2)$ vanishes and $p : Y \to X$ is like above. Then the torus $T = p^{-1}(M)$ is primitively embedded in Y and therefore $[T] \in H_2(Y)$ is an infinite order class. If, moreover, X, F and M are chosen like in Proposition 1.3, then $T \subset Y$ is c-embedded.*

Recall that the product formula [4]

$$SW_{Y_K} = SW_Y \cdot \Delta_K(t), \quad \text{where } t = \exp(2[T])$$

expresses the Seiberg-Witten invariants (combined in a single polynomial) of the manifold Y_K, obtained by an FS-surgery, in terms of the Seiberg-Witten invariants of Y and the Alexander polynomial, $\Delta_K(t)$, of K.

This formula implies that the basic classes of Y_K can be expressed as $\pm\beta + 2n[T]$, where $\pm\beta \in H_2(Y)$ are the basic classes of Y and $|n| \leq \deg(\Delta_K(t))$, are the degrees of the non-vanishing monomials in $\Delta_K(t)$. So, if $[T]$ has infinite order, then the manifolds $Y(K,\phi) \cong Y_{K\#K}$ differ from each other by their SW-invariants, and moreover, by the numbers of their basic classes, for an infinite family of knots K, since the number of the basic classes is determined by the number of the terms in $\Delta_{K\#K} = (\Delta_K)^2$ (one can take any family of knots with Alexander polynomials of distinct degrees). □

1.5. A generalization

More generally, one can produce "fake algebraic curves" under the following conditions.

Theorem 1.6. *Assume that F is a non-singular connected curve in a simply connected complex surface X, which admits a deformation degenerating F into an irreducible curve $F_0 \subset X$, with a singularity of the type X_9, such that the fundamental group $\pi_1(X \smallsetminus F_0)$ is abelian. Then there exists an infinite family of surfaces $F_{K,\phi} \subset X$ homeomorphic to F and realizing the same homology class as F, having the same fundamental group of the complement, but with the smoothly non-equivalent pairs $(X, F_{K,\phi})$.*

Recall that X_9-singularity is a point where 4 non-singular branches meet pairwise transversally. Nori's theorem [7] gives conditions under which $\pi_1(X \smallsetminus F_0)$ must be abelian. For instance, it is so if A_0 has no other singularities except X_9 and $A \circ A > 16$.

Remark 1.2. The claim of Theorem 1.6 holds also if F_0 has a more complicated then X_9 singularity, provided the group $\pi_1(X \smallsetminus F_0)$ is abelian.

2. Commutativity of the fundamental group throughout the knotting

Lemma 2.1. *The assumptions of Proposition 1.2 imply that $\pi_1(X \smallsetminus (F \cup M)) = \pi_1(X \smallsetminus F)$ is cyclic with a generator presented by a loop around F.*

Proof. The Alexander duality in X combined with the exact cohomology sequence of a pair $(X, F \cup M)$ gives

$$H_1(X \smallsetminus (F \cup M)) \cong H^3(X, F \cup M) = H^2(F \cup M)/i^* H^2(X)$$

where $i : F \cup M \to X$ is the inclusion map. If F is oriented and $F \smallsetminus \partial M$ is connected, then the Mayer-Vietoris Theorem yields $H^2(F \cup M) \cong H^2(F) \cong \mathbb{Z}$, and thus $H_1(X \smallsetminus (F \cup M)) \cong H_1(X \smallsetminus F)$ is cyclic with a generator presented by a loop around F. The same property holds for the fundamental groups of $X \smallsetminus (F \cup M)$ and $X \smallsetminus F$, since they are abelian by the assumption of Proposition 1.2. □

Proof of Proposition 1.2. Put $X_0 = \mathrm{Cl}(X \smallsetminus U)$. Then $\partial X_0 = \partial U \cong S^1 \times S^2$ and $\partial U \smallsetminus F$ is a deformational retract of $U \smallsetminus (F \cup M)$, so

$$\pi_1(X_0 \smallsetminus F) = \pi_1(X \smallsetminus (F \cup M))$$

Since this group is cyclic and is generated by a loop around F, the inclusion homomorphism $h : \pi_1(\partial U \smallsetminus F) \to \pi_1(X_0 \smallsetminus F)$ is epimorphic and thus $\pi_1(X_0 \smallsetminus F) = \pi_1(\partial U \smallsetminus F)/k$, where k is the kernel of h.

Applying the Van Kampen theorem to the triad $(X_0\smallsetminus F, U\smallsetminus F_{K,\phi}, \partial U\smallsetminus F)$, we conclude that $\pi_1(X\smallsetminus F_{K,\phi})\cong\pi_1(U\smallsetminus F_{K,\phi})/j(k)$, where $j:\pi_1(\partial U\smallsetminus F)\to\pi_1(U\smallsetminus F_{K,\phi})$ is the inclusion homomorphism. Furthermore, in the splitting

$$\pi_1(U\smallsetminus F_{K,\phi})\cong\pi_1(S^1\times(D^3\smallsetminus f_K))\cong\mathbb{Z}\times\pi_1(D^3\smallsetminus f_K)$$

factorization by $j(k)$ kills the first factor $\mathbb{Z}$ and adds some relations to $\pi_1(D^3\smallsetminus f_K)$, one of which effects to $\pi_1(D^3\smallsetminus f_K)$ as if we attach a 2-cell along a loop, m_b, going once around the band b_K (to see it, note that factorization by k leaves only one generator of $\pi_1(\partial D^3\smallsetminus f_K)=\pi_1(S^2\smallsetminus\{4\text{pts}\})$). Attaching such a 2-cell effects to π_1 as connecting together a pair of the endpoints of f_K, which transforms f_K into an arc (see Figure 2). This arc is unknotted and thus factorization by $j(k)$ makes $\pi_1(D^3\smallsetminus f_K)$ cyclic and leaves $\pi_1(X\smallsetminus F_{K,\phi})$ isomorphic to $\pi_1(X_0\smallsetminus F)\cong\pi_1(X\smallsetminus(F\cup M))\cong\pi_1(X\smallsetminus F)$. □

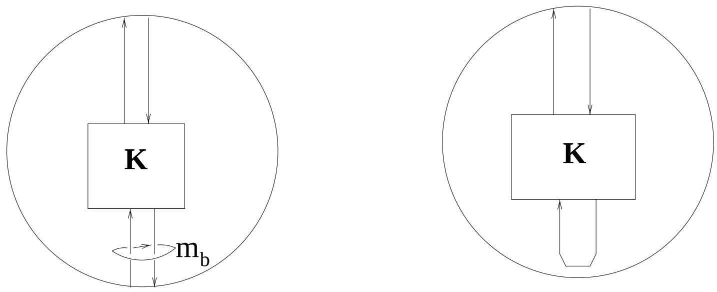

FIGURE 2. Gluing a 2-cell along m_b effects as transforming f_K into an unknotted arc

Proof of Proposition 1.3. All the assumptions of Proposition 1.2 except the last two are obviously satisfied. It is well known that $\mathbb{C}A\smallsetminus\mathbb{R}A$ splits for a maximal nest curve $\mathbb{C}A$ into a pair of connected components permuted by the complex conjugation, and thus, $\mathbb{C}A\smallsetminus\partial M$ is connected, provided $\partial M\subsetneq\mathbb{R}A$, which is the case for $d\geq 5$. So, it is only left to check that the group $\pi_1(\mathbb{C}\mathrm{P}^2\smallsetminus(\mathbb{C}\mathrm{A}\cup\mathrm{M}))$ is abelian.

There are several ways to check it. For instance, one can refer to my old work [1] containing computation of the homotopy type of $\mathbb{C}\mathrm{P}^2\smallsetminus(\mathbb{C}\mathrm{A}\cup\mathbb{R}\mathrm{P}^2)$ and, in particular, of its fundamental group (see also §4 in [3]). This computation concerns a real curve $\mathbb{C}A\subset\mathbb{C}\mathrm{P}^2$ if it is an *L-curve*, i.e., $\mathbb{C}A$ can be obtained by a non-singular perturbation from a curve $\mathbb{C}A_0=\mathbb{C}L_1\cup\dots\mathbb{C}L_d$ splitting into d real lines, $\mathbb{C}L_i$, in a generic position. The maximal nest curves, $\mathbb{C}A\subset\mathbb{C}\mathrm{P}^2$, can be easily constructed as L-curves, and the result of [1] gives a presentation $\pi=\pi_1(\mathbb{C}\mathrm{P}^2\smallsetminus(\mathbb{C}\mathrm{A}\cup\mathbb{R}\mathrm{P}^2))=\langle\mathrm{a},\mathrm{b}\,|\,\mathrm{a}^\mathrm{d}\mathrm{b}^\mathrm{d}=1\rangle$, where a, b are represented by loops around the two connected components of $\mathbb{C}A\smallsetminus\mathbb{R}A$. More specifically, a basis point and these loops can be taken on the conic $C=\{x^2+y^2+z^2=0\}\subset\mathbb{C}\mathrm{P}^2$, which have the real point set empty. The group $\pi_1(\mathbb{C}\mathrm{P}^2\smallsetminus(\mathbb{C}\mathrm{A}\cup\mathrm{M}))$ is obtained from π by adding the relations corresponding to puncturing the components R_i, $0\leq i\leq k$,

$i \neq 1$, of $\mathbb{RP}^2 \smallsetminus \mathbb{R}\mathrm{A}$ (here $d = 2k$ or $d = 2k + 1$). Such a relation (as we puncture R_i) is $a^{d-i}b^i = b^{d-i}a^i = 1$, see [1], or §4 in [3]. A pair of the relations for $i = 2$ and $i = 3$ implies that $a = b$.

The arguments from [1] and [3] relevant to the above calculation are briefly summarized in the Appendix. □

Remark 2.1. It follows from the proof above that $\pi_1(\mathbb{CP}^2 \smallsetminus (\mathbb{C}\mathrm{A} \cup \mathrm{M}))$ is not abelian and $\mathbb{C}A \smallsetminus \partial M$ is not connected for a maximal nest quartic, $\mathbb{C}A$.

3. The double surgery in the double covering

Proof of Proposition 1.4. The proof is based on the following two observations. First, we notice that $Y(K, \phi)$ is obtained from Y by a pair of FS-surgeries along the tori parallel to T, then we notice that such pair of surgeries is equivalent to a single FS-surgery along T. The both observations are corollaries of Lemma 2.1 in [5], so, I have to recall first the construction from [4], [5].

An FS-surgery [4] on a 4-manifold X along a torus $T \subset X$, with the self-intersection $T \circ T = 0$, via a knot $K \subset S^3$ is defined as a *fiber sum* $X\#_{T=S^1\times m_K} S^1 \times M_K$, that is an amalgamated connected sum of X and $S^1 \times M_K$ along the tori T and $S^1 \times m_K \subset S^1 \times M_K$. Here M_K is a 3-manifold obtained by the 0-surgery along K in S^3, and m_K denotes a meridian of K (which may be seen both in S^3 and in M_K). Such a fiber sum operation can be viewed as a direct product of S^1 and the corresponding 3-dimensional operation, which I call *S^1-fiber sum*.

More precisely, S^1-fiber sum $X\#_{K=L}Y$ of oriented 3-manifolds X and Y along oriented framed knots $K \subset X$ and $L \subset Y$ is the manifold obtained by gluing the complements $\mathrm{Cl}(\mathrm{X} \smallsetminus \mathrm{N(K)})$ and $\mathrm{Cl}(\mathrm{Y} \smallsetminus \mathrm{N(L)})$ of tubular neighborhoods, $N(K)$, $N(L)$, of K and L via a diffeomorphism $f : \partial N(K) \to \partial N(L)$ which identifies the longitudes of K with the longitudes of L preserving their orientations, and the meridians of K with the meridians of L reversing the orientations. As it is shown in Lemma 2.1 of [5], tying a knot K in an arc in D^3 can be interpreted as a fiber sum $D^3\#_{m=m_K}M_K$, where m is a meridian around this arc. The meridians m and m_K are endowed here with the 0-framings (0-framing of a meridian makes sense as a meridian lies in a small 3-disc). To understand this observation, it is useful to view an S^1-fiber sum with M_K as surgering a tubular neighborhood, $N(m)$, of m and replacing it by the complement, $S^3 \smallsetminus N(K)$ of a tubular neighborhood, $N(K)$, of K, so that the longitudes of m are glued to the meridians of K and the meridians of m to the longitudes of K. The framing of an arc in D^3 is preserved under such a fiber sum, so tying a knot in the band $b \subset D^3$ is equivalent to taking an S^1-fiber sum with M_K along a meridian m_b around b.

The double covering over D^3 branched along f is a solid torus, $N \cong S^1 \times D^2$, and the pull back of m_b splits into a pair of circles, $m_1, m_2 \subset N$, parallel to $m = S^1 \times \{0\}$. Therefore, $Y(K, \phi)$ is obtained from Y by performing FS-surgery twice, along the tori

$$T_i = S^1 \times m_i \subset p^{-1}(U) \cong S^1 \times N, \quad i = 1, 2$$

The following Lemma implies that this gives the same result as a single FS-surgery along $T = p^{-1}(M)$ via the knot $K\#K$. □

Lemma 3.1. *For any pair of knots, K_1, K_2, the manifold*

$$M_{K_1}\#_{m_{K_1}=m_1}N\#_{m_2=m_{K_2}}M_{K_2}$$

obtained by taking an S^1-fiber sum twice, is diffeomorphic to $N\#_{m=m_K}M_K$, for $K = K_1\#K_2$, via a diffeomorphism identical on ∂N.

Proof. A solid torus N can be viewed as the complement $N = S^3 - N'$ of an open tubular neighborhood N' of an unknot, so that m, m_1, m_2 represent meridians of this unknot. Taking a fiber sum of S^3 with M_{K_i} along $m_i = m_{K_i}$ is equivalent to knotting N' in S^3 via K_i. So, performing S^1-fiber sum twice, along m_1 and m_2, we obtain the same result as after taking fiber sum along m once, via $K = K_1\#K_2$. □

Remark 3.1. The above additivity property can be equivalently stated as

$$M_{K_1}\#_{m_{K_1}=m_{K_2}}M_{K_2} \cong M_{K_1\#K_2}$$

Proof of Proposition 1.5. Lemma 2.1 implies that, in the assumptions of Proposition 1.2, $\pi_1(Y \smallsetminus (F \cup T))$ is a cyclic group with a generator represented by a loop around F. Thus, $\pi_1(Y \smallsetminus T) = 0$ and, by the Alexander duality, $H_3(Y, T) = H^1(Y \smallsetminus T) = 0$, which implies that $[T] \in H_2(Y)$ has infinite order.

To check that T is c-embedded it is enough to observe that there exists a pair of vanishing cycles on T, or more precisely, a pair of D^2-membranes, $D_1, D_2 \subset Y$, on T, having (-1)-framing and intersecting at a unique point $x \in T$, so that $[\partial D_1], [\partial D_2]$ form a basis of $H_1(T)$. In the setting of Proposition 1.3, $Y \to \mathbb{C}\mathrm{P}^2$ is a double covering branched along a maximal nest curve $\mathbb{C}A$ and T is a connected component of the real part of Y (with respect to a certain real structure on Y lifted from $\mathbb{C}\mathrm{P}^2$). Two nodal degenerations of $\mathbb{C}A$ shown on the top part of Figure 3 give nodal degenerations of the double covering Y.

In the first of the degenerations of $\mathbb{C}A$, a node appears as an oval O_1 is collapsed into a point. In the second degeneration a crossing-like node can be seen as the fusion point of the ovals O_1 and O_2. Existence of such degenerations for our explicitly constructed curve $\mathbb{C}A$ is known and trivial. Another simple observation (which is obvious for quartics and thus follows for any maximal nest curve of a higher degree) is that our pair of nodal degenerations can be united into one cuspidal degeneration. This means in particular that the two vanishing cycles in Y intersect transversally at a single point.

Furthermore, our complex vanishing cycles in Y can be chosen conj-invariant. Being a (-2)-sphere, each of such complex cycles is divided by its real pair into a pair of (-1)-discs. Choosing one disc from each pair, we obtain D_1 and D_2 that we need.

It is easy to view these (-2)-spheres and the (-1)-disks explicitly. First, note that R_0 is a (-1)-membrane on $\mathbb{C}A$ and $p^{-1}(R_0)$ is the first of the conj-symmetric vanishing cycles. The (-1)-disk D_1 is any of its halves. Furthermore, there is another (-1)-disk membrane, Q on $\mathbb{C}A$ corresponding to the second nodal degeneration. It can be chosen conj-invariant and then is split by $Q \cap \mathbb{R}\mathrm{P}^2$ into semi-discs $Q = Q_1 \cup Q_2$ permuted by conj. Q_i is bounded by the arcs $Q \cap \mathbb{R}\mathrm{P}^2$ and $Q_i \cap \mathbb{C}A$. The disk D_2 is any of the discs $p^{-1}(Q_i)$. □

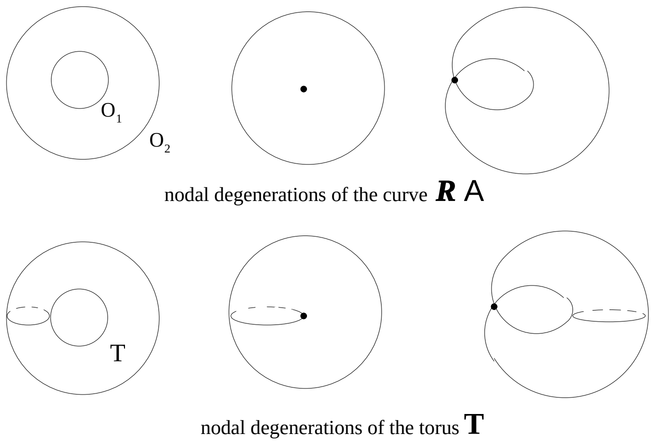

FIGURE 3. Nodal degenerations of $\mathbb{R}A$ providing (-1)-framed D^2-membranes on $T\ f_K$ into an unknotted arc

4. The case of d-fold branched covering

Consider as before a maximal nest curve, $\mathbb{C}A \subset \mathbb{C}\mathrm{P}^2$, of degree $d \geq 2$, and $\mathbb{C}A_{K,\phi}$ obtained from $\mathbb{C}A$ via an annulus rim-surgery along R_1, but now let us denote by $p : Y \to \mathbb{C}\mathrm{P}^2$ and $Y(K,\phi) \to \mathbb{C}\mathrm{P}^2$ the *d-fold coverings* branched along $\mathbb{C}A$ and $\mathbb{C}A_{K,\phi}$ respectively. Consider a d-fold covering $N \to D^3$ branched along f. The pull-back of m_b consists of d circles, $m_1, \dots, m_d$, which are cyclically ordered. Using a homeomorphism $(D^3, f) \cong (D^2 \times [0,1], \{z_1, z_2\} \times [0,1])$, where $\{z_1, z_2\} \subset \mathrm{Int}(\mathrm{D}^2)$, we present N as $F \times [0,1]$, where F is a sphere with d holes. The circles m_i go around these holes. An annulus rim-surgery in $\mathbb{C}\mathrm{P}^2$ along $m_b \times S^1 \subset D^3 \times S^1$, is covered by d copies of FS-surgery along the tori $T_i = m_i \times S^1 \subset N \times S^1$.

The following observation implies that the Fintushel-Stern formula for Seiberg-Witten invariants can be applied in this setting.

Proposition 4.1. *Each of the tori T_i is primitively c-embedded in the complement of the others.*

Proof. A pair of (-1)-disc membranes, D_1^i, D_2^i, on each of T_i is constructed like in the proof of Proposition 1.5. Namely, $p^{-1}(R_0)$ consists of d disks which yield the disks D_1^i, that are glued along $\{\mathrm{pt}\} \times S^1 \subset m_i \times S^1$.

Furthermore, $p^{-1}(Q_1)$ splits also into d disks, $Q_1^1, \dots, Q_1^d$. Let us choose their orientations induced from a fixed orientation of Q_1 and cyclically order in accord with the ordering of T_i, then the unions $Q_1^i \cup (-Q_1^{i+1})$ provide the required discs D_2^i, which are glued along $m_i \times \{\mathrm{pt}\}$. More precisely, D_1^i are the parts of the components of $p^{-1}(R_0)$ bounded by the intersections of the components with the tori T_i, whereas D_1^i are obtained from $Q_1^i \cup (-Q_1^{i+1})$ by a small shift making them membranes on T_i. □

Next, we observe that there exists only one linear dependence relation between the classes $[T_i] \in H_2(Y)$.

Proposition 4.2. *The inclusion map $H_2(\bigcup_i T_i) \to H_2(Y)$ has kernel $\mathbb{Z}$ generated by the relation $\Sigma_{i=1}^d [T_i] = 0$. Here T_i are oriented uniformly in accord with some fixed orientation of $m_b \times S^1$.*

Proof. It is enough to show that $\pi_1(Y \smallsetminus (N \times S^1)) = 0$, since it implies that $H_3(Y, N \times S^1) \cong H^1(Y \smallsetminus (N \times S^1)) = 0$ and thus the inclusion map $H_2(N \times S^1) \to H_2(Y)$ is monomorphic. The first inclusion map in the composition $H_2(\bigcup_i T_i) \to H_2(N \times S^1) \to H_2(Y)$ that we analyze, is just $H_1(\partial F) \otimes H_1(S^1) \to H_1(F) \otimes H_1(S^1)$, and has kernel $H_2(F, \partial F) \otimes H_1(S^1) \cong \mathbb{Z}$, as stated in the Proposition.

Now note that $p^{-1}(R_1)$ is a deformational retract (spine) of $N \times S^1$, so it is enough to check the triviality of $\pi_1(Y \smallsetminus (p^{-1}(R_1))$. This triviality follows from that $\pi_1(\mathbb{C}\mathrm{P}^2 \smallsetminus (\mathbb{C}\mathrm{A} \cup \mathrm{R}_1))$ is $\mathbb{Z}/d$, with a generator represented by a loop around $\mathbb{C}A$ (say, by the computation in [1] reproduced in the Appendix), and thus $\pi_1(Y \smallsetminus p^{-1}(\mathbb{C}A \cup R_1)) = 0$. □

Proposition 4.2 together with the Fintushel-Stern formula [4] guarantees that the Seiberg-Witten invariants of $Y(K, \phi)$ are distinct for some sequence of knots K with increasing degrees of $\Delta_K(t)$.

Proof of Theorem 1.6 The case of a primitive class $[F] \in H_2(X)$ is considered in [5]. More precisely, the assumptions in Theorem 1.1 in [5] are satisfied because our condition on the fundamental group yields that $\pi_1(X \smallsetminus F)$ is abelian and thus trivial, existence of an irreducible deformation of F implies that $F \circ F \geq 0$, and X_9-degeneration guarantees that F is not a rational curve.

If $[F]$ is divisible by $d \geq 2$, then we consider a d-fold covering, $p : Y \to X$, branched along F and perform an annulus rim-surgery on F along a membrane M defined as follows. Consider a local topological model of the singularity X_9, defined in $\mathbb{C}^2$ by the equation $(x^2+y^2)(x^2+2y^2) = 0$, and a model of its perturbation, $(x^2+y^2-4\varepsilon)(x^2+2y^2-\varepsilon) = \delta$, where $\varepsilon, \delta \in \mathbb{R}$, $0 \ll \delta \ll \varepsilon \ll 1$. The real locus of a perturbed singularity contains a pair of ovals which bound together in $\mathbb{R}^2$ an annulus that we take as M.

The assumptions of Theorem 1.6 imply those of Proposition 1.2. Namely, irreducibility of F_0 implies that $F \smallsetminus \partial M$ is connected and commutativity of $\pi_1(X \smallsetminus F_0)$ implies commutativity of $\pi_1(X \smallsetminus (F \cup M))$ via Van Kampen theorem. Moreover, the singularity X_9 provides the topological picture that was used in the above proof of Theorem 1.1, in the case of d-fold covering. Namely, X_9 yields the both (-1)-disk membranes that were used to show that the Fintushel and Stern formula can be applied to Y. □

Remark 4.1. Note that to apply the formula [5] it is not required that $b_2^+(Y) > 1$. Nevertheless, it is so, because $b_2^+(Y) \geq d$, which can be proved by observing d linearly independent pairwise orthogonal classes in $H_2(Y)$, having non-negative squares. One of these classes is $[F]$, and the other $(d-1)$ come from $p^{-1}(M)$, due to Proposition 4.2 (each of these $(d-1)$ classes has self-intersection 0).

5. Appendix: The topology of $\mathbb{C}\mathrm{P}^2 \setminus (\mathbb{R}\mathrm{P}^2 \cup \mathbb{C}\mathrm{A})$ for L-curves $\mathbb{C}A$

Let $\mathbb{C}A_0 = \mathbb{C}L_1 \cup \dots \cup \mathbb{C}L_d \subset \mathbb{C}\mathrm{P}^2$ denote the complex point set of a real curve of degree d splitting into d lines, $\mathbb{C}L_i$. Put $\widetilde{V} = C \cap \mathbb{C}A_0$, where C is the conic from the proof of Proposition 1.3. Our first observation is that $C \setminus \widetilde{V}$ is a deformational retract of $\mathbb{C}\mathrm{P}^2 \setminus (\mathbb{R}\mathrm{P}^2 \cup \mathbb{C}\mathrm{A}_0)$, and moreover, the latter complement is homeomorphic to $(C \setminus \widetilde{V}) \times \mathrm{Int}(\mathrm{D}^2)$. To see it, it suffices to note that $\mathbb{C}\mathrm{P}^2 \setminus \mathbb{R}\mathrm{P}^2$ is fibered over C with a 2-disc fiber, each fiber being a real semi-line, that is a connected component of $\mathbb{C}L \setminus \mathbb{R}\,L$ for some real line $\mathbb{C}L \subset \mathbb{C}\mathrm{P}^2$, where $\mathbb{R}L = \mathbb{C}L \cap \mathbb{R}\,\mathrm{P}^2$. This fibering maps a semi-line into its intersection point with C.

It is convenient to view the quotient C/conj of the conic C by the complex conjugation as the projective plane, $\widehat{\mathbb{R}\mathrm{P}}^2$, dual to $\mathbb{R}\mathrm{P}^2 \subset \mathbb{C}\mathrm{P}^2$, since each real line, $\mathbb{C}L$, intersects C in a pair of conjugated points. If we let $V = \{l_1, \dots, l_d\} \subset \widehat{\mathbb{R}\mathrm{P}}^2$ denote the set of points l_i dual to the lines $\mathbb{R}L_i \subset \mathbb{R}\mathrm{P}^2$, then $\widetilde{V} = q^{-1}(V)$, where $q : C \to C/\mathrm{conj}$ is the quotient map.

The information about a perturbation of $\mathbb{C}A_0$ is encoded in a *genetic graph of a perturbation*, $\Gamma \subset \widehat{\mathbb{R}\mathrm{P}}^2$. The graph Γ is a complete graph with the vertex set V, whose edges are line segments. Note that there exist two topologically distinct perturbations of a real node of $\mathbb{R}A_0$ at $p_{ij} = \mathbb{R}L_i \cap \mathbb{R}L_j$, as well as there exist two line segments in $\widehat{\mathbb{R}\mathrm{P}}^2$ connecting the vertices $l_i, l_j \in V$. Let $\mathbb{R}A$ denotes a real curve obtained from $\mathbb{R}A_0$ by a sufficiently small perturbation. Then the edge of Γ connecting l_i and l_j contains the points dual to those lines passing through $p_{i,j}$ which do not intersect $\mathbb{R}A$ locally, in a small neighborhood of $p_{i,j}$.

The complement $\mathbb{C}\mathrm{P}^2 \setminus (\mathbb{C}\mathrm{A} \cup \mathbb{R}\mathrm{P}^2)$ turns out to be homotopy equivalent to a 2-complex obtained from $C \setminus \widetilde{V}$ by adding 2-cells glued along a figure-eight shaped loops along the edges of $\widetilde{\Gamma} = q^{-1}(\Gamma) \subset C$. Such 2-cells identify pairwise certain generators of $\pi_1(C \setminus \widetilde{V})$ "along the edges" of $\widetilde{\Gamma}$ (cf. [3] for details). This easily implies that the group $\pi_1(\mathbb{C}\mathrm{P}^2 \setminus (\mathbb{C}\mathrm{A} \cup \mathbb{R}\mathrm{P}^2))$ is generated by a pair of elements, a and b, represented by a pair of loops in $C \setminus \widetilde{V}$ around a pair of conjugated vertices of $\widetilde{V}$.

For example, for a maximal nest curve, the graph Γ is contained in an affine part of $\widehat{\mathbb{R}\mathrm{P}}^2$, i.e., has no common points with some line in $\widehat{\mathbb{R}\mathrm{P}}^2$, namely, with a line dual to a point inside the inner oval of the nest. Therefore, the graph $\widetilde{\Gamma}$ splits into two connected components separated by a big circle in C. A loop around any vertex of $\widetilde{V}$ from one of these components represents a, and a loop around a vertex from the other component represents b. It is trivial to observe also the relation $a^d b^d = 1$ (which is indeed a unique relation in the case of maximal nest curves).

As we puncture $\mathbb{R}\mathrm{P}^2$ at a point $x \in \mathbb{R}\mathrm{P}^2 \setminus \mathbb{R}\mathrm{A}_0$, we attach a 2-cell to $C \setminus \widetilde{V}$ along the big circle $S_x \subset C$ dual to x. If x moves across a line $\mathbb{R}L_i$, then S_x moves across the pair of points $q^{-1}(l_i)$. Since a small perturbation and puncturing are located at distinct points of $\mathbb{C}\mathrm{P}^2$ and can be done independently, it is not difficult to see that if we choose $x \in R_i$ (in the case of a maximal nest curve $\mathbb{C}A$), then the big circle S_x cuts C into the hemispheres, one of which

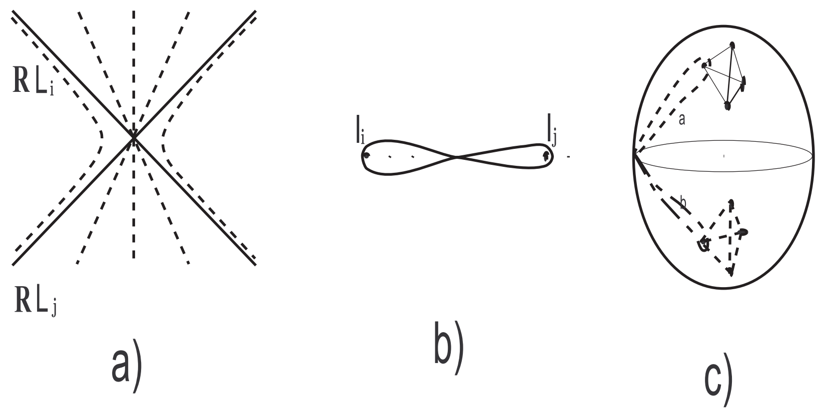

FIGURE 4. a) A perturbation of a real node; the dashed lines are dual to the points of an edge of Γ; b) A figure-eight loop along an edge of Γ; c) The loops in $C \smallsetminus \widetilde{V}$ representing generators "a" and "b"

contains i vertices from one component of $\widetilde{\Gamma}$ and $d-i$ vertices from the other component. This gives relations $a^i b^{d-i} = a^{d-i} b^i = 1$.

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