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On the Semi-Markovian Random Walk Process with Reflecting and Delaying Barrriers

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Abstract

In this paper, a semi-Markovian random walk process $X(t)$ with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$-level and the first falling moment of the process into the delaying barrier, (γ), are considered. Some probability characteristics of γ, such as its distribution function, moment generating function and expected value are calculated.

Key Words: Semi-Markovian random walk, reflecting barrier, delaying barrier, expected value.

1. Introduction

In recent years, random walks with one or two barriers are being used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, matematical biology etc. Many good monographs in this field exist in literature (see for example, [2], [3], [6], [7], [8] etc.).

In particular, a number of very interesting problems of stock control, queues and reliability theories can be expressed by means of random walks with two barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite volumes or queueing systems with finite waiting time or sojourn time by means of random walks with two delaying barriers. Furthermore, the functioning of stochastics systems with spare equipment can be given by random walks with two barriers, one of them is delaying while the other being of any barrier type.

Numerious studies have been done about random walks with two barriers because of their practical and theoretical importance (see, for example [1], [4], [5], [7] etc.). But most of these studies are associated with random walk processes with either delaying or absorbing barriers. Semi-Markovian random walk processes with two barriers in which one or both of them are reflecting barrier have not been considered well enough in literature. For this reason, although there are some studies in this subject (see for example

[3], [8], etc.) these studies are especially associated with finite state space random walks.

In this study, a semi-Markovian random walk with reflecting and delaying barriers that has a denumerable state space is constructed mathematically and main probability characteristics of a boundary functional of this process are considered. This process expresses the following physical model.

The Model. Assume that we observe random motion of a particle, initially at the position $X_0 \in [0,\beta], \beta > 0$, in a stripe bounded by two barriers; one of them is lying on the zero-level as reflecting and the other is lying on β-level as delaying. Furthermore, assume that this motion proceeds according to the following rules: After staying at the position X_0 for random duration ξ_1, the particle goes to position $X_0 = \eta_1$. If $X_0 + \eta_1 > \beta$, then the particle will be kept at the position $X_1 = \beta$, since there is delaying barrier at β-level. If $X_0 + \eta_1 \in [0,\beta]$, then the particle will be at the position $X_1 = X_0 + \eta_1$. Since there is a reflecting barrier at zero-level, when $X_0 + \eta_1 < 0$ the particle will be reflected from this barrier as long as $|X_0 + \eta_1|$. In this case, if $|X_0 + \eta_1| \leq \beta$ then the particle will be kept at the position $X_1 = |X_0 + \eta_1|$ and if $|X_0 + \eta_1| > \beta$ then the particle will be at the position β, so that the particle will be kept at the position $X_1 = \min\{\beta; |X_0 + \eta_1|\}$.

After staying at position X_1 for random duration ξ_2, again it will jump to position $X_2 = \min\{\beta; |X_1 + \eta_2|\}$ according to the above mentioned rules. Thus at the end of n-th jump, the particle will be at position $X_n = \min\{\beta; |X_{n-1} + \eta_n|\}$, $n \geq 1$.

2. Construction of the Process

Suppose $\{(\xi_i, \eta_i) : i = 1, 2, \dots\}$ is a sequence of identically and independently distributed pairs of random variables, defined on any probability space $(\Omega, \Im, P)$ such that ξ_i's are positive valued, i.e., $P\{\xi_i > 0\} = 1$. Also let us denote the distribution function of ξ_1 and η_1 by $\Phi(t)$ and $F(x)$, respectively, i.e.,

$$\Phi(t) = P\{\xi_1 \leq t\}, F(x) = P\{\eta_1 \leq x\}, \quad t \geq 0, \quad x \in (-\infty, +\infty). \tag{2.1}$$

First, let us construct the following sequences of random variables:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad Y_n = \sum_{i=1}^{n} \eta_i, \quad n \geq 1 \ : \quad Y_0 = T_0 = 0. \tag{2.2}$$

Note that $\{T_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ form a renewal process and a random walk, respectively. Now, we can construct the desired stochastic process $X(t)$:

$$X(t) = X_n, \quad \text{if} \quad t \in [T_n, T_{n+1}], \tag{2.3}$$

where

$$X_n = \min\Big\{\beta; \big|X_{n-1} + \eta_n\big|\Big\}, n \geq 1; X_0 \in [0, \beta].$$

This process forms a semi-Markovian random walk with reflecting barrier on the zero-level and delaying barrier on β-level. (For the semi-Markovian random walk process see Feller [3], Spitzer [8].).

Let us denote by γ the first falling moment of the process $X(t)$ into the delaying barrier. Now, we construct this random variable mathematically. Therefore, firstly we define the integer valued random variable v as follows:

$$v = \min\{n \geq 1 : X_{n-1} + \eta_n > \beta\}, \quad X_0 \in [0, \beta].$$

By using the random variable v we can construct γ as

$$\gamma = \sum_{i=1}^{v} \xi_i. \tag{2.4}$$

Note that γ is important from a scientific and practical point of view and it is an important boundary functional of the process $X(t)$. γ is the first moment in which the warehouse with finite volume becomes full. This random variable play an important role in solving of most probability problems arising in control of random levels of stocks in warehouse which is functioning according to the process $X(t)$. For this reason, the consideration with detailed of random variable γ seems very interesting from scientific and practical point of view.

Our aim in this study is to express the distribution function, and expected value of random variable γ by the probability characteristics of renewal process $\{T_n\}$ and random walk $\{Y_n\}$. Note that the main probability characteristics of renewal process and random walk are studied with detail in various scientific studies (for example see [3], [8], etc.). Let us state the following probability characteristics of random walk $\{Y_n\}$ and renewal process $\{T_n\}$:

$$\begin{aligned}
a_n(z) &= P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n\}, n \geq 1, a(z) = (a_n(z)),\\
c_n(z; v) &= P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n-1; z + Y_n < v\}, n \geq 1, c(z; v) = (c_n(z; v)),\\
c_n(z; dv) &= d_v(c_n(z; v)), n \geq 1, c(z; dv) = (c_n(z; dv)),\\
\Phi_n(t) &= P\{T_n \leq t\}, \Delta\Phi_n(t) = \Phi_n(t) - \Phi_{n+1}(t), n \geq 0,
\end{aligned}$$

where $z \in [0, \beta], v < 0$ and $a_0(z) = 1; c_0(z; v) = 0$.

Moreover, for any two sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$, the n-th term of the convolution of the sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ is defined as follows:

$$(\alpha * \beta)_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}, \quad n \geq 0.$$

3. Distribution Function of γ

Firstly, let us calculate the distribution function of γ, as distribution functions are the most important characteristic of random variables. Associated with this, we can state the following theorem.

Theorem 3.1 *If ξ_1 and η_1 are independent random variables in the initial sequence of random pairs mentioned above, then in terms of the probability characteristics of renewal process $\{T_n\}$ and random walk $\{Y_n\}$ the distribution function of γ is as follows:*

$$P_z\{\gamma \leq t\} = \Phi(t) - \sum_{n=1}^{\infty} g(n; z) \Delta\Phi_n(t), \tag{3.1}$$

where

$$g(n; z) = a_n(z) + \sum_{m=1}^{n} \int_{-\beta}^{0} \cdots (m) \cdots \int_{-\beta}^{0} \left(\left(\sum_{k=1}^{m} * c(|v_{k-1}|; dv_k) \right) * a(|v_m|) \right)_n, \quad n \geq 1$$

and $|v_0| = z$.

Proof. For the event A, introduce the notation $P_z\{A\} = P\{A|X(0) = z\}$. Denote by $G(t; z)$ the following conditional probability:

$$G(t; z) = P_z\{\gamma > t\} = P\{\gamma > t | X(0) = z\}, z \in [0, \beta], t > 0.$$

Then, applying the total probability formula we can write

$$G(t; z) = P_z\{\gamma > t\} = \sum_{n=0}^{\infty} P_z\{T_n \leq t < T_{n+1}; \gamma > t\}. \tag{3.2}$$

Substituting formula (2.4) for γ in (3.2), we get

$$G(t;z) = \sum_{n=0}^{\infty} P_z \left\{ T_n \le t < T_{n+1}; \sum_{i=1}^{v} \xi_i > t \right\}. \tag{3.3}$$

If we consider the definition of the random variable v, $G(t;z)$ can be rewritten as follows:

$$\begin{aligned} G(t;z) &= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} P_z \left\{ v = k; T_n \le t < T_{n+1}; \sum_{i=1}^{k} \xi_i > t \right\} \\ &= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} P_z \{ v = k; T_n \le t < T_{n+1}; T_k > t \}. \end{aligned} \tag{3.4}$$

Since the random variables $T_k, k \ge 1$, form a monotonically increasing sequence, the event $\{w : T_k > t\}, k \ge n+1$, implies the event $\{w : T_{n+1} > t\}$, that is, for any $k \ge n+1$ and $n \ge 0$ we get $\{w : T_{n+1} > t\} \subseteq \{w : T_k > t\}$. Therefore, we can write (3.4) as follows:

$$G(t;z) = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} P_z \{ v = k; T_n \le t < T_{n+1} \}. \tag{3.5}$$

The indepence of ξ_1 and η_1 implies the indepence of ξ_i and η_i for all $i \ge 1$. Therefore, we can write

$$\begin{aligned} G(t;z) &= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} P_z \{ v = k; T_n \le t < T_{n+1} \} \\ &= \sum_{n=0}^{\infty} P_z \{ v > n \} P \{ T_n \le t < T_{n+1} \}. \end{aligned} \tag{3.6}$$

On the other hand, it is easy to show that

$$P\{T_n \le t < T_{n+1}\} = \Phi_n(t) - \Phi_{n+1}(t) = \Delta\Phi_n(t). \tag{3.7}$$

Now, let

$$g(n;z) = P_z\{v > n\}, n \geq 0, \tag{3.8}$$

and let us express $g(n;z)$ by means of the probability characteristics of random walk $\{Y_n\}, n \geq 0$. For this, we define a random variable $v_0(n)$ with integer values, such that $v_0(n)$ denotes the number of reflections of the markovian chain $\{X_n\}, n \geq 1$, from the lower barrier during the first n-step. Now, we construct this random variable mathematically. Assume that

$$v_0(n) = \sum_{k=1}^{n} \chi_k, n \geq 1; \quad \chi_k = \begin{cases} 1, X_{k-1} + \eta_k < 0 \\ 0, X_{k-1} + \eta_k \geq 0 \end{cases} \quad k \geq 1. \tag{3.9}$$

Using the definition of $v_0(n)$ and applying the total probability formula we can write $g(n;z)$ as follows:

$$g(n;z) = P_z\{v > n\} = \sum_{m=0}^{n} P_z\{v_0(n) = m; \quad v > n\}. \tag{3.10}$$

We calculate every term in (3.10) seperately. In the special case where $m = 0$, it is easy to see that

$$\begin{aligned} P_z\{v_0(n) = 0; v > n\} &= P\{z + Y_1 \in [0,\beta], \dots, z + Y_n \in [0,\beta]\} \\ &= P\{z + Y_i \in [0,\beta]; 1 \leq i \leq n\} = a_n(z). \end{aligned} \tag{3.11}$$

We now consider the case $m = 1$. Here, we have

$$\begin{aligned} P_z\{v_0(n) = 1; v > n\} &= \int_{-\beta}^{0} \sum_{k=1}^{n} P\{z + Y_i \in [0,\beta]; 1 \leq i \leq k-1; z + Y_k \in dv\} \\ &\quad P\{|v| + Y_i \in [0,\beta]; 1 \leq i \leq n-k\} \\ &= \int_{-\beta}^{0} \sum_{k=0}^{n} c_k(z;dv) a_{n-k}(|v|) \end{aligned}$$

because $c_0(z;v) = 0$ for every $v < 0$ and $z \in [0, \beta]$. Thus, using the convolution of the sequences, we get

$$P_z\{v_0(n) = 1; v > n\} = \int_{-\beta}^{0} (c(z; dv) * a(|v|))_n. \tag{3.12}$$

To discover the general formula of $P\{v_0(n) = m; v > n\}$ let us consider the case of $m = 2$.

$$\begin{aligned} P_z\{v_0(n) = 2; v > n\} &= \int_{-\beta}^{0}\int_{-\beta}^{0} \sum_{2 \le k_1+k_2 \le n} P\left\{z + Y_i \in [0,\beta]; 1 \le i \le k_1 - 1; z + Y_{k_1} \in dv_1\right\} \\ &\cdot P\left\{|v_1| + Y_i \in [0,\beta]; 1 \le i \le k_2 - 1; |v_1| + Y_{k_2} \in dv_2\right\} \\ &\cdot P\{|v_2| + Y_i \in [0,\beta]; 1 \le i \le n - k_1 - k_2\} \\ &= \int_{-\beta}^{0}\int_{-\beta}^{0} \sum_{2 \le k_1+k_2 \le n} c_{k_1}(z; dv_1) c_{k_2}(|v_1|; dv_2) a_{n-k_1-k_2}(|v_2|). \end{aligned} \tag{3.13}$$

If we consider $c_0(z;v) = 0$ for every $v < 0$ and $z \in [0, \beta]$, then we get

$$P_z\{v_0(n) = 2; v > n\} = \int_{-\beta}^{0}\int_{-\beta}^{0} (c(z; dv_1) * c(|v_1|; dv_2) * a(|v_2|))_n. \tag{3.14}$$

Analogously, it is possible to prove that for every $m = 1, 2, \ldots, n$

$$\begin{aligned} &P_z\{v_0(n) = m; v > n\} \\ &\quad= \int_{-\beta}^{0} \cdots (m) \cdots \int_{-\beta}^{0} (c(z; dv_1) * c(|v_1|; dv_2) \\ &\quad\quad * \cdots * c(|v_{m-1}|; dv_m) * a(|v_m|))_n \\ &\quad= \int_{-\beta}^{0} \cdots (m) \cdots \int_{-\beta}^{0} \left(\left(\sum_{k=1}^{m} * c(|v_{k-1}|; dv_k) \right) * a(|v_m|) \right)_n, \end{aligned} \tag{3.15}$$

where $|v_0| = z \in [0, \beta]$. Substituting (3.11) and (3.15) in (3.10) we obtain

$$g(n;z) = a_n(z) + \sum_{m=1}^{n} \int_{-\beta}^{0} \cdots (m) \cdots \int_{-\beta}^{0} \left(\left(\sum_{k=1}^{m} *c(|v_{k-1}|; dv_k) \right) * a(|v_m|) \right)_n \tag{3.16}$$

for every $n \geq 1$. Furthermore, it is easy to see that $g(0;z) = a_0(z) = 1$. Thus, the following formula is obtained for $G(t;z)$:

$$\begin{aligned} G(t;z) &= \sum_{n=0}^{\infty} g(n;z)\Delta\Phi_n(t) \\ &= 1 - \Phi(t) + \sum_{n=1}^{\infty} g(n;z)\Delta\Phi_n(t). \end{aligned} \tag{3.17}$$

Hence the distribution function of γ can be given as follows:

$$P_z\{\gamma \leq t\} = 1 - G(t;z) = \Phi(t) - \sum_{n=1}^{\infty} g(n;z)\Delta\Phi_n(t). \tag{3.18}$$

Thus, the proof is completed. □

4. Moment Generating Function of γ

It is required to calculate the initial and central moments of random variables at the near of the distribution functions. In order to express these moments it is sufficient to calculate the moment generating function of random variables. Therefore, an explicit expression is given for the moment generating function of γ in this section. Let us denote by $\tilde{G}(\lambda; z)$, the moment generating function of γ and define it as follows:

$$\tilde{G}(\lambda; z) = \int_{0}^{\infty} e^{-\lambda t} G(t;z)dt, \lambda > 0,$$

In order to express the function $\tilde{G}(\lambda; z)$ by means of the probability characteristics of renewal process $\{T_n\}$ and random walk $\{Y_n\}$, let us introduce the following notations:

$$\varphi(\lambda) = E[\exp\{-\lambda\xi_i\}], \quad \tilde{a}(s;z) = \sum_{n=0}^{\infty} s^n a_n(z),$$

$$\tilde{c}(s;z;v) = \sum_{n=0}^{\infty} s^n c_n(z;v), \quad \tilde{c}(s;z;dv) = d_v(\tilde{c}(s;z;v)),$$

where $|s| \leq 1, \lambda > 0,\ z \in [0,\beta]$ and $v < 0$.

Now, we state the main results of this section in the following theorem.

Theorem 4.2 *Under conditions of Theorem 1, in terms of the probability characteristics of renewal process $\{T_n\}$ and random walk $\{Y_n\}$, the moment generating function of the random variable $\gamma, \tilde{G}(\lambda;z)$, can be given as follows:*

$$\tilde{G}(\lambda,z) = \frac{1-\varphi(\lambda)}{\lambda}\left[\tilde{a}(\varphi(\lambda);z) + \sum_{m=1}^{\infty}\int_{-\beta}^{0}\cdots(m)\cdots\int_{-\beta}^{0}\prod_{k=1}^{m}\tilde{c}(\varphi(\lambda);|v_{k-1}|;dv_k)\tilde{a}(\varphi(\lambda);|v_m|)\right],$$

where $|v_0| = z \in [0,\beta]$.

Proof. Assume that

$$\tilde{g}(s;z) = \sum_{n=0}^{\infty} s^n g(n;z) \tag{4.1}$$

for every $s, |s| \leq 1$, and let us calculate $\tilde{g}(s;z)$.

$$\begin{aligned}
\tilde{g}(s;z) &= \sum_{n=0}^{\infty} s^n g(n;z) \\
&= a_0(z) + \sum_{n=1}^{\infty} s^n a_n(z) + \sum_{n=1}^{\infty} s^n \sum_{m=1}^{n}\int_{-\beta}^{0}\cdots(m)\cdots\int_{-\beta}^{0}\left(\left(\prod_{k=1}^{m} *c(|v_{k-1}|;dv_k)\right) * a(|v_m|)\right)_n \\
&= \tilde{a}(s;z) + \sum_{m=1}^{\infty}\int_{-\beta}^{0}\cdots(m)\cdots\int_{-\beta}^{0}\prod_{k=1}^{m}\tilde{c}(s;|v_{k-1}|;dv_k)\tilde{a}(s;|v_m|).
\end{aligned} \tag{4.2}$$

On the other hand, it is easy to show that the Laplace transform of $\Delta\Phi_n(t), n = 0, 1, 2, \ldots$ is as follows:

$$\int_0^\infty e^{-\lambda t}\Delta\Phi_n(t)dt = \int_0^\infty e^{-\lambda t}[\Phi_n(t) - \Phi_{n+1}(t)]dt$$
$$= \frac{1}{\lambda}\left[\varphi^n(\lambda) - \varphi^{n+1}(\lambda)\right] = \frac{1-\varphi(\lambda)}{\lambda}\varphi^n(\lambda).$$

Applying the Laplace transform to $G(t; z)$ with respect to t in (3.17) and considering the above equality we have

$$\tilde{G}(\lambda; z) = \int_0^\infty e^{-\lambda t}\left[1 - \Phi(t) + \sum_{n=1}^\infty g(n; z)\Delta\Phi_n(t)\right]dt$$
$$= \frac{1-\varphi(\lambda)}{\lambda}\sum_{n=0}^\infty (\varphi(\lambda))^n g(n; z).$$

Since $|\varphi(\lambda)| \leq 1$ for any $\lambda \geq 0$, substituting $\varphi(\lambda)$ instead of s in (4.2), we get

$$\tilde{G}(\lambda; z) = \frac{1-\varphi(\lambda)}{\lambda}\tilde{g}(\varphi(\lambda); z)$$
$$= \frac{1-\varphi(\lambda)}{\lambda}\left[\tilde{a}(\varphi(\lambda); z) + \sum_{m=1}^\infty \int_{-\beta}^0 \cdots (m) \cdots \int_{-\beta}^0 \prod_{k=1}^m \tilde{c}(\varphi(\lambda); |v_{k-1}|; dv_k)\tilde{a}(\varphi(\lambda); |v_m|)\right].$$

Thus, the proof is complet. □

In general, in many practical problems, the calculation of the expected values of the random variable γ is required. Therefore, it is obtained the explicit expression for expected value of γ as result of Theorem 2. With this in mind, let us introduce the following notations:

$$A(z) = \sum_{n=0}^\infty a_n(z), C(z; v) = \sum_{n=0}^\infty c_n(z; v), C(z; dv) = d_v C(z; v).$$

Denote by $E_z[\gamma]$, the conditional expectation of γ under the condition that $X_0 \in [0, \beta]$.

Corollary 4.1 *In addition to the conditions of Theorem 1, if* $E[\xi_1 > \infty,\ E[\gamma] < \infty$, *then* $E_z[\gamma]$ *can be given as follows:*

$$E_z[\gamma] = E[\xi_1]\left[A(z) + \sum_{m=1}^{\infty} \int_{-\beta}^{0} \cdots (m) \cdots \int_{-\beta}^{0} \prod_{k=1}^{m} C(|v_{k-1}|; dv_k) A(|v_m])\right], \tag{4.3}$$

where $|v_0| = z \in [0, \beta]$.

Proof. If we approach the limit as $\lambda \to 0$ in the formula for $\tilde{G}(\lambda; z)$, then we get

$$\lim_{\lambda \to 0} \tilde{G}(\lambda; z) = \int_{0}^{\infty} P_z\{\gamma > t\} dt = E_z[\gamma]$$

from the left side and

$$\lim_{\lambda \to 0} \tilde{G}(\lambda; z) = E[\xi_1]\left[\tilde{a}(1; z) + \sum_{m=1}^{\infty} \int_{-\beta}^{0} \cdots (m) \cdots \int_{-\beta}^{0} \sum_{k=1}^{m} \tilde{c}(1; |v_{k-1}|; dv_k) \tilde{a}(1; |v_m|)\right]$$

from the right side respectively. On the other hand, we get

$$\tilde{a}(1; z) = \sum_{n=0}^{\infty} a_n(z) = A(z) \text{ and } \tilde{c}(1; z; dv) = \sum_{n=0}^{\infty} c_n(z; dv) = C(z; dv).$$

Substituting these expression in (4.3) we have

$$E_z[\gamma] = E[\xi_1]\left[A(z) + \sum_{m=1}^{\infty} \int_{-\beta}^{\infty} \cdots (m) \cdots \int_{-\beta}^{0} \prod_{k=1}^{m} C(|v_{k-1}|; dv_k) A(|v_m|)\right].$$

Thus, the proof is completed. □

Note. If the conditions of Corollary 1 are satisfied, then we calculate the moment of γ of higher order by using the formula

$$E_z[\gamma^k] = k(-1)^{k-1} \lim_{\lambda \to 0} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} [\tilde{G}(\lambda; z)], \quad k \geq 2,$$

thus proving that the expectation of γ^k exists and finite.

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