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On Modified Baskakov Operators on Weighted Spaces*

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Abstract

The author presents a modification of the Baskakov operator for the intervals $[0, b_n]$, where b_n is an increasing sequence of positive numbers with either finite or infinite limit. Convergence properties of such an operator for continuous and differentiable functions in weighted space are established.

Key Words: Baskakov Operator, Weighted Space, Linear Positive Operator

1. Introduction

Baskakov [2] introduced the sequence of linear operators $\{L_n\}$,
$L_n : C[0, \infty) \to C[0, A]$, defined by

$$L_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \varphi_n^{(k)}(x) \frac{(-x)^k}{k!}. \tag{1}$$

Many results on convergence of these operators are known. Moreover, there are some in which the derivatives of the Baskakov operator converge to the derivatives of functions. But all of them are convergence conditions on a finite interval (for example, [1], [7], [8]).

The aim of this paper is to study convergence properties of the modified Baskakov operators in weighted spaces when the interval of convergence grows as $n \to \infty$. For this we will use the weighted Korovkin type theorems, proved by A. D. Gadzhiev [3], [4]. Now, we give Gadzhiev's results in weighted spaces. We use the same notation as in [3].

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Let $\rho(x) = 1 + x^2, -\infty < x < \infty$ and B_ρ be the set of all functions f defined on the real axis satisfying the condition $|f(x)| \leq M_f \rho(x)$, where M_f is a constant depending only on f. B_ρ is a normed space with the norm $||f||_\rho = \sup_{x \in \mathbf{R}} \frac{|f(x)|}{\rho(x)}, f \in B_\rho$. C_ρ denotes the subspace of all continuous functions in B_ρ and C_ρ^k denotes the subspace of all functions $f \in C_\rho$ with $\lim_{|x| \to \infty} f(x)/\rho(x) = k$, where k is a constant depending on f.

Theorem A. Let $\{T_n\}$ be a sequence of linear positive operators taking C_ρ into B_ρ and satisfying the conditions

$$\lim_{n \to \infty} \|T_n(t^\nu, x) - x^\nu\|_\rho = 0, \nu = 0, 1, 2.$$

Then, for any function $f \in C_\rho^k$,

$$\lim_{n \to \infty} \|T_n f - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \backslash C_\rho^k$ such that

$$\lim_{n \to \infty} \|T_n f^* - f^*\|_\rho \geq 1.$$

Applying Theorem A to the operators

$$T_n(f; x) = \begin{cases} V_n(f; x), & x \in [0, a_n] \\ f(x), & x \notin [0, a_n] \end{cases}$$

one then also has the following theorem.

Theorem B ([5]). Let $\{a_n\}$ be a sequence with $\lim_{n \to \infty} a_n = \infty$ and $\{V_n\}$ be a sequence of linear positive operators taking $C_{\rho, [0, a_n]}$ into $B_{\rho, [0, a_n]}$. If for $\nu = 0, 1, 2$

$$\lim_{n \to \infty} \|V_n(t^\nu, x) - x^\nu\|_{\rho, [0, a_n]} = 0,$$

then for any function $f \in C_{\rho, [0, a_n]}^k$

$$\lim_{n \to \infty} \|V_n f - f\|_{\rho, [0, a_n]} = 0,$$

where $B_{\rho, [0, a_n]}, C_{\rho, [0, a_n]}$ and $C_{\rho, [0, a_n]}^k$ denote the same as B_ρ, C_ρ and C_ρ^k, respectively, but the functions are taken on $[0, a_n]$ instead of real axis $\mathbf{R}$ and the norm is taken as

$$\|f\|_{\rho, [0, a_n]} = \sup_{0 \leq x \leq a_n} \frac{|f(x)|}{\rho(x)}.$$

We now give a modification of the Baskakov operator.

Let $\{b_n\}$ be a sequence of positive numbers which has finite or infinite limit and $\{\varphi_n\}$ be a sequence of functions φ_n satisfying the following conditions.

(i) φ_n is analytic on the interval $[0, b_n]$ for each positive integer n;

(ii) $\varphi_n(0) = 1$ for every positive n;

(iii) $(-1)^k \varphi_n^{(k)}(x) \geq 0$ for every positive integer n, $x \in [0, b_n]$ and for every nonnegative integer k;

(iv) There exists an integer m such that for all positive integers k and for some nonnegative integer $n + m$

$$\varphi_n^{(k)}(x) = -n\varphi_{n+m}^{(k-1)}(x)(1 + \alpha_{k,n}(x))$$

where

$$\alpha_{k,n}(0) = O(\frac{1}{n^k}), k = 1, 2, ... \tag{2}$$

and

$$\frac{\varphi_n^{(k)}(0)}{n^k} = (-1)^k + O(\frac{1}{n}). \tag{3}$$

If we replace the interval $[0, A]$ by $[0, b_n]$ in (1) and take φ_n's satisfying (i)-(iv) then we call L_n the modified Baskakov operator.

Note that, if $\lim_{n\to\infty} b_n = A$, then we obtain the Baskakov operators, defined in [2].

2. Convergence in Weighted Spaces

Theorem 1 *For any $f \in C^k_{\rho,[0,b_n]}$ we have $\|L_n f - f\|_{\rho,[0,b_n]} \to 0$ as $n \to \infty$.*

Proof. Since the Baskakov operators L_n satisfy

$$L_n(1; x) = 0, L_n(t; x) = -x\frac{\varphi_n^{'}(0)}{n}, L_n(t^2; x) = \frac{1}{n^2}\left(x^2\varphi_n^{''}(0) - x\varphi_n^{'}(0)\right), x \geq 0,$$

we have $|L_n(\rho; x)| \leq M\rho(x)$,where M is a constant. Thus $\|L_n(f, x)\|_{\rho,[0,b_n]}$is uniformly bounded on $[0, b_n]$. Hence $\{L_n\}$ is the sequence of linear positive operators taking $C_{\rho,[0,b_n]}$ into $B_{\rho,[0,b_n]}$.

Clearly, we can write

$$\lim_{n\to\infty} \|L_n(1,x)-1\|_{\rho,[0,b_n]} = 0.$$

Using properties (iv) and (ii) we get

$$\begin{aligned} L_n(t,x) &= \tfrac{1}{n}(-x\varphi_n^{'}(0)) \\ &= x\varphi_{n+m}(0)(1+\alpha_{1,n}(0)) \\ &= x(1+\alpha_{1,n}(0)), \end{aligned}$$

and by (2)

$$\begin{aligned} \sup_{x\in[0,b_n]} \frac{|L_n(t,x)-x|}{1+x^2} &= \alpha_{1,n}(0) \sup_{x\in[0,b_n]} \frac{x}{1+x^2} \\ &= O(\frac{1}{n}) \sup_{x\in[0,b_n]} \frac{x}{1+x^2}. \end{aligned}$$

Hence we obtain

$$\lim_{n\to\infty} \|L_n(t,x)-x\|_{\rho,[0,b_n]} = 0.$$

Also, the properties (iv) and (ii) show that

$$\begin{aligned} L_n(t^2,x) &= \tfrac{1}{n^2}\left(x^2\varphi_n^{''}(0) - x\varphi_n^{'}(0)\right) \\ &= \tfrac{n+m}{n}x^2(1+\alpha_{1,n+m}(0)(1+\alpha_{2,n}(0)) \\ &\quad +\tfrac{1}{n}x(1+\alpha_{1,n}(0)). \end{aligned}$$

From (2) we can write

$$\begin{aligned} \sup_{x\in[0,b_n]} \tfrac{\left|L_n(t^2,x)-x^2\right|}{1+x^2} &= \left|\tfrac{n+m}{n}-1\right| \sup_{x\in[0,b_n]} \tfrac{x^2}{1+x^2} \\ &\quad +\tfrac{n+m}{n}O(\tfrac{1}{n}) \sup_{x\in[0,b_n]} \tfrac{x^2}{1+x^2} \\ &\quad +O(\tfrac{1}{n}) \sup_{x\in[0,b_n]} \tfrac{x}{1+x^2}. \end{aligned}$$

Thus

$$\lim_{n\to\infty}\left\|L_n(t^2,x)-x^2\right\|_{\rho,[0,b_n]}=0.$$

Therefore, the desired result follows from Theorem B. □

We now want to find the rate of convergence of the sequence of operators $\{L_n(f;x)\}$. It is known that the usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta\to 0$, on infinite interval. Now, we define a weighted modulus of continuity $\Omega_n(f,\delta)$ which tends to zero as $\delta\to 0$ on infinite interval. A similar definition of the modulus of continuity can be found in [6].

Let

$$\Omega_n(f;\delta)=\sup_{|h|\le\delta,x\in[0,b_n]}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)},$$

for each $f\in C^k_{\rho,[0,b_n]}$. We call $\Omega_n(f;\delta)$ the weighted modulus of continuity of the function f on the space $C^k_{\rho,[0,b_n]}$. Since $f\in C^k_{\rho,[0,b_n]}$, there exists a positive real number x_0 such that

$$\begin{aligned}\Omega_n(f,\delta) &\le \sup_{0\le x\le x_0,|h|\le\delta}|f(x+h)-f(x)|\\ &+ \sup_{x_0\le x\le b_n}\left|\tfrac{f(x+h)}{1+(x+h)^2}-k\right|\\ &+ \delta k\sup_{x_0\le x\le b_n}\tfrac{2x+\delta}{1+x^2}\\ &+ \sup_{x_0\le x\le b_n}\left|\tfrac{f(x)}{1+x^2}-k\right|\\ &< \omega(f,\delta)+2\delta k+\varepsilon,\end{aligned}$$

where $\omega(f;\delta)$ is the usual first modulus of continuity of f on the interval $[0,x_0]$ and $\lim\limits_{\delta\to0}\omega(f;\delta)=0$ since f is uniformly continuous on $[0,x_0]$. Consequently,

$$\lim_{\delta\to0}\Omega_n(f;\delta)=0,\text{for every } f\in C^k_{\rho,[0,b_n]}.$$

It is easily seen that

$$\Omega_n(f;m\delta)\le 2m(1+\delta^2)\Omega_n(f;\delta)\text{for any positive integer } m.$$

Thus for any $\lambda > 0$, $\Omega_n(f;\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega_n(f;\delta)$. This property of weighted modulus of continuity $\Omega_n(f,\delta)$ and its definition show that for every $f \in C^k_{\rho,[0,b_n]}$ and $x, t \in [0, b_n]$

$$|f(t) - f(x)| \leq (1+x^2)(1+(t-x)^2)\Omega_n(f;|t-x|)$$

and consequently

$$|f(t) - f(x)| \leq 2\left(\frac{|t-x|}{\delta_n} + 1\right)(1+\delta_n^2)\Omega_n(f;\delta_n)(1+x^2)(1+(t-x)^2). \tag{4}$$

Theorem 2 *If $f \in C^k_{\rho,[0,b_n]}$, then the inequality*

$$\sup_{x\in[0,b_n]} \frac{|L(f;x) - f(x)|}{(1+x^2)^3} \leq K\Omega_n\left(f; n^{-1/4}\right) \tag{5}$$

holds for a sufficiently large n, where K is a constant independent of n.

Proof. Denoting $\varphi_n^{(k)}(x)\frac{(-x)^k}{k!}$ by $P_{k,n}(x)$ we get

$$L_n(f,x) - f(x) = \sum_{k=0}^{\infty}\left[f\left(\frac{k}{n}\right) - f(x)\right]P_{k,n}(x)$$

since

$$\sum_{k=0}^{\infty} P_{k,n}(x) = 1.$$

From (4) we can write

$$|L_n(f,x) - f(x)| \leq 2(1+x^2)(1+\delta_n^2)\Omega_n(f;\delta_n)\sum_{k=0}^{\infty} P_{k,n}(x)S(x) \tag{6}$$

where $S(x) = \left(1 + \frac{|x-\frac{k}{n}|}{\delta_n}\right)\left(1 + \left(x - \frac{k}{n}\right)^2\right)$. Since

$$S(x) \leq \begin{cases} 2(1+\delta_n^2) & \text{if} \left|\frac{k}{n} - x\right| \leq \delta_n \\ 2(1+\delta_n^2)\frac{\left(\frac{k}{n}-x\right)^4}{\delta_n^4} & \text{if} \left|\frac{k}{n} - x\right| \geq \delta_n \end{cases}$$

we obtain for all $x \in [0, b_n]$ and $\frac{k}{n} \in [0, \infty)$

$$S(x) \leq 2(1+\delta_n^2)\left\{1+\frac{\left(\frac{k}{n}-x\right)^4}{\delta_n^4}\right\}. \tag{7}$$

Using the inequality (7) in (6) we obtain

$$\begin{aligned} |L_n(f,x)-f(x)| &\leq 16(1+x^2)\Omega_n(f,\delta_n) \\ &\times \left\{1+\tfrac{1}{\delta_n^4}\sum_{k=0}^{\infty} P_{k,n}(x)\left(\tfrac{k}{n}-x\right)^4\right\}. \end{aligned} \tag{8}$$

Since

$$\sum_{k=0}^{\infty} P_{k,n}(x)\left(\tfrac{k}{n}-x\right)^4 = x^4\left(\frac{\varphi_n^{(4)}(0)}{n^4} - 4\frac{\varphi_n^{'''}(0)}{n^3} + 6\frac{\varphi_n^{''}(0)}{n^2} - 4\frac{\varphi_n^{'}(0)}{n} + 1\right)$$
$$-x^3\left(6\frac{\varphi_n^{'''}(0)}{n^4} + 12\frac{\varphi_n^{''}(0)}{n^3} + 6\frac{\varphi_n^{'}(0)}{n^2}\right) - x^2\left(6\frac{\varphi_n^{''}(0)}{n^4} - 4\frac{\varphi_n^{'}(0)}{n^3}\right) + 2x\frac{\varphi_n^{'}(0)}{n^4},$$

and considering the properties (iv) and (ii) and by simple calculations, this can be written in the form

$$\sum_{k=0}^{\infty} P_{k,n}(x)\left(\frac{k}{n}-x\right)^4 = A_{k,n}^4 x^4 + A_{k,n}^3 x^3 + A_{k,n}^2 x^2 + A_{k,n}^1 x, \tag{9}$$

where

$$\begin{aligned} A_{k,n}^4 &= \tfrac{n(n+m)(n+2m)(n+3m)}{n^4}(1+\alpha_{4,n}(0))(1+\alpha_{3,n+m}(0)) \\ &\quad \times(1+\alpha_{2,n+2m}(0))(1+\alpha_{1,n+3m}(0)) \\ &\quad -4\tfrac{n(n+m)(n+2m)}{n^3}(1+\alpha_{3,n}(0))(1+\alpha_{2,n+m}(0))(1+\alpha_{1,n+2m}(0)) \\ &\quad -4\tfrac{n(n+m)(n+2m)}{n^3}(1+\alpha_{3,n}(0))(1+\alpha_{2,n+m}(0))(1+\alpha_{1,n+2m}(0)) \end{aligned}$$

$$\begin{aligned} A_{k,n}^3 &= 6\tfrac{n(n+m)(n+2m)}{n^4}(1+\alpha_{3,n}(0))(1+\alpha_{2,n+m}(0)) \\ &\quad 6\tfrac{n(n+m)(n+2m)}{n^4}(1+\alpha_{3,n}(0))(1+\alpha_{2,n+m}(0)) \\ &\quad \times(1+\alpha_{1,n+m}(0)) + 6\tfrac{(1+\alpha_{1,n}(0))}{n^2} + 6\tfrac{(1+\alpha_{1,n}(0))}{n} \end{aligned}$$

$$A_{k,n}^2 = 9\frac{n+m}{n^3}(1+\alpha_{2,n}(0))(1+\alpha_{1,n+m}(0)) + 8\frac{1+\alpha_{1,n}(0)}{n^2}$$

$$A_{k,n}^1 = 13\frac{1+\alpha_{1,n}(0)}{n^3} + 12\frac{1+\alpha_{1,n}(0)}{n^2}.$$

Using condition (2) we see that

$$A_{k,n}^j = O(\frac{1}{n}), j = 1,2,3 \text{ and } 4.$$

Substituting the equality (9) in (8) we get

$$|L_n(f,x) - f(x)| \leq 16(1+x^2)\Omega_n(f;\delta_n)\left\{1 + \frac{1}{\delta_n^4}O(\frac{1}{n})\left(x^4+x^3+x^2+x\right)\right\}.$$

Choosing $\delta_n = n^{-1/4}$ we can write

$$|L_n(f,x) - f(x)| \quad \leq \quad 16(1+x^2)\Omega_n(f;\delta_n)\left\{1+x^4+x^3+x^2+x\right\}.$$

Therefore, we obtain $\sup\limits_{x\in[0,b_n]} \frac{|L_n(f;x)-f(x)|}{(1+x^2)^3} \leq K\Omega_n\left(f, n^{-1/4}\right).$ □

Remark 1 *As it is seen in Theorem 1,* $L_n(f;x)$ *converges to* $f(x)$ *in the weighted space,* $C_{\rho,[0,b_n]}$*, whose weight function is* $(1+x^2)$*. But, in Theorem 2 we obtained the rate of convergence for the modified Baskakov operator only in the weighted space* $C_{\rho^3,[0,b_n]}, \rho(x) = 1+x^2$*. Hence it is still an open problem to obtain rate of convergence in the case of the weight function* $(1+x^2)^\alpha, 1 \leq \alpha < 3$ *without any extra condition on function* f*.*

3. Approximation by the derivatives $L_n^{(r)}(f;x)$

Let $C_\rho^{(r)}[0,\infty)$ denotes the set of r times continuously differentiable function on $[0,\infty)$ belonging to $C_{\rho,[0,b_n]}$.

If $\varphi_n(x)$'s satisfy (i)-(iv) and $f \in C_\rho^{(r)}[0,\infty)$, then obviously the series

$$\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\varphi_n^{(k)}(x)\frac{(-x)^k}{k!}$$

is infinitely differentiable on $[0,b_n]$ and

$$L_n^{(r)}(f,x) \quad = \quad \sum_{k=0}^{\infty}(-1)^r \triangle_{n^{-1}}^r f\left(\tfrac{k}{n}\right)\varphi_n^{(k+r)}(x)\tfrac{(-x)^k}{k!} \tag{10}$$

where $\Delta_{n^{-1}}$ denotes the difference operator of the function f with step $1/n$ and $\Delta_{n^{-1}}^r$ is the r-th iterate of this operator [7].

By the mean-value theorem, we can write

$$\Delta^r f\left(\frac{k}{n}\right) = \frac{1}{n^r} f^{(r)}\left(\frac{k+\theta_k r}{n}\right), 0<\theta_k<1.$$

Then, the series (10) has the form

$$L_n^{(r)}(f,x) = \sum_{k=0}^{\infty}(-1)^r f^{(r)}\left(\frac{k+\theta_k r}{n}\right)\varphi_n^{(k+r)}(x)\frac{(-x)^k}{k!}. \tag{11}$$

Theorem 3 *Let $\{\varphi_n\}$ be a sequence of functions satisfying (i)-(iv). If $f \in C_\rho^{(r-1)}[0,\infty)$ and its r-th derivative, $f^{(r)}$, belongs to the Lipschitz class with exponent α and the constant M this is denote by $f^{(r)} \in Lip_\alpha M$, that is*

$$\left|f^{(r)}(x) - f^{(r)}(t)\right| \le M\left|x-t\right|^{\alpha}, 0<\alpha\le 1 \text{ for any } x,t\ge 0,$$

then

$$\lim_{n\to\infty}\sup_{x\in[0,b_n]}\frac{\left|L_n^{(r)}(f;x)-f^{(r)}(x)\right|}{1+x^{\alpha}} = 0.$$

Proof. Let $L_n^{(r)}(f,x)$ be the operators defined in (11). Consider the inequality

$$\begin{aligned}\left|L_n^{(r)}(f;x)-f^{(r)}(x)\right| \quad &\le \quad \sum_{k=0}^{\infty}\left|f^{(r)}\left(\tfrac{k+\theta_k r}{n}\right)-f^{(r)}(x)\right|\tfrac{(-1)^r}{n^r}\varphi_n^{(k+r)}(x)\tfrac{(-x)^k}{k!} \\ &+ \quad \left|\tfrac{(-1)^r}{n^r}\varphi_n^{(r)}(0)-1\right|\left|f^{(r)}(x)\right|.\end{aligned} \tag{12}$$

Since $f^{(r)} \in \mathrm{Lip}_\alpha M$, we get

$$\left| f^{(r)}\left(\frac{k+\theta_k r}{n}\right) - f^{(r)}(x)\right| \leq M \left|\frac{k+\theta_k r}{n} - x\right|^\alpha$$

and

$$\begin{aligned} \left| f^{(r)}(x)\right| &\leq \max\left(M, \left|f^{(r)}(0)\right|\right)(1+x^\alpha) \\ &= M_f(1+x^\alpha) \end{aligned}$$

where M_f is a constant depending on f. Thus, we have

$$\begin{aligned} \left|L_n^{(r)}(f;x) - f^{(r)}(x)\right| &\leq M \sum_{k=0}^{\infty} \left|\tfrac{k+\theta_k r}{n} - x\right|^\alpha \tfrac{(-1)^r}{n^r} \varphi_n^{(k+r)}(x) \tfrac{(-x)^k}{k!} \\ &+ M_f \left|\tfrac{(-1)^r}{n^r}\varphi_n^{(r)}(0) - 1\right|(1+x^\alpha). \end{aligned}$$

Applying the Hölder's inequality, we obtain

$$\begin{aligned} \left|L_n^{(r)}(f;x) - f^{(r)}(x)\right| &\leq M \left(\sum_{k=0}^{\infty} \left(\tfrac{k+r}{n} - x\right)^2 \tfrac{(-1)^r}{n^r} \varphi_n^{(k+r)}(x) \tfrac{(-x)^k}{k!}\right)^{\alpha/2} \\ &+ M_f \left|\tfrac{(-1)^r}{n^r}\varphi_n^{(r)}(0) - 1\right|(1+x^\alpha). \end{aligned}$$

Using property (iv) we can write

$$\begin{aligned} &\left|L_n^{(r)}(f;x) - f^{(r)}(x)\right| \leq M \tfrac{(-1)^r}{n^r}\left[\tfrac{x}{n}(1+\alpha_{2,n}(0))\right. \\ &\left. + \tfrac{m+n}{n}(1+\alpha_{2,n}(0))(1+\alpha_{1,n+m}(0))x^2\right]^{\alpha/2} \\ &+ M_f\left|\tfrac{(-1)^r}{n^r}\varphi_n^{(r)}(0) - 1\right|(1+x^\alpha). \end{aligned}$$

Therefore, considering the conditions (3) and (2), we obtain the result. □

As a consequence of this theorem we also have the following theorem.

Theorem 4 *Let* $f \in C_\rho^{(r-1)}[0,\infty)$ *and let* $f^{(r)} \in Lip_\alpha M$*. Then*

$$\sup_{x\in[0,b_n]} \frac{\left|L_n^{(r)}(f;x) - f^{(r)}(x)\right|}{1+x^\alpha} = O\left(\frac{1}{n}\right), 0 < \alpha \leq 1$$

holds for a sufficiently large n.

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