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Formula for the Highly Regularized Trace of the Sturm-Liouville Operator with Unbounded Operator Coefficients Having Singularity

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Abstract

In this work, a formula for the n^{th} regularized trace of the Sturm-Liouville operator with unbounded operator coefficients having singularity is obtained

1. Introduction

The regularized trace of the scalar differential operators has been calculated by I. M. Gelfand and B. M. Levitan [1], L. A. Dikiy [2], C. J. Halberg and V. A. Kramer [3] among other works. The list of published work on this subject can be found in B. M. Levitan and I. S. Sargsyan [4], I. C. Fulton and S. A. Pruess [5]. The regularized trace of differential operators with unbounded operator coefficients has been computed in [6, 8, 9, 10].

In this study, the n ($n \in N$) regularized trace of Sturm-Liouville operator with unbounded operator coefficients having singularity has been calculated in the interval [0,1].

Let H be a separable Hilbert Space. Denote by $H_1 = L_2(H : [0,1])$ the Hilbert space of all vector-valued measurable functions y(t) (t$\in [0,1]$) with values in H and such that

$$\int_0^1 \|y(t)\|^2 dt < \infty.$$

The scalar product vectors y(t), z(t) in H_1 is defined by

$$(y,z)_1=\int_0^1 (y(t),z(t))dt.$$

Let us write

$$l_0(y)=-y^{''}(x)+\frac{v^2-\frac{1}{4}}{x^2}y(x)+Ay(x)$$

$$l(y)=-y^{''}(x)+\frac{v^2-\frac{1}{4}}{x^2}y(x)+Ay(x)+Q(x)y(x),\quad v\geq\frac{1}{2},$$

where A is an operator from $D(A)$ to H which satisfies the following conditions such that $\overline{D(A)}=H$:

$$A=A^*\geq I,\quad A^{-1}\in\sigma_\infty(H).$$

Expressions $l_0(y)$ and $l(y)$ form two operators L and L_0 in space H_1:

$D(L_0^/)=\{y(x)\in H_1: y(x)\in D(A), y''(x),\ Ay(x)$ continuous in [0,1] according to norm in H $y(0)=y(1)=0$ and $l_0(y)\in H_1\}$

$$L_0^/y=l_0(y),\quad y\in D(L_0^/).$$

Let us denote $\overline{L_0}=L_0$, where the overbar symbol shows closure of the operator. It can be shown that L_0 is a self-adjoint operator in H_1. Operator $L=L_0+Q(x)$ ($Q(x)=Q^*(x)$ and is bounded in H_1: see condition 2)) with domain $D(L)=D(L_0)$ is a self-adjoint operator in H_1.

Let us accept that the operator function $Q(x)$ in the expression $l(y)$satisfies the following conditions:

1) $Q(x)$ has a weak derivative of the second order in [0,1]. For $\forall x\in[0,1]$, $Q^{(i)}(x)$ (i=0,1,2) are self-adjoint operator from H to H and $A^{n-1}Q^{(i)}(x)\in\sigma_1(H)$;

2) The functions $||A^{n-1}Q^{(i)}(x)||_{\sigma_1(H)}$ (i=0,1,2) are bounded and measurable in [0,1];

3) For $\forall f \in H$, $\int\limits_0^1 (Q(x)f,f)dx = 0$;

4) For $\forall f \in H$ and $x \in [0,\delta]$, there is an operator $B \in \sigma_1(H)$ such that

$$|(Q^{(i)}(x)f,f)| \quad \leq \quad |(Bf,f)|.$$

Here, $\sigma_1(H)$ is the space of kernel operators from H to H as in [11], and $(.,.)$ denotes the inner product in H. We denote the inner product by $(.,.)_1$ and the norm of operator by $||.||_1$ in H_1.

In this study, we will use the following inequalities proved in [11]:

$$|trB_1| \leq ||B_1||_{\sigma_1(H)}$$

$$||B_1B_2||_{\sigma_1(H)} \leq ||B_1||_{\sigma_1(H)}.||B_2||$$

$$||B_2B_1||_{\sigma_1(H)} \leq ||B_1||_{\sigma_1(H)}.||B_2||,$$

where $B_1 \in \sigma_1(H)$ and $B_2 \in L(H,H)$.

Note that the operator L_0 has a pure discrete spectrum. Let $\mu_1 \leq \mu_2 \leq ....$ be the eigenvalues of this operator and $\psi_1(x), \psi_2(x), ...$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, every eigenvalue is repeated according to multiplicity number. Since Q is a self-adjoint bounded operator from H_1 to H_1, $L = L_0 + Q$ is a self-adjoint operator in H_1 which satisfies $D(L) = D(L_0)$ and it has a pure discrete spectrum. Let $\lambda_1 \leq \lambda_2 \leq$ ...be the eigenvalues of this operator. Moreover, let $\gamma_1 \leq \gamma_2 \leq$ ...be the eigenvalues of the operator A from $D(A)$ to H and $\varphi_1, \varphi_2, ...$ be orthonormal eigenvectors corresponding to these eigenvalues.

If $\gamma_i \sim ai^\alpha$ while $i \to \infty$ (a>0, α>2), then it is proved in [7] ,[8] (and, respectively, for γ=1/2; γ ≥1/2) that

$$\mu_m, \lambda_m \sim dm^{\frac{2\alpha}{2+\alpha}} \text{ while } m \to \infty, (d > 0) \tag{1}$$

is satisfied. By using this formula, we can show that the sequence $\{\mu_m\}_{m=1}^\infty$ has a subsequence $\{\mu_{m_p}\}_{p=1}^\infty$ such that

$$\mu_k - \mu_{m_p} \geq d_1 \left(k^{\frac{2\alpha}{2+\alpha}} - m_p^{\frac{2\alpha}{2+\alpha}} \right), \quad (k = m_p, m_p + 1, m_p + 2, ...), \tag{2}$$

where d_1 is a positive constant. Let R_λ^0, R_λ be the resolvents of L_0 and L, respectively. From (1), if $\alpha > 2$ and $\lambda \neq \lambda_k, \mu_k$ $(k = 1, 2, ...)$then the series $\sum\limits_{k=1}^{\infty} |\mu_k - \lambda|^{-1}$ and $\sum\limits_{k=1}^{\infty} |\lambda_k - \lambda|^{-1}$ are convergent. Therefore R_λ^0 and R_λ are the kernel operators. In this case, from [11] it is known that

$$tr(R_\lambda - R_\lambda^0) = trR_\lambda - trR_\lambda^0 = \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right).$$

If we multiply this equation with $\frac{\lambda^n}{2\pi i}$ and integrate along the circle

$$|\lambda| = b_p = 2^{-1}(\mu_{m_p} + \mu_{m_p+1})$$

then we obtain

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda^n tr(R_\lambda - R_\lambda^0) d\lambda = -\sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \frac{\lambda^n}{\lambda - \lambda_k} d\lambda - \frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \frac{\lambda^n}{\lambda - \mu_k} d\lambda \right\}. \tag{3}$$

It can be shown that for large values of p,

$$\{\lambda_k, \mu_k\}_{k=1}^{m_p} \subset K(0, b_p) = \{\lambda : |\lambda| < b_p\}. \tag{4}$$

Here, $\lambda_k, \mu_k \notin \overline{K(0, b_p)} = \{\lambda : |\lambda| \leq b_p\}, (k \geq m_p + 1)$. From (3) and (4) we find

$$\sum_{k=1}^{m_p} (\lambda_k^n - \mu_k^n) = -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda^n tr(R_\lambda - R_\lambda^0) d\lambda. \tag{5}$$

It is known that $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$. From here, for any natural number $N \geq 2$, we obtain

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{N} (-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N+1} R_\lambda (QR_\lambda^0)^{N+1}.$$

If we substitute this expression into (5) then we find

$$\sum_{k=1}^{m_p} (\lambda_k^n - \mu_k^n) = \frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda^n tr\left[\sum_{j=1}^{N} (-1)^{j+1} R_\lambda^0 (QR_\lambda^0)^j + (-1)^N R_\lambda (QR_\lambda^0)^{N+1}\right] d\lambda$$

$$= \sum_{j=1}^{N} B_{pj} + B_p^{(N)} \quad . \tag{6}$$

Here,

$$B_{pj} = \frac{(-1)^{j+1}}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda^n tr\big[R_\lambda^0 (QR_\lambda^0)^j\big] d\lambda, \quad (j = 1, 2, ..., N) \tag{7}$$

$$B_p^{(N)} = \frac{(-1)^N}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda^n tr\big[R_\lambda (QR_\lambda^0)^{N+1}\big] d\lambda. \tag{8}$$

For every natural number j, it can be shown that the operator function $(QR_\lambda^0)^j$ is analytic according to the norm in $\sigma_1(H_1)$ in the resolvent region $\rho(L_0)$of the operator L_0. Moreover,

$$tr\big[R_\lambda^0 (QR_\lambda^0)^j\big] = tr\big[(QR_\lambda^0)^{j-1} Q (R_\lambda^0)^2\big] = tr\left[(QR_\lambda^0)^{j-1} \frac{d}{d\lambda}(QR_\lambda^0)\right],$$

$$tr\left\{\frac{d}{d\lambda}\big[(QR_\lambda^0)^j\big]\right\} = j tr\left[(QR_\lambda^0)^{j-1} \frac{d}{d\lambda}(QR_\lambda^0)\right].$$

From the last two relations one obtains

$$tr\big[R_\lambda^0(QR_\lambda^0)^j\big]=\frac{1}{j}tr\bigg\{\frac{d}{d\lambda}\big[(QR_\lambda^0)^j\big]\bigg\}.$$

If this expression is substituted in (7), then

$$\begin{aligned}B_{pj}&=\frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_p}\lambda^n tr\bigg\{\frac{d}{d\lambda}\big[(QR_\lambda^0)^j\big]\bigg\}d\lambda\\&=\frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_p}tr\bigg\{\frac{d}{d\lambda}\big[\lambda^n(QR_\lambda^0)^j\big]-n\lambda^{n-1}(QR_\lambda^0)^j\bigg\}d\lambda\end{aligned}$$

is found. So if we note that

$$tr\bigg\{\frac{d}{d\lambda}\big[\lambda^n(QR_\lambda^0)^j\big]\bigg\}=\frac{d}{d\lambda}\big\{tr\big[\lambda^n(QR_\lambda^0)^j\big]\big\}$$

then

$$B_{pj}=\frac{(-1)^j n}{2\pi ij}\int\limits_{|\lambda|=b_p}\lambda^{n-1}tr\big[(QR_\lambda^0)^j\big]d\lambda+\quad\frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_p}\frac{d}{d\lambda}\big\{tr\big[\lambda^n(QR_\lambda^0)^j\big]\big\}d\lambda$$

is obtained. On the other hand, by using (4) we obtain

$$\int\limits_{|\lambda|=b_p}\frac{d}{d\lambda}\big\{tr\big[\lambda^n(QR_\lambda^0)^j\big]\big\}d\lambda=0.$$

Therefore,

$$B_{pj}=\frac{(-1)^j n}{2\pi ij}\int\limits_{|\lambda|=b_p}\lambda^{n-1}tr\big[(QR_\lambda^0)^j\big]d\lambda \tag{9}$$

or

$$B_{pj} = (-1)^j j^{-1} n \sum_{k=1}^{m_p} Res\big[\lambda^{n-1} tr(QR_\lambda^0)^j\big]. \tag{10}$$

From (6) and (10), we obtain

$$\sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^{N} (-1)^j j^{-1} \underset{\lambda=\mu_k}{Res} \left[\lambda^{n-1} tr(QR_\lambda^0)^j\right] \right\} = B_{p1} + B_{pN}. \tag{11}$$

Let $\beta_1 < \beta_2 < ....,$ be the non-negative roots of the Bessel function $J_\nu(x)$. The eigenvalues of the operator L_0 and its eigenvectors corresponding to these eigenvalues are of the respective forms

$$\beta_q^2 + \gamma_i, \quad (q, i = 1, 2, ...)$$

$$\sqrt{2x} \frac{J_\nu(\beta_q x)}{J_{\nu+1}(\beta_q)} \varphi_i, \quad (q, i = 1, 2, ...).$$

Therefore, the eigenvalues $\mu_1 \leq \mu_2 \leq ...$of the operator L_0

$$\mu_k = \beta_{q_k}^2 + \gamma_{i_k} \quad (k = 1, 2, ...)$$

and the orthonormal eigenvectors $\psi_k(x)$ corresponding to these eigenvalues can be in the form

$$\psi_k(x) = \sqrt{2x} \frac{J_\nu(\beta_{q_k} x)}{J_{\nu+1}(\beta_{q_k})} \varphi_{i_k} \quad (k = 1, 2, ...).$$

From formula (9) we have

$$B_{p1} = \frac{-n}{2\pi i} \int\limits_{|\lambda| = b_p} \lambda^{n-1} tr\big[(QR_\lambda^0)\big] d\lambda.$$

Moreover, $tr(QR_\lambda^0) = \sum\limits_{k=1}^{\infty} (QR_\lambda^0 \psi_k, \psi_k)_1 = \sum\limits_{k=1}^{\infty} (\mu_k - \lambda)^{-1} (Q\psi_k, \psi_k)_1$.

Hence,

$$\begin{aligned}
B_{p1} &= n\sum_{k=1}^{\infty} (Q\psi_k, \psi_k)_1 \frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \frac{\lambda^{n-1}}{\lambda - \mu_k} d\lambda \\
&= n\sum_{k=1}^{m_p} \mu_k^{n-1} (Q\psi_k, \psi_k)_1 \\
&= n\sum_{k=1}^{m_p} (\beta_{qk}^2 + \gamma_{i_k})^{n-1} (Q\psi_k, \psi_k)_1 \\
&= n\sum_{k=1}^{m_p} \sum_{l=0}^{n-1} C_{n-1}^{p} \beta_{qk}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \\
&= n\sum_{k=1}^{m_p} \sum_{l=0}^{n-2} C_{n-1}^{l} \beta_{qk}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 + n\sum_{k=1}^{m_p} \gamma_{i_k}^{n-1} . (Q\psi_k, \psi_k)_1
\end{aligned} \tag{12}$$

From (11) and (12) we find

$$\sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n\sum_{j=2}^{N} (-1)^j j^{-1} \underset{\lambda=\mu_k}{Res} \left[\lambda^{n-1} tr(QR_\lambda^0)^j \right] - n\sum_{l=0}^{n-2} C_{n-1}^{l} \beta_{qk}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\}$$

$$= n\sum_{k=1}^{m_p} \gamma_{i_k}^{n-1} (Q\psi_k, \psi_k)_1 + \beta_p^{(N)}, \tag{13}$$

where the indices q_k and i_k are the natural numbers such that $\mu_k = \beta_{q_k}^2 + \gamma_{i_k}$ ($\mu_1 \le \mu_2 \le \ldots$ and $k = 1, 2, \ldots$). In the following we will compute the limit

$$\lim_{p\to\infty} \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n\sum_{j=2}^{N} (-1)^j j^{-1} \underset{\lambda=\mu_k}{Res} \left[\lambda^{n-1} tr(QR_\lambda^0)^j \right] - n\sum_{l=0}^{n-2} C_{n-1}^{l} \beta_{qk}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\}.$$

This limit will be called the nth regularized trace of the operator L.

Theorem 1. *If the conditions 1)-4) are satisfied then*

$$\sum_{q=1}^{\infty}\sum_{i=1}^{\infty}\left|\gamma_i^{n-1}\int_0^1 2x\frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}(Q(x)\varphi_i,\varphi_i)dx\right|<\infty.$$

Proof. Let $f_i(x)=(Q(x)\varphi_i,\varphi_i)$. Then

$$\begin{aligned}&\int_0^1 2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}(Q(x)\varphi_i,\varphi_i)dx=\int_0^1 2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}f_i(x)dx\\ &\int_0^1 2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}[f_i(x)-f_i(0)]dx+f_i(0)=\int_0^{\beta_q^{-1+\varepsilon}} 2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}[f_i(x)-f_i(0)]dx\\ &+\int_{\beta_q^{-1+\varepsilon}}^1 2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}[f_i(x)-f_i(0)]dx+f_i(0)\end{aligned}\tag{14}$$

where $\varepsilon\in(0,\frac{1}{2})$. It has been shown in [8] that the asymptotic formula

$$2x\frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}=1+\sin(2\beta_q x-\nu\pi)-\frac{Cos(2\beta_q x-\nu\pi)}{8\beta_q x}(4\nu^2-1)+O((\beta_q x)^{-2})\tag{15}$$

is satisfied for the large value of q, where $x\in[\beta_q^{-1+\varepsilon},1)$.

From (14) and (15) we write

$$\begin{aligned}&\sum_{q=1}^{\infty}\sum_{i=1}^{\infty}\gamma_i^{n-1}\left|\int_0^1 2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}f_i(x)dx\right|\le\ const\cdot\sum_{q=1}^{\infty}\sum_{i=1}^{\infty}\left[\left|\int_0^{\beta_q^{-1+\varepsilon}}|f_i(x)-f_i(0)|2x\tfrac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}dx\right|+\right.\\ &\left|\int_{\beta_q^{-1+\epsilon}}^1(f_i(x)-f_i(0))(1+\sin(2\beta_q x-\nu\pi))dx+f_i(0)\right|+\\ &\left.\left|\int_{\beta_q^{-1+\epsilon}}^1(f_i(x)-f_i(0))\tfrac{\cos(2\beta_q x-\nu\pi)}{\beta_q x}dx\right|+\int_{\beta_q^{-1+\varepsilon}}^1\tfrac{|f_i(x)-f_i(0)|}{(\beta_q x)^2}dx\right]\gamma_i^{n-1}.\end{aligned}\tag{16}$$

By using the inequality $|\sqrt{x}J_\nu(x)|<C$ and conditions 1), 4) for the first integral on the right hand side of (16), we get

$$\gamma_i^{n-1}\left|\int_0^{\beta_q^{-1+\varepsilon}}(f_i(x)-f_i(0))2x\frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}dx\right|\le Const.\gamma_i^{n-1}\int_0^{\beta_q^{-1+\varepsilon}}|f_i(x)-f_i(0)|dx$$
$$=Const.\gamma_i^{n-1}\int_0^{\beta_q^{-1+\varepsilon}}|f_i^{'}(\xi x)|xdx=Const.\int_0^{\beta_q^{-1+\varepsilon}}|(A^{n-1}Q'(\xi x)\varphi_i,\varphi_i)|xdx \tag{17}$$
$$\le Const.|(B\varphi_i,\varphi_i)|\beta_q^{-2+2\varepsilon}$$

From the conditions 1^0)-4^0) and by using the asymptotic formula $\beta_q=\left(q+\frac{\nu}{2}-\frac{1}{4}\right)\pi+O(q^{-1})$ we obtain

$$\gamma_i^{n-1}\left|\int_{\beta_q^{-1+\epsilon}}^{1}(f_i(x)-f_i(0))(1+\sin(2\beta_q x-\nu\pi))dx+f_i(0)\right|=\gamma_i^{n-1}\left|\int_0^1(f_i(x)-f_i(0))dx\right.$$
$$\left.-\int_0^{\beta_q^{-1+\varepsilon}}|f_i(x)-f_i(0)|dx+\int_{\beta_q^{-1+\epsilon}}^{1}(f_i(x)-f_i(0))\sin(2\beta_q x-\nu\pi)dx+f_i(0)\right|$$
$$\le\gamma_i^{n-1}\int_0^{\beta_q^{-1+\varepsilon}}|f_i'(\xi x)|xdx+\gamma_i^{n-1}\left|\frac{\cos(2\beta_q x-\nu\pi)(f_i(1)-f_i(0))}{2\beta_q}\right.$$
$$\left.-\frac{\cos(2\beta_q^{\varepsilon}-\nu\pi)(f_i(\beta_q^{-1+\varepsilon})-f_i(0))}{2\beta_q}-\frac{1}{2\beta_q}\int_{\beta_q^{-1+\epsilon}}^{1}f_i'(x)\cos(2\beta_q x-\nu\pi)dx\right|$$
$$\le Const.\left[|(B\varphi_i,\varphi_i)|\beta_q^{-2+2\varepsilon}+\gamma_i^{n-1}\beta_q^{-2}\left(|f_i(0)|+|f_i(1)|\right)+|(B\varphi_i,\varphi_i)|\beta_q^{-2+\varepsilon}\right]$$
$$+\frac{\gamma_i^{n-1}}{2\beta_q^2}\left|f_i'(1)\sin(2\beta_q-\nu\pi)-f_i'(\beta_q^{-1+\varepsilon})\sin(2\beta_q^{\varepsilon}-\nu\pi)\right.$$
$$\left.-\int_{\beta_q^{-1+\epsilon}}^{1}f_i''(x)\sin(2\beta_q x-\nu\pi)dx\right|\le Const\left[|(B\varphi_i,\varphi_i)|\beta_q^{-2+2\varepsilon}\right.$$
$$\left.+\gamma_i^{n-1}\beta_q^{-2}\left(|f_i(0)|+|f_i(1)|+|f_i'(1)|\right)+\gamma_i^{n-1}\beta_q^{-2}\int_0^1|f_i''(x)|dx\right]. \tag{18}$$

Now, let us give a bound the third integral on the right hand side of (16). First we write

$$\left|\int_{\beta_q^{-1+\varepsilon}}^{1}(f_i(x)-f_i(0))\frac{Cos(2\beta_q x-\nu\pi)}{\beta_q x}dx\right|=\left|\int_{\beta_q^{-1+\varepsilon}}^{\delta}(f_i(x)-f_i(0))\frac{Cos(2\beta_q x-\nu\pi)}{\beta_q x}dx+\right.$$
$$\left.+\int_\delta^1(f_i(x)-f_i(0))\frac{Cos(2\beta_q x-\nu\pi)}{\beta_q x}dx\right|.$$

From (1),(2) and (4) we have

$$\gamma_i^{n-1}\left|\int\limits_{\beta_q^{-1+\varepsilon}}^{1}(f_i(x)-f_i(0))\frac{\cos(2\beta_q x-\nu\pi)}{\beta_q x}dx\right| \leq \left|\frac{\gamma_i^{n-1}}{\beta_q^2}\frac{\sin(2\beta_q x-\nu\pi)}{x}(f_i(x)-f_i(0))\right|_{\beta_q^{-1+\varepsilon}}^{\delta}$$
$$-\frac{\gamma_i^{n-1}}{\beta_q^2}\int\limits_{\beta_q^{-1+\varepsilon}}^{\delta}\left(\frac{f_i'(x)}{x}-\frac{f_i(x)-f_i(0)}{x^2}\right)\sin(2\beta_q x\nu\pi)dx|$$
$$+\left|\frac{\gamma_i^{n-1}}{\beta_q^2}\frac{\sin(2\beta_q x-\nu\pi)}{x}(f_i(x)-f_i(0))\right|_{\delta}^{1}-\frac{\gamma_i^{n-1}}{\beta_q^2}\int\limits_{\delta}^{1}\left(\frac{f_i'(x)}{x}-\frac{f_i(x)-f_i(0)}{x^2}\right)\sin(2\beta_q x-\nu\pi)dx\Bigg|$$
$$\leq Const.|(B\varphi_i,\varphi_i)|\beta_q^{-2}+Const.\gamma_i^{n-1}\beta_q^{-2}\left[|f_i(0)|+|f_i(1)|+\int\limits_0^1(|f_i(x)|+|f_i'(x)|)dx\right]. \tag{19}$$

Finally let us give a bound for the forth integral on the right hand side of (16).

$$\begin{aligned}
&\gamma_i^{n-1}\int\limits_{\beta_q^{-1+\varepsilon}}^{1}\frac{|f_i(x)-f_i(0)|}{(\beta_q x)^2}dx=\gamma_i^{n-1}\int\limits_{\beta_q^{-1+\varepsilon}}^{\delta}\frac{|f_i(x)-f_i(0)|}{(\beta_q x)^2}dx\\
&+\gamma_i^{n-1}\int\limits_{\delta}^{1}\frac{|f_i(x)-f_i(0)|}{(\beta_q x)^2}\leq|(B\varphi_i,\varphi_i)|\beta_q^{-2}\ln(\delta\beta_q^{1-\varepsilon})\\
&+Const.\gamma_i^{n-1}\beta_q^{-2}\left[|f_i(0)|+\int\limits_0^1|f_i(x)|dx\right]\\
&\leq Const.\gamma_i^{n-1}\beta_q^{-2+\varepsilon}+Const.\gamma_i^{n-1}\beta_q^{-2}\left[|f_i(0)|+\int\limits_0^1|f_i(x)|dx\right].
\end{aligned} \tag{20}$$

From the relations (16),(17),(18),(19) and (20) we obtain

$$\sum_{q=1}^{\infty}\sum_{i=1}^{\infty}\left|\gamma_i^{n-1}\int\limits_0^1 2x\frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}f_i(x)dx\right|\leq Const.\left(\sum_{q=1}^{\infty}\beta_q^{-2+2\varepsilon}\right)\sum_{i=1}^{\infty}|(B\varphi_i,\varphi_i)|$$
$$+Const.\sum_{q=1}^{\infty}\beta_q^{-2}\sum_{i=1}^{\infty}\gamma_i^{n-1}\left[|f_i(0)|+|f_i(1)|+|f_i'(1)|+\int\limits_0^1(|f_i(x)|+|f_i'(x)|+|f_i''(x)|)\,dx\right]. \tag{21}$$

By assumption, since $B\in\sigma_1(H)$ and $\left\|A^{n-1}Q^{(j)}(x)\right\|_{\sigma_1(H)}\leq Const.$,$(j=0,1,2)$, we have

$$\sum_{i=1}^{\infty}|(B\varphi_i,\varphi_i)|\leq\|B\|_{\sigma_1(H)}. \tag{22}$$

and

$$\begin{aligned}
&\sum_{i=1}^{\infty}\gamma_i^{n-1}\int_0^1\left(|f_i(x)|+|f_i'(x)|+|f_i''(x)|\right)dx\\
&=\lim_{N\to\infty}\int_0^1\left[\sum_{i=1}^{N}\gamma_i^{n-1}\left(|f_i(x)|+|f_i'(x)|+|f_i''(x)|\right)\right]dx \qquad (23)\\
&\le\int_0^1\left[\sum_{i=1}^{\infty}\gamma_i^{n-1}\left(\sum_{j=0}^{2}|\,f_i^{(j)}(x)\,|\right)\right]dx\le\int_0^1\left[\sum_{j=0}^{2}\left\|A^{n-1}Q^{(j)}(x)\right\|_{\sigma_1(H)}\right]dx<Const.
\end{aligned}$$

If we take into account $\lim\limits_{q\to\infty}\frac{\beta_q}{\pi q}=1$, then from (21),(22) and (23) we obtain

$$\sum_{q=1}^{\infty}\sum_{i=1}^{\infty}\left|\gamma_i^{n-1}\int_0^1 2x\frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}(Q(x)\varphi_i,\varphi_i)dx\right|<Const.$$

Thus the theorem is proved. □

The main result of this work is given in the following theorem.

Theorem 2. *If the conditions 1)-4) are satisfied and* $\lim\limits_{i\to\infty}\frac{\gamma_i}{ai^{\alpha}}=1$ $(a>0,\alpha>2)$, *then for the* n^{th} *regularized trace of the operator* L *the formula*

$$\begin{aligned}
&\lim_{p\to\infty}\sum_{k=1}^{m_p}\left\{\lambda_k^n-\mu_k^n-n\sum_{j=2}^{N}(-1)^j j^{-1}\underset{\lambda=\mu_k}{Res}\left[\lambda^{n-1}tr(QR_\lambda^0)^j\right]\right.\\
&\left.-n\sum_{l=0}^{n-2}C_{n-1}^l\beta_{q_k}^{2(n-1-l)}\gamma_{i_k}^l(Q\psi_k,\psi_k)_1\right\}\\
&=-\tfrac{n}{4}\left[2\nu.tr\left(A^{n-1}Q(0)\right)+tr(A^{n-1}Q(1))\right]
\end{aligned}$$

is satisfied, where $N=\left[\left|\frac{2\alpha n+\alpha+6}{\alpha-2}\right|\right]+1$.

Proof. From theorem 1, we observe that the sequence $\sum\limits_{k=1}^{\infty}\gamma_{i_k}^{n-1}(Q\psi_k,\psi_k)_1$ is convergent and

$$\sum_{k=1}^{\infty}\gamma_{i_k}^{n-1}(Q\psi_k,\psi_k)_1=\sum_{i=1}^{\infty}\sum_{q=1}^{\infty}\gamma_i^{n-1}\int_0^1 2x.\frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)}(Q(x)\varphi_i,\varphi_i)_1dx.$$

From these and the equality

$$\sum_{q=1}^{\infty}\int_{0}^{1}2x\frac{J_{\nu}^{2}(\beta_q x)}{J_{\nu+1}^{2}(\beta_q)}(Q(x)\varphi_i,\varphi_i)dx=\frac{1}{4}[2\nu(Q(0)\varphi_i,\varphi_i)+(Q(1)\varphi_i,\varphi_i)]$$

proved in [8], we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\gamma_{i_k}^{n-1}(Q\psi_k,\psi_k)_1&=-\frac{1}{4}\left[2\nu\sum_{i=1}^{\infty}\gamma_i^{n-1}(Q(0)\varphi_i,\varphi_i)+\sum_{i=1}^{\infty}\gamma_i^{n-1}(Q(1)\varphi_i,\varphi_i)\right]\\&=-\frac{1}{4}\big[2\nu.tr(A^{n-1}Q(0))+tr(A^{n-1}Q(1))\big]\end{aligned}\tag{24}$$

By making use of inequality (2), it can be proved that the inequalities

$$\left\|R_\lambda^0\right\|_{\sigma_1(H_1)}\leq Const.m_p^{1-\delta}$$

$$\left\|R_\lambda^0\right\|_1\leq Const.m_p^{-\delta}$$

$$\left\|R_\lambda\right\|_1\leq Const.m_p^{-\delta}$$

are satisfied on the circle $|\lambda|=b_p$where $\delta=\frac{\alpha-2}{\alpha+2}$. Now, let us give a bound for $B_p^{(N)}$ by using these inequalities. From the (8), we find

$$\begin{aligned}B_p^{(N)}|&\leq\frac{1}{2\pi}\int\limits_{|\lambda|=b_p}|\lambda|^n\big|tr\big[R_\lambda(QR_\lambda^0)^{N+1}\big]\big||d\lambda|\\&\leq b_p^n\int\limits_{|\lambda|=b_p}\left\|R_\lambda(QR_\lambda^0)^{N+1}\right\|_{\sigma_1(H_1)}|d\lambda||\end{aligned}$$

$$\leq b_p^n \int\limits_{|\lambda|=b_p} \|R_\lambda\|_1 \left\|(QR_\lambda^0)^{N+1}\right\|_{\sigma_1(H_1)} |d\lambda|$$

$$\leq b_p^n \int\limits_{|\lambda|=b_p} \|R_\lambda\|_1 \left\|(QR_\lambda^0)^{N}\right\|_1 \left\|QR_\lambda^0\right\|_{\sigma_1(H_1)} |d\lambda|$$

$$\leq b_p^n \int\limits_{|\lambda|=b_p} \|R_\lambda\|_1 \|Q\|_1^N \left\|R_\lambda^0\right\|_1^N \|Q_1\| \left\|R_\lambda^0\right\|_{\sigma_1(H_1)} |d\lambda|$$

$$\leq Const.b_p^{n+1} m_p^{-(N+1)\delta} m_p^{1-\delta}.$$

Taking into account $b_p \leq Const.m_p^{1+\delta}$, one then obtains

$$\left|B_p^{(N)}\right| \leq Const.m_p^{(n+1)(1+\delta)-(N+1)\delta+1-\delta}.$$

Hence, if

$$N = \left[\left|\delta^{-1}(n+2+n\delta-\delta)\right|\right] + 1 = \left[\left|\frac{2\alpha n+\alpha+6}{\alpha-2}\right|\right] + 1$$

then

$$\lim_{p\to\infty} B_p^{(N)} = 0.$$

From (13), (24) and this last equality, we obtain

$$\begin{aligned} &\lim_{p\to\infty} \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^{N} (-1)^j j^{-1} \operatorname*{Res}_{\lambda=\mu_k} \left[\lambda^{n-1} tr(QR_\lambda^0)^j\right] \right. \\ &\left. - n \sum_{l=0}^{n-2} C_{n-1}^l \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\} \\ &= -\tfrac{n}{4}\left[2\nu.tr\left(A^{n-1}Q(0)\right) + tr(A^{n-1}Q(1))\right] \end{aligned}$$

Thus, the proof is done. □

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