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Some Characterization of Curves of Constant Breadth in $\mathbf{E}^n$ Space

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Abstract

In this paper, the concepts concerning the space of constant breadth were extended to E^n-space. An approximate solution of the equation system which belongs to this curve was obtained. Using this solution vectorial expression of the curves of constant breadth was obtained. The relation $\int_0^{2\pi} \widetilde{f}(s)\, ds = 0$ between the curvatures of curves of constant breadth in E^n was obtained.

Key Words and Phrases: Curvature, Constant Breadth, Integral Characterization of Curve

1. Introduction

Curves of constant breadth were introduced by L.Euler [3]. F. Reuleaux gave a method obtaining some curves of constant breadth and has found use in the kinematics of machinary [11]. Some authors have obtained the geometric properties of plane curves of constant breadth [2], [7].

W.Blascke defined the curve of constant breadth on the sphere [1] and M. Fujivara had obtained a problem to determine whether there exist "space curve of constant breadth" or not, and he defined "breadth" for space curves and obtained these curves on a surface of constant breadth [4]. Ö. Köse presented some concepts for space curves of constant breadth [8]. M. Sezer investigated differential equations characterizing space curves of constant breadth and gave a criterion for these curves [12]. A.R. Forsyt had given the theory of curves in E^4 [5]. The curves of constant breadth were extented to the E^4-space and some characterizations were obtained by [9].

Definition 1 *In E^n Euclidean space, if a normal hyperplane on a $X(s) = P$ point of a simple closed (C) curve meets (C) curve at single Q point other then P point, Q point is called opposit point of P.*

In E^n Euclidean space, if the distance between opposite points of simple closed C curve is constant, this curve is called the curve of constant breadth.

In this paper, this kind of curves were extented to E^n-space and some characterizations were obtained.

2. The Curves of Constant Breadth

Let $\vec{X} = \vec{X}(s)$ be a simple closed curve in E^n-space. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other then P. We call the point Q the opposite point of P. We consider a curve having parallel tangents $\vec{V}_1$ and $\vec{V}_1^*$ in opposite directions at the opposite points X and X^* of the curve. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented by the quation

$$\vec{X}^*(s) = \vec{X}(s) + \sum_{i=1}^{n} m_i(s)V_i(s), \tag{1}$$

where $\vec{X}$ and $\vec{X}^*$ are opposite points and $\vec{V}_i$ denote the Frenet-Serret frame in E^n-space. We have from equation (1)

$$\begin{aligned}
\frac{dX^*}{ds} &= \frac{dX^*}{ds^*}\frac{ds^*}{ds} = V_1^*\frac{ds^*}{ds} \\
&= (1 + \frac{dm_1}{ds} - m_2k_1)V_1 + (\frac{dm_2}{ds} + m_1k_1 - m_3k_2)V_2 \\
&+ (\frac{dm_3}{ds} + m_2k_2 - m_4k_3)V_3 + (\frac{dm_4}{ds} + m_3k_3 - m_5k_4)V_4 + \cdots \qquad (2) \\
&+ (\frac{dm_{n-1}}{ds} + m_{n-2}k_{n-2} - m_nk_{n-1})V_{n-1} + (\frac{dm_n}{ds} + m_{n-1}k_{n-1})V_n,
\end{aligned}$$

where k_i is the curvatures of the curve. Since $\vec{V_1^*} = -\vec{V_1}$, we obtain

$$\left.\begin{array}{l} -\quad \frac{ds^*}{ds} = 1 + \frac{dm_1}{ds} - m_2 k_1 \\ \frac{dm_2}{ds} + m_1 k_1 - m_3 k_2 = 0 \\ \frac{dm_3}{ds} + m_2 k_2 - m_4 k_3 = 0 \\ \frac{dm_4}{ds} + m_3 k_3 - m_5 k_4 = 0 \\ \quad\vdots \\ \frac{dm_{n-1}}{ds} + m_{n-2} k_{n-2} - m_n k_{n-1} = 0 \\ \frac{dm_n}{ds} + m_{n-1} k_{n-1} = 0. \end{array}\right\} \tag{3}$$

If we call ϕ as the angle between the tangent of the curve (C) at point $X(s)$ with a given fixed direction and consider $\frac{d\phi}{ds} = k_1$, we can rewrite equation (3) as

$$\left.\begin{array}{l} \frac{dm_1}{d\phi} = m_2 - f(\phi) \\ \frac{dm_2}{d\phi} = -m_1 + \rho k_2 m_3 \\ \frac{dm_3}{d\phi} = -\rho k_2 m_2 + \rho k_3 m_4 \\ \quad\vdots \\ \frac{dm_{n-1}}{d\phi} = -\rho k_{n-2} m_{n-2} + \rho k_{n-1} m_n \\ \frac{dm_n}{d\phi} = -\rho k_{n-1} m_{n-1}, \end{array}\right\} \tag{4}$$

where $f(\phi) = \rho + \rho^*$, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1^*}$ denote the radii of curvatures at X and X^*, respectively. If $\frac{dm_i}{d\phi} = m_i'$, then equation (4) can be written as

$$\left.\begin{array}{l} m_1' = m_2 - f \\ m_2' = -m_1 + \rho k_2 m_3 \\ m_3' = -\rho k_2 m_2 + \rho k_3 m_4 \\ \quad\vdots \\ m_{n-1}' = -\rho k_{n-2} m_{n-2} + \rho k_{n-1} m_n \\ m_n' = -\rho k_{n-1} m_{n-1} \end{array}\right\} \tag{5}$$

where the functions ρ and k_i are a function of ϕ and

$$\left.\begin{array}{l} m_2 = h_1^{-1}(m_1' + h_0 f) \\ m_3 = h_2^{-1}(m_2' + h_1 m_1) \\ m_4 = h_3^{-1}(m_3' + h_2 m_2) \\ \quad\vdots \\ m_{n-1} = h_{n-2}^{-1}(m_{n-2}' + h_{n-3} m_{n-3}) \\ m_n = h_{n-1}^{-1}(m_{n-1}' + h_{n-2} m_{n-2}) \end{array}\right\} \tag{6}$$

where $h_0 = 1$ and $h_i^{-1} = \frac{1}{\rho k_i}$, $\quad (i = 1, 2, 3, \ldots, n-1)$. From first equation of equation system (6) we obtain

$$m_2' = h_1^{-1} m_1'' + (h_1^{-1})' m_1' + (h_1^{-1} f)'.$$

Here if $a_{22} = h_1^{-1}$, $\quad a_{21} = (h_1^{-1})'$ and $A_1 = (h_1^{-1} f)'$, then

$$m_2' = a_{22} m_1'' + a_{21} m_1' + A_1 \tag{7}$$

having second derivative this will be

$$m_2'' = a_{22} m_1^{(3)} + (a_{22}' + a_{21}) m_1^{(2)} + a_{21}' m_1^{(1)} + A_1'. \tag{8}$$

If we put equation (8) in the derivation of second equation of equation system (6), we get the following equation

$$\begin{aligned} m_3' \quad &= h_2^{-1} a_{22} m_1^{(3)} + [(h_2^{-1} a_{22})' + h_2^{-1} a_{21}] m_1^{(2)} \\ &+ [(h_2^{-1} a_{21})' + h_2^{-1} h_1] m_1^{(1)} + h_2^{-1} h_1 m_1 + (h_2^{-1} A_1)'. \end{aligned} \tag{9}$$

After using abbreviations $a_{33} = h_2^{-1} a_{22}, a_{32} = (h_2^{-1} a_{22})' + h_2^{-1} a_{21}, a_{31} = (h_2^{-1} a_{21})' + h_2^{-1} h_1, a_{30} = a_{20}', a_{20} = h_2^{-1} h_1, A_2 = (h_2^{-1} A_1)'$ the equation (9) is shortened as below

$$m_3' = a_{33} m_1^{(3)} + a_{32} m_1^{(2)} + a_{31} m_1^{(1)} + a_{30} m_1 + A_2. \tag{10}$$

Second derivation of this equation will be

$$m_3'' = a_{33} m_1^{(4)} + (a_{33}' + a_{32}) m_1^{(3)} + (a_{32}' + a_{31}) m_1^{(2)} + (a_{31}' + a_{30}) m_1^{(1)} + a_{30}' m_1 + A_2'. \tag{11}$$

If we put equation (11) in the derivation of third equation of equation system (6), we get following equation

$$\begin{aligned} m_4' &= h_3^{-1}a_{33}m_1^{(4)} + [(h_3^{-1}a_{33})' + h_3^{-1}a_{32}]m_1^{(3)} \\ &+[(h_3^{-1}a_{32})' + h_3^{-1}a_{31} + h_3^{-1}h_2a_{22}]m_1^{(2)} \\ &+[(h_3^{-1}a_{31})' + h_3^{-1}a_{30} + h_3^{-1}h_2a_{21} + h_3^{-1}h_2h_1^{-1}]m_1^{(1)} \\ &+(h_3^{-1}a_{30})'m_1 + (h_3^{-1}A_2)' + h_3^{-1}h_2A_1 + h_3^{-1}h_2h_1^{-1}f. \end{aligned} \tag{12}$$

Here if we do following abbreviations

$$\begin{aligned} a_{44} &= h_3^{-1}a_{33}, a_{43} = (h_3^{-1}a_{33})' + h_3^{-1}a_{32}, a_{42} = (h_3^{-1}a_{32})' + h_3^{-1}a_{31} + h_3^{-1}h_2a_{22} \\ a_{41} &= (h_3^{-1}a_{31})' + h_3^{-1}a_{30} + h_3^{-1}h_2a_{21} + h_3^{-1}h_2h_1^{-1}, a_{40} = (h_3^{-1}a_{30})' \\ A_3 &= (h_3^{-1}A_2)' + h_3^{-1}h_2A_1 + h_3^{-1}h_2h_1^{-1}f, \end{aligned}$$

the equation (12) is turned to

$$m_4' = a_{44}m_1^{(4)} + a_{43}m_1^{(3)} + a_{42}m_1^{(2)} + a_{41}m_1^{(1)} + a_{40}m_1 + A_3. \tag{13}$$

If this process is going on like that, then derivation of m_n can be written by the power of derivation of m_1 as following

$$\begin{aligned} m_n' &= h_{n-1}^{-1}a_{(n-1)(n-1)}m_1^{(n)} + [(h_{n-1}^{-1}a_{(n-1)(n-1)})' \\ &+h_{n-1}^{-1}a_{(n-1)(n-2)}]m_1^{(n-1)} + \cdots + [(h_{n-1}^{-1}a_{(n-1)0})' + \cdots \\ &+(h_{n-1}^{-1}h_{n-2})'h_{n-3}^{-1}h_{n-4}h_{n-5}^{-1}\cdots h_4h_3^{-1}a_{30}]m_1 + (h_{n-1}^{-1}A_{n-2})' \\ &+h_{n-1}^{-1}h_{n-2}A_{n-3} + (h_{n-1}^{-1}h_{n-2})'h_{n-3}^{-1}h_{n-4}h_{n-5}^{-1}\cdots h_3^{-1}h_2h_1^{-1}A_0 \end{aligned} \tag{14}$$

where $A_0 = f$. If $a_0, a_1, \ldots, a_n$ stand for coefficients of $m_1^{(n)}, m_1^{(n-1)}, \ldots, m_1', m_1$ respectively, then the equation (14) will be

$$a_0m_1^{(n)} + a_1m_1^{(n-1)} + \cdots + a_{n-1}m_1' + a_nm_1 = \widetilde{f} \tag{15}$$

where

$$\begin{aligned} \widetilde{f} &= -[(h_{n-1}^{-1}A_{n-2})' + h_{n-1}^{-1}h_{n-2}A_{n-3} + \cdots \\ &+(h_{n-1}^{-1}h_{n-2})'h_{n-3}^{-1}h_{n-4}h_{n-5}^{-1}\cdots h_5^{-1}h_4h_3^{-1}h_2h_1^{-1}A_0]. \end{aligned}$$

If we choose $\widetilde{a_i} = -\frac{a_i}{a_0}, (i = 1, 2, \ldots, n)$, then equation (15) will be written as

$$m_1^{(n)} = \widetilde{a_1}m_1^{(n-1)} + \cdots + \widetilde{a_n}m_1 + \frac{\widetilde{f}(\phi)}{a_0(\phi)}.$$

Having successively integration, we obtain

$$\begin{aligned}
m_1^{(n-1)} &= \int_0^\phi (\widetilde{a_1}(s)m_1^{(n-1)} + \cdots + \widetilde{a_n}(s)m_1)\,ds + \int_0^\phi \tfrac{\widetilde{f}(s)}{a_0(s)}\,ds\\
m_1^{(n-2)} &= \int_0^\phi (\phi - s)(\widetilde{a_1}(s)m_1^{(n-1)} + \cdots + \widetilde{a_n}(s)m_1)\,ds + \int_0^\phi (\phi - s)\tfrac{\widetilde{f}(s)}{a_0(s)}\,ds\\
m_1^{(n-3)} &= \int_0^\phi \tfrac{(\phi-s)^2}{2!}(\widetilde{a_1}(s)m_1^{(n-1)} + \cdots + \widetilde{a_n}(s)m_1)\,ds + \int_0^\phi \tfrac{(\phi-s)^2}{2!}\tfrac{\widetilde{f}(s)}{a_0(s)}\,ds\\
&\vdots\\
m_1 &= \int_0^\phi \tfrac{(\phi-s)^n}{n!}(\widetilde{a_1}(s)m_1^{(n-1)} + \cdots + \widetilde{a_n}(s)m_1)\,ds + \int_0^\phi \tfrac{(\phi-s)^n}{n!}\tfrac{\widetilde{f}(s)}{a_0(s)}\,ds.
\end{aligned}$$

Here it is imposible to get m_1 directly, but by using an approximate solution of system as [10], we get the following solution

$$m_{1,k} = \int_0^\phi \frac{(\phi - s)^n}{n!}(\widetilde{a_1}(s)m_{1,k-1}^{(n-1)} + \cdots + \widetilde{a_n}(s)m_{1,k-1})\,ds + \int_0^\phi \frac{(\phi - s)^n}{n!}\frac{\widetilde{f}(s)}{a_0(s)}\,ds.$$

and then

$$\begin{aligned}
m'_{1,k} &= \int_0^\phi \tfrac{(\phi-s)^{n-1}}{(n-1)!}(\widetilde{a_1}(s)m_{1,k-1}^{(n-1)} + \cdots + \widetilde{a_n}(s)m_{1,k-1})\,ds + \int_0^\phi \tfrac{(\phi-s)^{n-1}}{(n-1)!}\tfrac{\widetilde{f}(s)}{a_0(s)}\,ds\\
&\vdots\\
m_{1,k}^{(n)} &= \widetilde{a_1}(\phi)m_{1,k-1}^{(n-1)} + \cdots + \widetilde{a_n}(\phi)m_{1,k-1} + \tfrac{\widetilde{f}(\phi)}{a_0(\phi)}.
\end{aligned}$$

We now calculate how $m_{1,k}$ approximate to m_1. If $\varepsilon_{1,k} = \|m_1 - m_{1,k}\|$, then

$$\begin{aligned}
\varepsilon_{1,k} &\leq \int_0^\phi \tfrac{(\phi-s)^n}{n!}[\|\widetilde{a_1}\|\varepsilon_{1,k-1}^{(n-1)} + \cdots + \|\widetilde{a_n}\|\varepsilon_{1,k-1}]\,ds\\
&\vdots\\
\varepsilon_{1,k}^{(n-1)} &\leq \int_0^\phi [\|\widetilde{a_1}\|\varepsilon_{1,k-1}^{(n-1)} + \cdots + \|\widetilde{a_n}\|\varepsilon_{1,k-1}]\,ds\\
\varepsilon_{1,k}^{(n)} &\leq \|\widetilde{a_1}\|\varepsilon_{1,k-1}^{(n-1)} + \cdots + \|\widetilde{a_n}\|\varepsilon_{1,k-1}.
\end{aligned}$$

If $L_1 = max\{\|\widetilde{a_1}\|, \dots, \|\widetilde{a_n}\|\}$, then

$$\begin{array}{ll} \varepsilon_{1,k} & \leq L_1 \int_0^\phi \frac{(\phi-s)^n}{n!} (\sum_{j=0}^{n-1} \varepsilon_{1,k-1}^{(j)})\, ds \\ \\ \varepsilon'_{1,k} & \leq L_2 \int_0^\phi \frac{(\phi-s)^{n-1}}{(n-1)!} (\sum_{j=0}^{n-1} \varepsilon_{1,k-1}^{(j)})\, ds \\ \vdots & \\ \varepsilon_{1,k}^{(n-1)} & \leq L_{n-1} \int_0^\phi (\sum_{j=0}^{n-1} \varepsilon_{1,k-1}^{(j)})\, ds. \end{array}$$

We can sum up above equations side by side in order to obtain following equation

$$\sum_{j=0}^{n-1} \varepsilon_{1,k}^{(j)} \leq \int_0^\phi [L_1 \frac{(\phi-s)^n}{n!} + \cdots + L_{n-1}] \sum_{j=0}^{n-1} \varepsilon_{1,k-1}^{(j)})\, ds.$$

In this case, if we say $\delta_k = \sum_{j=0}^{n-1} \varepsilon_{1,k}^{(j)}$, then we can write

$$\delta_k \leq \int_0^\phi [\sum_{j=0}^{n-1} L_j \frac{(\phi-s)^{n-j}}{(n-j)!} \delta_{k-1}]\, ds$$

and if

$$\sum_{j=0}^{n-1} L_j \frac{(\phi-s)^{n-j}}{(n-j)!} \leq L$$

then we can write

$$\delta_k \leq L \int_0^\phi \delta_{k-1} ds.$$

we now show approximate of δ_k. To do this

$$\begin{array}{ll} \delta_1 & \leq L\delta_0\phi \\ \delta_2 & \leq L \int_0^\phi \delta_1 ds \leq L^2 \delta_0 \frac{\phi^2}{2!} \\ \vdots & \\ \delta_k & \leq \delta_0 \frac{(L\phi_0)^k}{k!} \quad (0 \leq \phi \leq \phi_0) \end{array}$$

thus

$$\sum_{j=0}^{n-1} \varepsilon_{1,k}^{(j)} \leq \delta_0 \frac{(L\phi_0)^k}{k!}.$$

Each term is smaller then $\delta_0 \frac{(L\phi_0)^k}{k!}$ since summation of all terms is smaller then $\delta_0 \frac{(L\phi_0)^k}{k!}$. That is,

$$\varepsilon_{1,k}^{(j)} \leq \delta_0 \frac{(L\phi_0)^k}{k!}$$

and then

$$\begin{array}{ll} \|m_1 - m_{1,k}\| & \leq \delta_0 \frac{(L\phi_0)^k}{k!} \\ \|m_1' - m_{1,k}'\| & \leq \delta_0 \frac{(L\phi_0)^k}{k!} \\ \vdots & \\ \|m_1^{(n-1)} - m_{1,k}\| & \leq \delta_0 \frac{(L\phi_0)^k}{k!} \end{array}$$

and

$$\varepsilon_{1,k}^{(n)} \leq L_n \delta_0 \frac{(L\phi_0)^k}{k!}$$

so

$$\|m_1^{(n)} - m_{1,k}^n\| \leq L_n \delta_0 \frac{(L\phi_0)^k}{k!}.$$

We take limit both side

$$\lim_{k\to\infty} \varepsilon_{i,k}^{(j)} \leq \lim_{k\to\infty} \delta_0 \frac{(L\phi_0)^k}{k!}$$

we get

$$\lim_{k\to\infty} m_{1,k}^{(j)} = m_1^{(j)} \quad (\quad j = 0, 1, 2, \cdots, n-1)$$

and

$$\lim_{k\to\infty} m_{1,k}^{(n)} = m_1^{(n)}.$$

We have that

$$\left.\begin{array}{ll} m_2 & = m_1' + f \\ m_3 & = h_2^{-1}(m_2' + h_1 m_1) \\ m_4 & = h_3^{-1}(m_3' + h_2 m_2) \\ \vdots & \\ m_n & = h_{n-1}^{-1}(m_{n-1}' + h_{n-2} m_{n-2}) \end{array}\right\} \tag{16}$$

and then

$$\begin{aligned}
\varepsilon_{2,k} &= \varepsilon'_{1,k} \\
\varepsilon_{3,k} &= k_2^{-1}\varepsilon'_{2,k} + k_1\varepsilon_{1,k} \\
\varepsilon_{4,k} &= k_3^{-1}\varepsilon'_{3,k} + k_2\varepsilon_{2,k} \\
&\vdots \\
\varepsilon_{n,k} &= k_{n-1}^{-1}\varepsilon'_{n-1,k} + k_{n-2}\varepsilon_{n-2,k}
\end{aligned}$$

Therefore, we can find the value of $m_2, m_3, \ldots, m_n$ from equation (16) since the value of m_1 is known. Thus, the error of $\varepsilon_{2,k}, \varepsilon_{3,k}, \cdots, \varepsilon_{n,k}$ can be found by depending on the error of $\varepsilon_{1,k}$. If $k \to \infty$, then $\vec{X}^*$ can be presented by $\vec{X}$ and its invarient. So the following theorem is proved.

Theorem 1 *Let $k_i(s)$, $(i = 1, 2, \ldots, n; 0 \leq s \leq L)$ be non-zero functions in the class Γ. Then $m_{i,k}(k = 0, 1, 2, \ldots)$ which obtained from (4) limits to the unique solution of system as following*

$$\|m_1^{(i)} - m_{1,k}^{(i)}\| \leq L_i\delta_0\frac{(L\phi_0)^k}{k!} \quad (i = 0, 1, \ldots, n; k = 0, 1 \ldots),$$

$$\|m_\ell^{(i)} - m_{\ell,k}^{(i)}\| \leq L_{\ell,i}^*\delta_0\frac{(L_\ell^*\phi_0)^k}{k!} \quad (\ell = 2, 3, \ldots, n),$$

where $L_i, L, L_{\ell,i}^*$ and L_ℓ^* are numbers which are known.

If the distance between the opposite points of (C) and (C^*) is constant, then

$$\|\alpha^* - \alpha\|^2 = m_1^2 + m_2^2 + \cdots + m_n^2 = k^2, k \in \Re.$$

Hence, we write

$$m_1\frac{dm_1}{d\phi} + m_2\frac{dm_2}{d\phi} + \cdots + m_n\frac{dm_n}{d\phi} = 0 \tag{17}$$

and then

$$m_1(\frac{dm_1}{d\phi} - m_2) = 0. \tag{18}$$

In this case, either $m_1 = 0$ or $\frac{dm_1}{d\phi} - m_2 = 0$ which means $f(\phi) = 0$. If $f(\phi) = 0$, then $\vec{X}^*$ is a transition of $\vec{X}$ with following constant vector

$$\ell = m_1 V_1 + m_2 V_2 + \cdots + m_n V_n. \tag{19}$$

If $m_1 = 0$, then from equation (14) we have

$$\begin{aligned} &(h_{n-1}^{-1} A_{n-2})' + h_{n-1}^{-1} h_{n-2} A_{n-3} + \cdots \\ &+ (h_{n-1}^{-1} h_{n-2})' h_{n-3}^{-1} h_{n-4} h_{n-5}^{-1} \cdots h_5^{-1} h_4 h_3^{-1} h_2 h_1^{-1} A_0 = 0. \end{aligned} \tag{20}$$

Thus, the function $f(\phi)$ satisfies the equation (20). $\vec{X}^*(0) = \vec{X}^*(2\pi)$ since the curve of constant breadth is closed. Accordingly, from (1) we can write

$$\begin{aligned} \alpha^*(0) &= \alpha(0) + \textstyle\sum_{i=2}^{n} m_i(0) V_i(0) = \alpha^*(2\pi) \\ &= \alpha(2\pi) + \textstyle\sum_{i=2}^{n} m_i(2\pi) V_i(2\pi) \end{aligned}$$

and finally $m_i(0) = m_i(2\pi)$.

Corollary 1 *In the equation*

$$m_{1,k} = \int_0^\phi \frac{(\phi - s)^n}{n!} (\widetilde{a_1}(s) m_{1,k-1}^{(n-1)} + \cdots + \widetilde{a_n}(s) m_{1,k-1})\, ds + \int_0^\phi \frac{(\phi - s)^n}{n!} \frac{\widetilde{f}(s)}{a_0(s)}\, ds$$

if we choose $m_{1,0} = 0$, then

$$m_{1,1} = \int_0^\phi \frac{(\phi - s)^n}{n!} \frac{\widetilde{f}(s)}{a_0(s)}\, ds.$$

Assume that $m_{1,k}$ limits for $k = 1$ to m_1. Then $m_{1,1} = 0 = m_1$ and then

$$\int_0^\phi \frac{(\phi - s)^n}{n!} \frac{\widetilde{f}(s)}{a_0(s)}\, ds = 0. \tag{21}$$

Corollary 2 *If we choose $m_{1,1} = 0$, then*

$$m_{1,2} = \int_0^\phi \widetilde{a_n}(s) ds + \int_0^\phi \frac{(\phi - s)^n}{n!} \frac{\widetilde{f}(s)}{a_0(s)}\, ds.$$

Assume that $m_{1,k}$ *limits for* $k = 2$ *to* m_1*. Then* $m_{1,2} = m_1 = 0$ *and then*

$$\int_0^\phi \widetilde{a_n}(s)ds = -\int_0^\phi \frac{(\phi - s)^n}{n!}\frac{\widetilde{f}(s)}{a_0(s)}\,ds.$$

By derivating

$$a_n(\phi) = a_0(\phi)\int_0^\phi \frac{(\phi - s)^{n-1}}{(n-1)!}\frac{\widetilde{f}(s)}{a_0(s)}\,ds$$

Corollary 3 *If we choose* $m_1 = 0$ *in equation (14),then*

$$m_n' = -\widetilde{f}$$

and

$$m_n = -\int_0^\phi \widetilde{f}ds.$$

We have $m_n(0) = m_n(2\pi)$*,so*

$$\int_0^{2\pi} \widetilde{f}(s)\,ds = 0.$$

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