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On the Centroid of the Prime Gamma Rings II

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Abstract

The aim of this paper is to study the properities of the extended centroid of the prime Γ-rings. Main results are the following theorems: (1) Let M be a simple Γ-ring with unity. Suppose that for some $a \neq 0$ in M we have $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then M is isomorphic onto the Γ-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring D and Γ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring D. Furthermore M is the Γ-ring of all $n \times n$ matrices over the field C_Γ. (2) Let M be a prime Γ-ring and C_Γ the extended centroid of M. If a and b are non-zero elements in $S = M\Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\beta, \gamma \in \Gamma$, then a and b are C_Γ-dependent. (3) Let M be prime Γ-ring, Q quotient Γ-ring of M and C_Γ the extended centroid of M. If q is non-zero element in Q such that $q\gamma_1 x\gamma_2 q\beta_1 y\beta_2 q = q\beta_1 y\beta_2 q\gamma_1 x\gamma_2 q$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then S is a primitive Γ-ring with minimal right (left) ideal such that $e\Gamma S$, where e is idempotent and $C_\Gamma \Gamma e$ is the commuting ring of S on $e\Gamma S$.

Key Words: Γ-division ring, Γ-field, extented centroid, central closure.

1. Introduction

Nobusawa [11] introduced the notion of a Γ-ring, more general than a ring. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

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Barnes [1], Luh [7] and Kyuno [4] studied the structure of Γ-rings and obtained various generalizations analogous of corresponding parts in ring theory. Öztürk and Jun [12] studied the extended centroid of a prime Γ-ring. As a continuation of [12], in this paper, we study further properities of the extended centroid of the prime Γ-rings.

2. Preliminaries

Let M and Γ be two abelian groups. If for all $x, y, z \in M$ and all $\alpha,\ \beta \in \Gamma$ the conditions

(i) $x\alpha y \in M$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)z = x\alpha z + x\beta z, x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call M a Γ-*ring.* By a *right* (resp. *left*) *ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left ideal, then we say that U is an *ideal* of M. For each a of a Γ-ring M the smallest right ideal containing a is called the *principal right ideal generated* by a and is denoted by $\langle a\rangle_r$. Similarly we define $\langle a\rangle_l$ (resp. $\langle a\rangle$), the *principal left* (resp. *two sided*) *ideal generated* by a. An ideal P of a Γ-ring M is said to be *prime* if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal Q of a Γ-ring M is said to be *semi-prime* if for any ideal U of M, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ-ring M is said to be *prime* (resp. *semi-prime*) if the zero ideal is prime (resp. semi-prime).

Theorem 2.1 ([4, Theorem 4]). *If M is a Γ-ring, the following conditions are equivalent:*

(i) *M is a prime Γ-ring.*

(ii) *If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.*

(iii) *If $\langle a\rangle$ and $\langle b\rangle$ are principal ideals in M such that $\langle a\rangle\Gamma\langle b\rangle = (0)$, then $a = 0$ or $b = 0$.*

(iv) *If A and B are right ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

(v) *If A and B are left ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

A Γ-ring M is said to be *simple* if $M\Gamma M \neq 0$ and M has no ideals other 0 and M itself. When a Γ-ring M has the descending (resp. ascending) chain condition for right ideals, it is abbreviated to M has *min-r condition* (resp. *max-r condition*). The terms *min-l condition* or *max-l condition* on a Γ-ring M are likewise defined. Let M be a Γ-ring and let F be the free group generated by $\Gamma \times M$. Then

$$A = \{\sum_i n_i(\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$$

is a subgroup of F. Let $R = F/A$ be the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Clearly, every element of R can be expressed as a finite sum $\sum_i [\gamma_i, x_i]$. Also it can be verified easily that $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ and $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication on R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j].$$

Then R forms a ring. If we define a composition on $M \times R$ into M by

$$a \sum_i [\gamma_i, x_i] = \sum_i a\gamma_i x_i, \ \forall a \in M, \ \forall \sum_i [\gamma_i, x_i] \in R$$

then M is a right R-module, and we call R the *right operator ring* of M. Similarly, we can define the *left operator ring* L of M. A Γ-ring M is said to be *right* (resp. *left*) *primitive* if it satisfies:

(i) the right (resp. left) operator ring of M is a right (resp. left) primitive ring

(ii) $M\Gamma x = 0$ (resp. $x\Gamma M = 0$) implies $x = 0$.

A Γ-ring M is said to be *two-sided primitive* (or simply, *primitive*) if it is both right and left primitive.

Theorem 2.2 ([7, Theorem 3.4]). *If M is a Γ-ring possessing minimal left (resp. right) ideal, then M is primitive if and only if it is prime.*

Theorem 2.3 ([7, Theorem 3.6]). *For a Γ-ring M with min-l condition, the following are equivalent:*

(i) *M is prime,*

(ii) *M is primitive,*

(iii) *M is simple.*

Theorem 2.4 ([7, Theorem 4.2]). *If M is a simple Γ-ring possessing minimal left (resp. right) ideals, then M is a direct sum of minimal left (resp. left) ideals.*

Theorem 2.5 ([5, Theorem 3.23]). *Let M be a semi-prime Γ-ring with min-r condition and let $M = I_1 \oplus I_2 \oplus \cdots \oplus I_m = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where $I_1, I_2, \cdots, I_m, J_1, J_2, \cdots, J_n$ are minimal right ideals. Then $m = n$.*

The integer $m = n$ in Theorem 2.5 is called the *right dimension* of the semi-prime Γ-ring with min-r condition and denoted by $dim(M_R)$. One can define the left dimension of a Γ-ring in a similar way. If M is simple, then M is semi-prime (see [5]). For an additive group G, denote by $G_{m,n}$ the additive group of all matrices over G. Let M be a Γ-ring M and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the sets of $m \times n$ matrices with entries from M and of $n \times m$ matrices with entries from Γ. For $(a_{ij}), (b_{ij}) \in M_{m,n}$ and $(\gamma_{ij}) \in \Gamma_{n,m}$, define $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}$. Then $M_{m,n}$ forms a $\Gamma_{n,m}$-ring.

Theorem 2.6 ([6, Theorem 4.2]). *Let M be a simple Γ-ring with min-r and min-l conditions and $\Gamma_0 = \Gamma/\kappa$, where $\kappa := \{\gamma \in \Gamma \mid M\gamma M = 0\}$. Then the Γ_0-ring M is isomorphic to the Γ'-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring D and Γ' is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring D and $m = dim(M_L)$ and $n = dim(M_R)$.*

Lemma 2.7 ([12, Lemma 3]). *Let M be a prime Γ-ring such that $M\Gamma M \neq M$ and quotient Γ-ring Q of M. Then, for each non-zero $q \in Q$ there is a non-zero ideal U of M such that $q(U) \subset M$.*

Lemma 2.8 ([12, p. 476]). *Let M be a prime Γ-ring such that $M\Gamma M \neq M$ and C_Γ the extended centroid of M. If a_i and b_i are non-zero elements of M such that $\sum a_i\gamma_i x\beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the a_i's (also a_i's) are linearly dependent over C_Γ. Moreover, if $a\gamma x\beta b = b\gamma x\beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a(\neq 0), b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda\alpha a$ for all $\alpha \in \Gamma$.*

3. Centroids

Let M be a prime Γ-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U,f) \mid \quad U(\neq 0) \text{ is an ideal of } M \text{ and } \\ f: U \to M \text{ is a right } M\text{-module homomorphism}\}.$$

Define a relation $\sim$ on $\mathcal{M}$ by $(U,f) \sim (V,g)$ if and only if $\exists W(\neq 0) \subset U \cap V$ such that $f = g$ on W. Since M is a prime Γ-ring, it is possible to find a non-zero W and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We then denote the equivalence class by $Cl(U,f) = \hat{f}$, where $\hat{f} := \{g : V \to M | (U,f) \sim (V,g)\}$, and denote by Q the set of all equivalence classes. Now we define an addition "+" on Q as follows:

$$\hat{f} + \hat{g} = Cl(U,f) + Cl(V,g) = Cl(U \cap V, f+g)$$

where $f+g : U \cap V \to M$ is a right M-module homomorphism. Then Q is an additive abelian group (see [12]). Since $M\Gamma M \neq M$ and since M is a prime Γ-ring, $M\Gamma M$ $(\neq 0)$ is an ideal of M. We can take the homomorphism $1_{M\Gamma} : M\Gamma M \to M$ as a unit M-module homomorphism. Note that $M\beta M \neq 0$ for all $0 \neq \beta \in \Gamma$ so that $1_{M\beta} : M\beta M \to M$ is non-zero M-module homomorphism. Denote

$$\mathcal{N} := \{(M\beta M, 1_{M\beta}) \mid 0 \neq \beta \in \Gamma\},$$

and define a relation "$\approx$" on $\mathcal{N}$ by $(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})$ if and only if $\exists W := M\alpha M(\neq 0) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on W. We can easily check that "$\approx$" is an equivalence relation on $\mathcal{N}$. Denote by $Cl(M\beta M, 1_{M\beta}) = \hat{\beta}$ the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\hat{\Gamma}$ the set of all equivalence classes of $\mathcal{N}$ with respect to $\approx$, that is,

$$\hat{\beta} := \{1_{M\gamma} : M\gamma M \to M \mid (M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})\}$$

and $\hat{\Gamma} := \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition "+" on $\hat{\Gamma}$ as follows:

$$\begin{aligned} \hat{\beta} + \hat{\delta} &= Cl(M\beta M, 1_{M\beta}) + Cl(M\delta M, 1_{M\delta}) \\ &= Cl(M\beta M \cap M\delta M, 1_{M\beta} + 1_{M\delta}) \end{aligned}$$

for every $\beta(\neq 0)$, $\delta(\neq 0) \in \Gamma$. Then $(\hat{\Gamma}, +)$ is an abelian group. Now we define a mapping $(-,-,-) : Q \times \hat{\Gamma} \times Q \to Q$, $(\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g}$, as follows:

$$\begin{aligned} \hat{f}\hat{\beta}\hat{g} &= Cl(U,f)Cl(M\beta M, 1_{M\beta})Cl(V,g) \\ &= Cl(V\Gamma M\beta M\Gamma U, f1_{M\beta}g) \end{aligned}$$

where

$$V\Gamma M\beta M\Gamma U = \{\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i \mid v_i \in V, u_i \in U, m_i, n_i \in M \text{ and } \alpha_i, \gamma_i \in \Gamma\}$$

is an ideal of M and $f1_{M\beta}g : V\Gamma M\beta M\Gamma U \to M$ which is given by

$$f1_{M\beta}g(\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i) = f(\sum g(v_i)\gamma_i m_i\beta n_i\alpha_i u_i)$$

is a right M-module homomorphism. Then Q is a $\hat{\Gamma}$-ring with unity. Noticing that the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$ is an isomorphism, we know that the $\hat{\Gamma}$-ring Q is a Γ-ring (see [12]). For purposes of convenience, we use q instead of $\hat{q} \in Q$.

Definition 3.1. Let M be a Γ-ring with unity. An element u in M is called a *unit* of M if it has a multiplicative inverse in M. If every nonzero emenet of M is a unit, we say that M is a Γ-*division ring.* A Γ-ring M is called a Γ-*field* if it is a commutative Γ-division ring.

Definition 3.2. The set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *extended centroid* of a Γ-ring M.

Lemma 3.3. *Let M be a prime Γ-ring. Then the extended centroid C_Γ of M is a Γ-field.*

Proof. Noticing that C_Γ is a commutative ring with unity, it is sufficient to show that every nonzero element of C_Γ is invertible. If $c(\neq 0) \in C_\Gamma$, then $c = Cl(U, \mu)$. Thus, by Lemma 2.7., there is a nonzero ideal U of M such that $\mu(U) \subset M$. Clearly, $0 \neq V = \mu(U)$ is an ideal of M. Since $U\Gamma M \subset U$, therefore $\mu(U)\Gamma M \subset \mu(U)$. Hence we can define a mapping $f : \mu(U) \to M$ by $f(\mu(u)) = u$ for all $u \in U$, and this is a right M-module homomorphism. In fact, let $v_1, v_2 \in V = \mu(U)$ and so there exists $u_1, u_2 \in U$ such that $v_1 = \mu(u_1)$ and $v_2 = \mu(u_2)$. It follows that

$$\begin{aligned} f(v_1 + v_2) &= f(\mu(u_1) + \mu(u_2)) \\ &= f(\mu(u_1 + u_2)) = u_1 + u_2 \\ &= f(\mu(u_1)) + f(\mu(u_2)) \\ &= f(v_1) + f(v_2). \end{aligned}$$

Now, for any $v \in V$, $m \in M$ and $\gamma \in \Gamma$, we have

$$f(v\gamma m) = f(\mu(u)\gamma m) = f(\mu(u\gamma m)) = u\gamma m = f(\mu(u))\gamma m = f(v)\gamma m.$$

Finally, considering $d = Cl(V, f)$, we get

$$\begin{aligned} d\gamma c &= Cl(V, f)Cl(M\gamma M, 1_{M\gamma})Cl(U, \mu) \\ &= Cl(U\Gamma M\gamma M\Gamma V, f1_{M\gamma}\mu) \\ &= Cl(U\Gamma M\gamma M\Gamma\mu(U), 1) = I. \end{aligned}$$

This completes the proof. □

Definition 3.4. For the extended centroid C_Γ of a prime Γ-ring M, we say that $S := M\Gamma C_\Gamma$ is the *central closure* of M.

Remark 3.5. For $a, b \in S$, if $a\Gamma S\Gamma b = 0$ then $a\Gamma M\Gamma C_\Gamma b = 0$ and so $a\Gamma M\Gamma b\Gamma M\Gamma a\Gamma C_\Gamma b = 0$. Since M is a prime Γ-ring, it follows that $a\Gamma M\Gamma b = 0$ or $a\Gamma C_\Gamma b = 0$ so $a = 0$ or $b = 0$. Thus S is a prime Γ-ring.

If M has a unit element, then $C_\Gamma = Z(S)$, the centre of S. If M is a simple Γ-ring with unity, then $Q = S = M$. Because the only non-zero ideal of M is M itself. In this case; M is its own central closure.

Throughout, we shall use M as a prime Γ-ring such that $M\Gamma M \neq M$.

Theorem 3.6. *Let C_Γ be the extended centroid of a prime Γ-ring M. If a is a nonzero element of M such that $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then $S = M\Gamma C_\Gamma$ is a primitive Γ-ring with minimal right (left) ideal and the commuting ring of S on this right (left) ideal is merely C_Γ itself.*

Proof. Let fixed $a\gamma_1 x\gamma_2 a$ element in the relation $(a\gamma_1 x\gamma_2 a)\beta_1 y\beta_2 a = a\beta_1 y\beta_2(a\gamma_1 x\gamma_2 a) = 0$ then, from Lemma 2.8 we get $a\gamma_1 x\gamma_2 a = \lambda(x)\alpha a$, where $\lambda(x) \in C_\Gamma$ and $\alpha \in \Gamma$ and for all $x \in M$. Similarly we also get $a\beta_1 y\beta_2 a = \lambda(y)\acute{\alpha}a$, where $\lambda(y) \in C_\Gamma$ and $\acute{\alpha} \in \Gamma$ and for all $y \in M$. Thus, since $a\beta_1 y\beta_2 a = \lambda(y)\acute{\alpha}a \in C_\Gamma\Gamma a$ we get $a\Gamma S\Gamma a \subset C_\Gamma\Gamma a$. Since $a \neq 0$ and S is prime Γ-ring, there is some $y_o \in S$ such that $a\beta_1 y_o\beta_2 a \neq 0$ for some $\beta_1, \beta_2 \in \Gamma$. Thus, $a\beta_1 y_o\beta_2 a = \lambda(y_o)\acute{\alpha}a$, where $0 \neq \lambda(y_o) \in C_\Gamma$. Similarly we get $a\gamma_1 x_o\gamma_2 a = \lambda(x_o)\alpha a$, where $0 \neq \lambda(x_o) \in C_\Gamma$. If $x_o = \lambda^{-1}(y_o)\alpha y_o$, then $a\gamma_1 x_o\gamma_2 a = a\gamma_1\lambda^{-1}(y_o)\alpha y_o\gamma_2 a = \lambda^{-1}(y_o)\alpha a\gamma_1 y_o\gamma_2 a = \lambda^{-1}(y_o)\alpha\lambda(y_o\alpha a = a$. Thus, let $e = a\gamma_1 x_o$. $e\gamma_2 e = (a\gamma_1 x_o)\gamma_2(a\gamma_1 x_o) = (a\gamma_1 x_o\gamma_2 a)\gamma_1 x_o = a\gamma_1 x_o = e$. From this we will have e idempotent. In this case; $e\Gamma S\Gamma e = (a\gamma x_o)\Gamma S\Gamma(a\gamma x_o) \subset C_\Gamma\Gamma(a\gamma x_o) = C_\Gamma\gamma e$. Thus $e\Gamma S$ is a minimal right ideal of S and $C_\Gamma\Gamma e$ is the commuting ring of S on $e\Gamma S$ by Lemma 3.3. Since S is prime Γ-ring and has a minimal right ideal. S is primitive Γ-ring by Theorem 2.2. □

Theorem 3.7. *Let M be a simple Γ-ring with unity. Suppose that for some $a \neq 0$ in M we have $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then M is isomorphic onto the Γ-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring D and Γ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring D. Furthermore M is the Γ-ring of all $n \times n$ matrices over the field C_Γ.*

Proof. Since M is simple Γ-ring we have $M = S$ and from Theorem 3.6 we get M has a minimal right (left) ideal of M. In this case, M is the sum of minimal right (left) ideals by Theorem 2.4, that is, M is the sum of minimal right ideals N_i, where $N_i = x_i\Gamma N$ (N is a non-zero minimal right ideal of M) for some $x_i \in M$. Also, since M has unit ($1 \in M$), $1 \in N_1 + ... + N_n$ for some n, we get $M = N_1 + ... + N_n$ and so M is the sum of a finite number of minimal right ideals, each of which is an irredicible right M-module. Thus M, as a M- module, has a composition serises. Thus M has min-r condition and so M is primitive Γ-ring by Theorem 2.3. In this case, by Theorem 3.6, the commuting ring of M on an irreducible module is $C_\Gamma = Z(M)$, the center of M. Thus, this finishes the proof of the theorem by Theorem 2.6. □

Theorem 3.8. *Let M be prime Γ-ring and C_Γ the extended centroid of M. If a and b are non-zero elements in $S = M\Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$, then a and b are C_Γ-dependent.*

Proof. Firstly, we assume that $a \neq 0$ and $b \neq 0$. Let U be a non- zero ideal of

M such that $a\Gamma U \subseteq M$ and $a\Gamma U \subseteq M$, and set $V = U\Gamma a\Gamma U = \{\sum x_i\gamma_i a\beta_i y_i \mid x_i, y_i \in U, \gamma_i, \beta_i \in \Gamma\}$. We define a mapping $f : V \to M$ defined by $v \mapsto f(v) = f(\sum x_i\gamma_i a\beta_i y_i) = \sum x_i\gamma_i b\beta_i y_i$, for all $x_i, y_i \in U$ and $\gamma_i, \beta_i \in \Gamma$. We suppose that $\sum x_i\gamma_i a\beta_i y_i = 0$. Then,

$$\begin{aligned} 0 &= b\alpha_i m\sigma_i \sum x_i\gamma_i a\beta_i y_i = \sum b\alpha_i(m\sigma_i x_i)\gamma_i a\beta_i y_i \\ &= \sum a\alpha_i(m\sigma_i x_i)\gamma_i b\beta_i y_i = a\alpha_i m\sigma_i \sum x_i\gamma_i b\beta_i y_i \end{aligned}$$

Thus, we get, for all $x_i, y_i \in U$ and $\gamma_i, \beta_i \in \Gamma$

$$a\Gamma M\Gamma(\sum x_i\gamma_i b\beta_i y_i) = 0$$

and so since $a \neq 0$ and M is prime Γ-ring we get $\sum x_i\gamma_i b\beta_i y_i = 0$. Therefore, f is well defined. Also, specially $f((x\gamma a\beta y)\alpha m) = x\gamma b\beta y\alpha m = f(x\gamma a\beta y)\alpha m$ for all $x, y \in U$ and $m \in M$ and $\gamma, \beta, \alpha \in \Gamma$ and so f is a M-module homomorphism. Let q denote the element of Q determined by f, that is, $q = Cl(V, f)$. Let p be any element of Q with $p(W) \subseteq M$ for some non-zero ideal W of M by Lemma 2.7. In this case,

$$\begin{aligned} & (f1_{M\alpha}p)(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i) \\ = \;& f(\sum p(w_i)\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i) \\ = \;& \sum p(w_i)\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i b\beta_i y_i \\ = \;& p(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i b\beta_i y_i) \\ = \;& p(1_{M\alpha}f(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i)) \\ = \;& (p1_{M\alpha}f)(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i) \end{aligned}$$

and so $q\alpha p = Cl(W\Gamma M\alpha M\Gamma V,\ f1_{M\alpha}P) = Cl(W\Gamma M\alpha M\Gamma V,\ P1_{M\alpha}f) = p\alpha q$. Thus, we get $q \in C_\Gamma$. For $\gamma,\ \beta,\ \alpha \in \Gamma$,

$$\begin{aligned} q\gamma(x\alpha a\beta y) &= Cl(V, f)Cl(M\gamma M,\ 1_{M\gamma})Cl(\acute{V},\ x\alpha a\beta y) \\ &= Cl(\acute{V}\Gamma M\gamma M\Gamma V,\ f1_{M\gamma}(x\alpha a\beta y)) \\ &= Cl(\acute{V}\Gamma M\gamma M\Gamma V,\ x\alpha a\beta y) \\ &= x\alpha b\beta y, \end{aligned}$$

Hence we have $(x\gamma q\alpha a - x\alpha b)\beta y = 0$ for all $x, y \in U$ and $\gamma, \beta, \alpha \in \Gamma$. Therefore, since M is prime Γ -ring we get $x\gamma q\alpha a - x\alpha b = 0$ for all $x, y \in U$ and $\gamma, \alpha \in \Gamma$. Now writing $\alpha + \gamma$ for in the previous equation we get, $x\gamma(q\gamma a - b) = 0$ for all $\gamma, \in \Gamma$ and $x \in U$. Thus, since M is prime Γ-ring, we get, $q\gamma a = b$ for all $\gamma \in \Gamma$ and so this completes the proof. □

Theorem 3.9. *Let M be prime Γ-ring, Q quotient Γ-ring of M and C_Γ the extended centroid of M. If q is non-zero element in Q such that $q\gamma_1 x\gamma_2 q\beta_1 y\beta_2 q = q\beta_1 y\beta_2 q\gamma_1 x\gamma_2 q$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then S is a primitive Γ-ring with minimal right (left) ideal such that $e\Gamma S$, where e is idempotent and $C_\Gamma \Gamma e$ is the commuting ring of S on $e\Gamma S$.*

Proof. If $q \in M$, then the proof finishes from Theorem 3.6. If $q \in Q$ then one can pick $a \in M$ such that $\acute{q} = q\alpha a$ is a non-zero element of M by Lemma 2.7. Also, $\acute{q}$ satisfies $\acute{q}\gamma_1 x\gamma_2 \acute{q}\beta_1 y\beta_2 \acute{q} = \acute{q}\beta_1 y\beta_2 \acute{q}\gamma_1 x\gamma_2 \acute{q}$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ and so this completes the proof. □

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