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p-Banach algebras with generalized involution and C^*-algebra structure

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Abstract

In this paper, we consider p-Banach algebras endowed with a generalized involution. We show that various C^*-like conditions force the algebra to be C^*-algebra under an equivalent norm.

Key words and phrases. p-Banach algebras, generalized involution, involutive anti-morphism, C^*-algebra.

Introduction

We know that a complex unitary commutative p-Banach algebra $(A, || \ ||_p)$, $0 < p \leq 1$, endowed with an involution $x \mapsto x^*$ such that $||x^*x||_p = ||x||_p^2$, for every $x \in A$, is a C^*-algebra for an equivalent norm [1]. Here, we consider complex p-Banach algebras, not necessarily commutative, endowed with generalized involution $x \mapsto x^*$. We show that the condition $||x||_p^2 \leq c||x^*x||_p$, for every $x \in A$ and some constant $c > 0$, implies that A is a C^*-algebra for a norm equivalent to $|| \ ||_p$. As a consequence, we obtain that any complex p-Banach algebra (non necessarily commutative) endowed with a generalized involution $x \mapsto x^*$ such that $||x^*x||_p = ||x||_p^2$, for every x in A, is in fact a C^*-algebra for the norm $\sqrt[p]{|| \ ||_p}$. We prove the same conclusion, with the condition $||h||_p \leq c\rho(h)^p$, for every $h \in H(A)$, in algebras where the spectrum of every hermitian element is real. The case of algebras satisfying $c||x||_p||x^*||_p \leq ||xx^*||_p$, for every $x \in N(A)$, is reduced to the previous one for we establish that the spectrum of every hermitian element is real. We do the same with the condition $||x||_p^2 \leq c\rho(xx^*)^p$, for every $x \in N(A)$. Finally, we show that $||u||_p \leq c$, for every $u \in U(A)$, implies that A is a C^*-algebra for a norm equivalent to $|| \ ||_p$. This norm is exactly $\sqrt[p]{|| \ ||_p}$ if, and only if, $||u||_p \leq 1$, for every u in the convex hull of $U(A)$. Notice that for an involutive anti-morphism, we show, in all cases, that the

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algebra is commutative.

1. Preliminaries

An involutive anti-morphism on a complex algebra A is a vector involution $x \mapsto x^*$ [3] such that $(xy)^* = x^*y^*$, for every x, y in A. A vector space involution $x \mapsto x^*$ is said to be a generalized involution if either it is an algebra involution or an involutive anti-morphism. As an example of an involutive anti-morphism which is not an algebra involution, we can consider the map $M = (a_{i,j})_{i,j} \longmapsto \overline{M} = (\overline{a_{i,j}})_{i,j}$, defined on $M_n(C)$ the algebra of square matrix of order n with complex coefficients. A p-normed algebra will be called p-homogeneous norm $||.||_p, 0 < p \leq 1$, satisfying $||xy||_p \leq ||x||_p \; ||y||_p$, for every $x, y \in A$. A complete p-normed algebra will be called p-Banach algebra. Notice that in [9], a p-normed algebra is necessarily complete which is not the case here. Let $(A, || \;\; ||_p)$, $0 < p \leq 1$, be a complex p-Banach algebra endowed with a generalized involution $x \mapsto x^*$. For $a \in A$, we define the real part of a, denoted Rea, by Re$a = \frac{1}{2}(a + a^*)$. An element a of A is said to be hermitian (resp., normal) if $a = a^*$ (resp., $a^*a = aa^*$. We designate by $H(A)$ (resp., $N(A)$) the set of hermitian (resp. normal) elements of A. In the unitary case, we say that a is unitary if $a^*a = aa^* = e$. The set of unitary elements of A will be denoted $U(A)$. We say that a p-Banach algebra with a generalized involution is hermitian if the spectrum of every hermitian element is real. We denote Ptàk's function by $| \;\; |$, that is, for every $a \in A, |a|^2 = \rho(aa)^*$, where ρ is the spectral radius i.e., $\rho(a) = \sup\{|\lambda| : \lambda \in Spa\}$. We define a C_p^*-algebra as being a complex p-Banach algebra $(A, || \;\; ||_p)$, $0 < p \leq 1$, endowed with a generalized involution $x \mapsto x^*$ such that $||x^*x||_p = ||x||_p^2$, for every $x \in A$. In a semi-simple involutive Banach algebra, Johnson's theorem [6] ensures the continuity of the involution. This result remains valid in the p-Banach case with a generalized involution. All algebras considered here will be associative and, except otherwise stated, over the complex field.

The following lemma has been obtained by J. W. M. Ford [5] in the Banach case. We give here, in our context, a proof which enlighten more the fact that the square root is hermitian.

Lemma 1.1 *Let $(A, || \;\; ||_p)$, $0 < p \leq 1$, be a unitary p-Banach algebra with a generalized involution $x \mapsto x^*$ and $h \in H(A)$ such that $Sph \subset \{\lambda \in C : \; Re\lambda > 0\}$. Then, there is $k \in H(A)$ such that $k^2 = h$.*

Proof. Considering, if necessary $A/RadA$ instead of A, we may suppose $x \mapsto x^*$ continuous. Put $\Omega = \{z \in C : z \notin R^-\}$. There is a holomorphic function f in Ω such that $f^2(\lambda) = \lambda$ and $f(1) = 1$. Let Γ a closed curve such that Sph is contained in its interior int(Γ) and int $(\Gamma) \cup \Gamma$ is contained in Ω. Put $k = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - h)^{-1} d\lambda$. Since homorphic functional calculus is an algebra morphism [9], we have

$$k^2 = \frac{1}{2\pi i} \int_\Gamma f^2(\lambda)(\lambda - h)^{-1} d\lambda = \frac{1}{2\pi i} \int_\Gamma \lambda(\lambda - h)^{-1} d\lambda = h.$$

It remains to show that k is hermitian. Let r and r' such that $0 < r < r'$, $Sph \subset D(r', r)$ and $\overline{D(r', r)} \subset \Omega$, where $D(r', r) = \{z \in C : |z - r'| > r\}$. We check that

$$k = \frac{1}{2\pi} \int_{|z - r'| = r} f(z) \mathrm{Re}[(z + h - 2r')(z - h)^{-1}] \frac{|dz|}{r}.$$

Moreover, since $x \mapsto x^*$ is continuous, we have

$$k^* = \frac{1}{2\pi} \int_{|z - r'| = r} f(\overline{z}) \mathrm{Re}[(\overline{z} + h - 2r')(\overline{z} - h)^{-1}] \frac{|d\overline{z}|}{r} = k.$$

□

Taking in account the fact that, in any p-Banach algebra A, we have $\rho(a)^p = \lim_n ||a^n||_p^{\frac{1}{n}}$, for every $a \in A$, we prove, as in [8], the following result.

Proposition 1.2 *Let $(A, ||\ \ ||_p)$, $0 < p \leq 1$, be a p-Banach algebra with a generalized involution $x \mapsto x^*$. The following assertions are equivalent.*

1) *A is hermitian.*

2) *There is $c > 0$ such that $\rho(a) \leq c|a|$, for every $a \in N(A)$.*

In case A is unitary, they are equivalent to

3) *There is $c > 0$ such that $\rho(u) \leq c$, for every $u \in U(A)$.*

Remark 1.3 *Let $(A, ||\ \ ||_p)$, $0 < p \leq 1$, be a p-Banach algebra with hermitian algebra involution $x \mapsto x^*$. We show, as in the Banach case, that Ptàk's function $|\ |$ is an algebra seminorm such that $\rho \leq |\ \ |$ and $|x|^2 = |x^* x|$, for every $x \in A$. According to theorem 3.2, (i) of [2], A is symetric. Moreover $Rad(A) = \{x \in A : |x| = 0\}$.*

Using theorem 3.10 of [9] and the fact that the quotient of a p-Banach algebra by a primitive ideal is a primitive p-Banach algebra, we extend theorem 4.8 of Kaplansky [7] to the p-Banach case as follows.

Theorem 1.4 *Any real semi-simple p-Banach algebra,* $0 < p \leq 1$, *in which every square is quasi-invertible, is necessarily commutative.*

2. Generalized involution and C^*-algebra structure

Any C_p^*-algebra (with an algebra involution) which is unitary and commutative is actually a C^*-algebra for an equivalent norm [1]. More general is the following result.

Proposition 2.1 *Let* $(A, ||\ \ ||_p)$, $0 < p \leq 1$, *be a p-normed algebra endowed with a generalized involution* $x \mapsto x^*$ *such that* $||x||_p^2 \leq c||x^*x||_p$, *for every* $x \in A$ *and a given* $c > 0$. *Then the completion* $\tilde{A}$ *of* A *is a* C^**-algebra for a norm equivalent to* $||\ \ ||_p$.

Proof. Notice that $x \mapsto x^*$ is continuous since $||x^*||_p \leq c||x||_p$, for every $x \in A$. We also may suppose A complete for the inequality $||x||_p^2 \leq c||x^*x||_p$ extends to the completion $\tilde{A}$. We consider first an algebra involution $x \mapsto x^*$. We have, in particular $||h||_p \leq c\rho(h)^p$, for every $h \in H(A)$. Whence $\rho(a)^{2p} \leq ||a||_p^2 \leq c||a^*a||_p \leq c^2|a|^{2p}$, for every $a \in A$. Hence A is hermitian, by proposition 1.2; from which it follows that $|\ \ |$ is an algebra seminorm such that $|x|^2 = |xx^*|$, for every $x \in A$. But $||x^*||_p \leq c||x||_p$ and hence $|x| \leq \sqrt[2p]{c}(\sqrt[p]{||x||_p})$, for every $x \in A$. Moreover $||x||_p^2 \leq c^2|x|^{2p}$. Whence $\sqrt[p]{||x||_p} \leq \sqrt[p]{c}|x|$, for every $x \in A$. So $|\ \ |$ is an algebra norm, on A, which is equivalent to $||\ \ ||_p$ and such that $(A, |\ \ |)$ is a C^*-algebra. Suppose now that $x \mapsto x^*$ is an involutive anti-morphism. We will show that, in this case, the algebra A is commutative. It is sufficient to consider the real p-Banach algebra $H(A)$. From hypothesis, one obtains $||a^n||_p^2 \leq c||(a^*a)^n||_p$, for every $a \in N(A)$. Whence $\rho(a) \leq |a|$, for every $a \in N(A)$. Hence, by proposition 1.2, $Sph \subset R$, for every $h \in H(A)$. Moreover $Rad(H(A)) = \{0\}$, since $||h||_p \leq c\rho(h)^p$, for every $h \in H(A)$. Whence, the commutativity of $H(A)$ by theorem 1.4. □

We have the following consequences.

Corollary 2.2 *Let* $(A, ||\ \ ||_p)$, $0 < p \leq 1$, *be a p-Banach algebra with a generalized involution* $x \mapsto x^*$. *If* $(A, ||\ \ ||_p)$ *is a* C_p^**-algebra, then* $(A, \sqrt[p]{||\ \ ||_p})$ *is a* C^**-algebra.*

Corollary 2.3 *Let* $(A, ||\ \ ||_p)$, $0 < p \leq 1$, *be a p-Banach algebra with a generalized involution* $x \mapsto x^*$. *The following assertions are equivalent.*

1) A is a C_p^*-algebra for a q-norm, $0 < q \leq 1$, equivalent to $|| \ \ ||_p$.

2) $(A, | \ \ |)$ is a C^*-algebra.

Proof. To show that **1)** implies **2)**, let $|| \ \ ||_q$ be a q-norm equivalent to $|| \ \ ||_p$ and such that $||x^*x||_q = ||x||_q^2$, for every $x \in A$. By corollary 2.2, $(A, \sqrt[q]{|| \ \ ||_q})$ is a C^*-algebra. Hence $\sqrt[q]{|| \ \ ||_q} = | \ \ |$. Conversely if $(A, | \ \ |)$ is a C^*-algebra, then $|x^*x|^p = |x|^{2p}$, for every $x \in A$. But $|| \ \ ||_p$ and $| \ \ |^p$ are complete algebra p-norms on A which is semi-simple, so they are equivalent. □

Remark 2.4 *Let $(A, || \ \ ||_p)$, $0 < p \leq 1$, be a p-Banach algebra with a generalized involution $x \mapsto x^*$ and $|| \ \ ||$ the gauge of the convex hull of $\{x \in A : ||x||_p \leq 1\}$. If $(A, || \ \ ||_p)$ is a C_p^*-algebra, then $(A, || \ \ ||)$ is a C^*-algebra (Corollary 2.2). Conversely if $(A, || \ \ ||)$ is a C_p^*-algebra, then, by corollary 2.3, A is a C_q^*-algebra for a q-norm equivalent to $|| \ \ ||_p$.*

Proposition 2.5 *Let $(A, || \ \ ||_p)$, $0 < p \leq 1$, be a p-Banach algebra with a hermitian generalized involution $x \mapsto x^*$ such that $||h||_p \leq c\rho(h)^p$, for every h in $H(A)$. Then A is a C^*-algebra for a norm equivalent to $|| \ \ ||_p$.*

Proof. First, $x \mapsto x^*$ is continuous for A is semi-simple. Let $\alpha > 0$ such that $||x^*||_p \leq \alpha ||x||_p$, for every $x \in A$. Suppose $x \mapsto x^*$ is an algebra involution. Since A is hermitian and semi-simple, Ptàk's function $| \ \ |$ is an algebra norm. Now, the inequality in hypotheses implies that, for $x = h + ik$, $(h, k \in H(A))$, we have $||x||_p \leq ||h||_p + ||k||_p \leq c(|h|^p + |k|^p) \leq 2c|x|^p$. But, we also have $|x|^{2p} = \rho(x^*x)^p \leq ||x^*x||_p \leq \alpha ||x||_p^2$. Hence $| \ \ |$ is equivalent to $|| \ \ ||_p$ and $(A, | \ \ |)$ is a C^*-algebra. For the case where $x \mapsto x^*$ in an involutive anti-morphism, we apply theorem 1.4, the algebra being semi-simple. □

Proposition 2.6 *Let $(A, || \ \ ||_p)$, $0 < p \leq 1$, be a p-Banach algebra with a generalized involution $x \mapsto x^*$ such that $c||x||_p||x^*||_p \leq ||xx^*||_p$, for every $x \in N(A)$ and some $c > 0$. Then A is a C^*-algebra for a norm equivalent to $|| \ \ ||_p$.*

Proof. The inequality in hypotheses implies $c||h||_p \leq \rho^p(h)$, for every $h \in H(A)$. Then for $x \in N(A)$ we obtain $c^2\rho(x)^{2p} \leq c^2||x||_p||x^*||_p \leq c||x^*x||_p \leq \rho(x^*x)^p = |x|^{2p}$. Hence A is hermitian, by proposition 1.2. To conclude, we apply proposition 2.5. □

Proposition 2.7 *Let* $(A, ||\ \ ||_p)$, $0 < p \leq 1$, *be a p-Banach algebra with a generalized involution* $x \mapsto x^*$ *such that* $||x||_p^2 \leq c\rho(xx^*)^p$, *for some* $c > 0$ *and every* $x \in N(A)$. *Then* A *is a* C^**-algebra for a norm equivalent to* $||\ \ ||_p$. *If moreover,* $||x||_p \leq |x|^p$, *for every* $x \in A$, *then* $(A, \sqrt[p]{||\ \ ||_p})$ *is a* C^**-algebra.*

Proof. By hypothesis, one has $\rho(x)^p \leq \sqrt{c}|x|^p$, for every $x \in N(A)$. Hence, A is hermitian, by proposition 1.2. We conclude by proposition 2.5, since $||h||_p \leq \sqrt{c}\rho(h)^p$, for every $h \in H(A)$. Now, if $||x||_p \leq |x|^p$, for every $x \in A$, then $||x||_p^2 \leq |x|^{2p} \leq ||x^*x||_p \leq ||x^*||_p||x||_p$. Hence $||x^*||_p = ||x||_p$. From which it follows that $|\ \ |^p = ||\ \ ||_p$. And so $(A, \sqrt[p]{||\ \ ||_p})$ is a C^*-algebra. □

Proposition 2.8 *Let* $(A, ||\ \ ||_p)$, $0 < p \leq 1$, *be a p-Banach algebra with an involutive anti-morphism* $x \mapsto x^*$ *such that* $||x^*x||_p = ||x^*||_p||x||_p$, *for every* $x \in A$. *Then* $(A, \sqrt[p]{||\ \ ||_p})$ *is a commutative* C^**-algebra.*

Proof. By proposition 2.6, A is a commutative C^*-algebra for a norm equivalent to $||\ \ ||_p$. The involution being continuous, let $\alpha > 0$ such that $||x^*||_p \leq \alpha||x||_p$, for every $x \in A$. The equality in hypotheses implies $||h||_p = \rho(h)^p$, for every $h \in H(A)$; from which it follows that $|x|^{2p} = \rho(x^*x)^p = ||x^*||_p||x||_p \leq \alpha||x||_p^2$, for every $x \in A$. But also $\rho(x) = |x|$ for A is commutative and hermitian. Whence $|x|^{pn} = \rho(x^n)^p = |x^n|^p \leq \sqrt{\alpha}||x||_p^n$, for every $n \in N$. Tending n to infinity, we get $|x|^p \leq ||x||_p$. Consequently $||x^*||_p = ||x||_p$. And we conclude by corollary 2.2. □

Proposition 2.9 *If* $(A, ||\ \ ||_p)$, $0 < p \leq 1$, *is a unitary p-Banach algebra with a generalized involution* $x \mapsto x^*$ *such that* $||u||_p \leq c$, *for some* $c > 0$ *and every* $u \in U(A)$, *then* A *is a* C^**-algebra for a norm equivalent to* $||\ \ ||_p$. *Moreover* $(A, \sqrt[p]{||\ \ ||_p})$ *is a* C^**-algebra if, and only if,* $||u||_p \leq 1$, *for every* u *in the convex hull of* $U(A)$.

Proof. Suppose $||u||_p \leq c$, for every $u \in U(A)$. Then $\rho(u)^p \leq ||u||_p \leq c$. Hence A is hermitian, by proposition 1.2. Now, for $h \in H(A)$ such that $\rho(h) < 1$, there is, by lemma 1.1, $k \in H(A)$ such that $k^2 = e - h^2$ and $kh = hk$. Put $u = h + ik$. We have $u \in U(A)$ and $||h||_p = \left|\left|\frac{u+u^*}{2}\right|\right|_p \leq 2^{1-p}c$. Hence $||h||_p \leq 2^{1-p}c\rho(h)^p$, for every $h \in H(A)$. By proposition 2.5, the algebra A is a C^*-algebra for a norm equivalent to $||\ \ ||_p$. For the rest of the proof is reduced to the case of an algebra involution. Concerning the last assertion, we show (classical proof) that the convex hull of the convex component of the unit, in $U(A)$, contains the set of elements x such that $\rho(x) < 1$ and $|x| < 1$. Hence, for $x \in A$ such that $|x| < 1$, one has $\rho(x) < 1$ (Remark 1.3). Whence $||x||_p \leq 1$ since x is in the convex hull of $U(A)$. We obtain $||x||_p \leq |x|^p$, for every $x \in A$. We conclude by proposition 2.7. □

Let $(A, ||\ \ ||_p)$ be a real p-Banach algebra with an involutive anti-morphism $x \longmapsto x^*$. The map $(x+iy)^\# = x^* - iy^*$ is an anti-morphism on complexification $A_C = A + iA$ of A. We give a version of proposition 2.6 in the real case.

Proposition 2.10 *Let $(A, ||\ \ ||_p)$, $0 < p \leq 1$, be a real p-Banach algebra with an involutive anti-morphism. If there is $c > 0$ such that $||x||_p^2 \leq c||x^*x + y^*y||_p$, for every x, y in A, then the complexification A_C, of A, is a commutative C^*-algebra.*

Proof. Let P by p-gauge of the p-absolutely convex hull of $\{x \in A : ||x||_p < 1 \times \{0\}$. P is a complete algebra-norm, on A_C, such that $\max(||a||_p, ||b||_p) \leq P(a+ib) \leq 2\max(||a||_p, ||b||_p)$, for every a, b in A. Now, for x and y in A, we have $||x||_p^2 \leq c||x^*x+y^*y||_p$ and $||y||_p^2 \leq c||x^*x + y^*y||_p$. Whence

$$P(x+iy)^2 \leq 4c||x^*x + y^*y||_p, \quad x, y \in A.$$

Since $(x+iy)^\#(x+iy) = x^*x + y^*y + i(x^*y - y^*x)$, we have $P((x+iy)^\#)P(x+iy) \leq 16c^2 P((x+iy)^\#(x+iy))$. And we conclude by proposition 2.6. □

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