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A General Fixed Point Theorem for Weakly Compatible Mappings in Compact Metric Spaces

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Abstract

A general fixed point theorem for weakly compatible mappings satisfying an implicit relation in compact metric spaces is proved generalizing the results by [1],[3],[13],[14] and others.

Key words and phrases: compact metric space,compatible mappings of type (A), compatible mappings of type (P), compatible mappings,weakly compatible mappings,implicit relation.

1. Introduction

Let S and T be self mappings of a metric space (X,d). Sessa [11] defines S and T to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all x in X. Jungck [2] defines S and T to be compatible if

$$\lim d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim Sx_n = \lim Tx_n = t$$

for some $t \in X$.Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible,but neither implications is reversible [12 ,Ex.1] and

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[2,Ex.2.2]. Recently,Jungck et al. [5] defines S and T to be compatible of type (A) if

$$\lim d(TSx_n, SSx_n) = 0$$

and

$$\lim d(STx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim Sx_n = \lim Tx_n = t$$

for some $t \in X$. Clearly, weakly commuting mappings are compatible of type (A). By [5 ,Ex.2.2] follows that the implication is not reversible. By [5 ,Ex.2.1 and 2.2] follows that the notions of compatible mappings and compatible mappings of type (A) are independent. In [10] the concept of compatible mappings of type (P) was introduced and compared with compatible mappings of type (A) and compatible mappings. S and T are compatible of type (P) if

$$\lim d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim Sx_n = \lim Tx_n = t$$

for some $t \in X$.

Lemma 1 [2] (resp. [5],[9]). Let f and g be compatible (resp. compatible of type (A),compatible of type (P)) self mappings on a metric space (X,d). If f(t)=g(t) for some $t \in X$, then fg(t)=gf(t).

Lemma 2 [5] (resp. [9]). Let $S, T : (X, d) \to (X, d)$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (A) (resp. compatible of type (P)).

In 1994 , Pant [6] introduced the notion of R-weakly commuting mappings. Two self mappings A and S of a metric space (X,d) are called R-weakly commuting at a point $x \in X$ if $d(ASx, SAx) \le Rd(Ax, Sx)$ for some $R > 0$. The mappings A and S are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that

$d(ASx, SAx) \leq Rd(Ax, Sx)$. It is proved in [7] that the notion of pointwise R-weak commutativity is equivalent to commutativity in coincidence points.
Recently,Jungck [4] defined S and T to be weakly compatible if Sx=Tx implies STx=TSx. Thus S and T are weakly compatible if and only if S and T are pointwise R-weakly commuting mappings. However as shown in [8] there exist weakly compatible mappings which are not compatible.
By Lemma 1 it follows that if S and T are compatible (resp. compatible of type (A),compatible of type (P)) then S and T are weakly compatible.
The following example from [8] is an example of a weakly compatible mappings which are not compatible of type (A) (resp. compatible of type (P)).
Let X=[2,20] with the usual metric. Define

T= $\begin{cases} 2 \text{ if } x = 2 \\ 12 + x \text{ if } 2 < x \leq 5 \\ x - 3 \text{ if } x > 5 \end{cases}$; S= $\begin{cases} 2 \text{ if } x \in 2 \cup (5, 20] \\ 8 \text{ if } 2 < x \leq 5 \end{cases}$

S and T are weakly compatible since they commute at their coincidence points. To see that S and T are not compatible of type (A)(resp. compatible of type (P)) let us consider a decreasing sequence $\{x_n\}$ such that

$$\lim x_n = 5.$$

Then $Tx_n = x_n - 3 \to 2; Sx_n = 2; STx_n = S(x_n - 3) = 8$ and
$TTx_n = T(x_n - 3) = 12 + x_n - 3 \to 14$,that is

$$\lim d(STx_n, TTx_n) = 6 \neq 0$$

and hence S and T are noncompatible of type (A). $SSx_n = S(2) = 2$ and

$$\lim d(SSx_n, TTx_n) = d(2, 14) = 12 \neq 0$$

and hence S and T are noncompatible of type (P).
Lemma 3. Two continuous self maps of a compact metric space are compatible (resp.compatible of type (A),compatible of type (P)) if and only if they are weakly compatible.

2. Implicit Relations

Let $\mathcal{F}^*$ be the set of real functions $F(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:
(F_1^*) : F is non increasing in variables t_5 and t_6,
(F_2^*) : For every $u \geq 0, v > 0$

$$(F_a^*) : F(u, v, v, u, u+v, 0) < 0 \text{ or}$$
$$(F_b^*) : F(u, v, u, v, o, u+v) < 0$$

we have $u < v$.
(F_3^*) : $F(u, u, o, o, u, u) \geq 0, \forall u > 0$.
Ex.1. $F(t_1, ..., t_6) = t_1 - max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$.
(F_1^*) : Obviously.
(F_2^*) : Let $u > 0$,$v > 0$ and $F(u, v, v, u, u+v, 0) = u - max\{u, v, \frac{1}{2}(u+v)\} < 0$. If $u \geq v$,then $u < u$, a contradiction. Thus $u < v$. If $u = 0, v > 0$,then $u < v$.
Similary,if $F(u, v, u, v, o, u+v) < 0$ then $u < v$.
(F_3^*) : $F(u, u, o, o, u, u) = 0, \forall u > 0$.
Ex.2: $F(t_1, ..., t_6) = t_1^2 - c_1 max\{t_2^2, t_3^2, t_4^2\} - c_2 max\{t_3t_5, t_4t_6\} - c_3t_5t_6$
where $c_1 + 2c_2 \leq 1, c_1 + c_3 \leq 1$ and $c_1, c_2, c_3 \geq 0$.
(F_1^*) : Obviously.
(F_2^*): Let $u > 0, v > 0$ and $F(u, v, v, u, u+v, o) = u^2 - c_1 max\{u^2, v^2\} - c_2 max\{v(u+v), 0\} < 0$. If $u \geq v$ then $u^2(1 - (c_1 + 2c_2)) < 0$, a contradiction. Thus $u < v$. If $u = 0, v > 0$, then $u < v$.
Similary, $F(u, v, u, v, o, u+v) < 0$ implies $u < v$.
(F_3^*) : $F(u, u, 0, 0, u, u) = u^2(1 - (c_1 + c_3)) \geq 0, \forall u > 0$.
Ex.3. $F(t_1, ..., t_6) = (1 + pt_2)t_1 - p max\{t_3t_4, t_5t_6\} - max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$
where $p > 0$.
(F_1^*) : Obviously.
(F_2^*) : Let $u > 0, v > 0$ and $F(u, v, v, u, u+v, o) = (1+pv)u - puv - max\{u, v, \frac{1}{2}(u+v)\} < 0$.
If $u \geq v$,then $u < u$, a contradiction. Hence $u < v$. If $u = 0, v > 0$, then $u < v$.
Similary, $F(u, v, u, v, o, u+v) < 0$ implies $u < v$.
(F_3^*) : $F(u, u, o, o, u, u) = (1 + pu)u - pu^2 - u = 0, \forall u > 0$.

Ex.4. $F(t_1, ..., t_6) = t_1 - max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6), b\sqrt{t_5 t_6}\}$, where $0 < b < 1$.
(F_1^*) : Obviously.
(F_2^*) : As in Ex.1.
(F_3^*) : $F(u, u, o, o, u, u) = u - max\{u, bu\} = u(1 - b) \geq 0, \forall u > 0$.
Ex.5. $F(t_1, ..., t_6) = t_1^3 - a t_1^2 t_2 - b t_1 t_3 t_4 - c t_5^2 t_6 - d t_5 t_6^2$, ,where $a, b, c, d \geq 0$ and $a+b+c+d < 1$.
(F_1^*) : Obviously.
(F_2^*) : Let $u > 0, v > 0$ and $F(u, v, v, u, u+v, 0) = u^3 - au^2v - bu^2v = u^2(u-(a+b)v) < 0$ which implies $u < (a+b)v < v$. If $u = 0, v > 0$ then $u < v$. Similary $F(u, v, u, v, o, u+v) < 0$ implies $u < v$.
(F_3) : $F(u, u, o, o, u, u) = u^3(1 - (a + c + d)) \geq 0, \forall u > 0$
Ex.6. $F(t_1, ..., t_6) = t_1^3 - c\frac{t_3^2 t_4^2 + t_5^2 t_6^2}{t_2 + t_3 + t_4 + 1}$, where $c \in (0, 1)$.
(F_1^*) : Obviously.
(F_2) : Let $u > 0, v > 0$ and $F(u, v, v, u, u, u + v, o) = u^3 - c\frac{u^2 v^2}{1+2v+u} < 0$. Then $u < \frac{cv^2}{2v+u+1} < cv < v$. If $u = 0, v > 0$ then $u < v$.
Similary, if $F(u, v, u, v, o, u + v) < 0$ then $u < v$.
(F_3) : $F(u, u, o, o, u, u) = u^3 \frac{(1-c)u+1}{u+1} > 0, \forall u > 0$.

3. Main Result

The following theorems are proved in [1] , [3] , [13] and [14] .

Theorem 1. [1]. Let (X,d) be a compact metric space and let S and T be continuous self maps of X satisfying
$(1)(1 + pd(x, y))d(Sx, Ty) < pmax\{d(x, Sx)d(y, Ty), d(x, Ty)d(y, Sx)\}$
$+ max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Sx))\}$
for all x,y in X for which the right hand side of (1) is positive, where $p \geq 0$. Then S and T have a unique common fixed point.

Theorem 2. [2]. Let A,B,S,T be continuous self mappings of a compact metric space with $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If $\{A, S\}$ and $\{B, T\}$ are compatible pairs and
(2) $d(Ax, By) < max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}(d(Ax, Ty) + d(By, Sx))\}$
for all x,y in X for which the right hand side of (2) is positive. Than A,B,S,T have a unique common fixed point.

Theorem 3. [13]. Let A,B,S and T be continuous self maps of a compact metric space (X,d) with $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If $\{A,S\}$ and $\{B,T\}$ are compatible pairs and

$$(3)\ d^2(Ax,By) < cmax\{d^2(Sx,Ax), d^2(Ty,By), d^2(Sx,Ty)\}, \frac{1}{2}(1-c)max\{d(Sx,Ax)d(Sx,By), d(Ax,Ty)d(By,Ty)\} + (1-c)d(Sx,By)d(Ty,Ax)$$

for all x,y in X for which the right hand side of (3) is positive, where $c \in (0,1)$. Then A,B,S and T have a common fixed point z.

Further, z is the unique common fixed point of A and S and of B and T.

Theorem 4. [14]. Let S and T be continuous self mappings of a compact metric space (X,d) satisfying inequality

$(4) d(Sx,Ty) < max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}d(x,Ty) + d(y,Sx)),$
$b\sqrt{d(x,Ty)d(y,Sx)}\}$

for all x,y in X for which the right hand side of (4) is positive, where $b > 0$. Then S and T have a common fixed point. Further, if $b < 1$, then the common fixed point is unique.

The purpose of this paper is to prove a general fixed point theorem for weakly compatible mappings in compact metric spaces which generalizes Theorems 1-4 and others.

Theorem 5. Let f,g,I,J be self maps of a compact metric space (X,d) such that:

$(a) f(X) \subset J(X)$ and $g(X) \subset I(X)$,

$(b) F(d(fx,gy), d(Ix,Jy), d(Ix,fx), d(Jy,gy), d(Ix,gy), d(Jy,fy)) < 0$

for all x,y in X for which one of d(Ix,Jy),d(Ix,fx),d(Jy,gy) is positive, where $F \in \mathcal{F}^*$

(c) The pair $\{f,I\}$ is compatible (resp. compatible of type (A),compatible of type (P)) and the pair $\{g,J\}$ is weakly compatible,

(d) The functions f and I are continuous,

then f,g,I and J have a unique common fixed point z. Further z is the unique common fixed point of f and I and of g and J.

Proof. Let $m = inf\{d(fx,Ix) : x \in X\}$. Since X is compact metric space there is a convergent sequence $\{x_n\}$ with

$$\lim x_n = x_0$$

in X such that

$$\lim d(Ix_n, fx_n) = m.$$

Since
$d(Ix_0, fx_0) \leq d(Ix_0, Ix_n) + d(Ix_n, fx_n) + d(fx_n, fx_0)$
then by continuity of f and I and

$$\lim x_n = x_0$$

we get $d(Ix_0, fx_0) \leq m$ and thus $d(Ix_0, fx_0) = m$.
Since $f(X) \subset J(X)$, there exists a point y_0 in X such that $Jy_0 = fx_0$ and thus $d(Ix_0, Jy_0) = m$. Suppose that $m > 0$. Then by (b) we have successively
$F(d(fx_0, gy_0), d(Ix_0, Jy_0), d(Ix_0, fx_0), d(Jy_0, gy_0), d(Ix_0, gy_0), d(Jy_0, fx_0)) < 0$
$F(d(Jy_0, gy_0), m, m, d(Jy_0, gy_0), d(Ix_0, Jy_0) + d(Jy_0, gy_0), 0) < 0$
$F(d(Jy_0, gy_0), m, m, d(Jy_0, gy_0), m + d(Jy_0, gy_0), 0) < 0$
By (F_a^*) follows that

$$(5) d(Jy_0, gy_0) < m.$$

Since $g(X) \subset I(X)$, then there is a point z_0 in X such that $Iz_0 = gy_0$ and thus $d(Iz_0, Jy_0) < m$. Since $d(Iz_0, fz_0) \geq m > 0$, by (b), we have
$F(d(fz_0, gy_0), d(Iz_0, Jy_0), d(Iz_0, fz_0), d(Jy_0, gy_0), d(Iz_0, gy_0), d(Jy_0, fz_0)) < 0$

$$F(d(Iz_0, fz_0), d(Jy_0, gy_0), d(Iz_0, fz_0), d(Jy_0, g(y_0)), 0, d(Jy_0, gy_0) + d(gy_0, fz_0)) < 0$$

$$F(d(Iz_0, fz_0), d(Jy_0, gy_0), d(Iz_0, fz_0), d(Jy_0, g(y_0)), 0, d(Jy_0, gy_0) + d(Iz_0, fz_0)) < 0$$

By (F_b^*) follows that

$$(6) d(Iz_0, fz_0) < d(Jy_0, gy_0).$$

Then, by (5) and (6) we obtain
$m \leq d(Iz_0, fz_0) < d(Jy_0, gy_0) < m$. Thus $m < m$, a contradiction.
Therefore, $m = 0$ which implies

$$(7) Ix_0 = Jy_0 = fx_0.$$

If $d(Jy_0, gy_0) > 0$, then by (b) we have successively
$F(d(fx_0, gy_0), d(Ix_0, Jy_0), d(Ix_0, fx_0), d(Jy_0, gy_0), d(Ix_0, gy_0), d(Jy_0, fx_0)) < 0$,
$F(d(Jy_0, gy_0), 0, 0, d(Jy_0, gy_0), d(Jy_0, gy_0), 0) < 0$
which implies by (F_a^*) that $d(Jy_0, gy_0) < 0$, a contradiction. Thus $d(Jy_0, gy_0) = 0$, which implies $Jy_0 = gy_0$. Therefore,

$$(8) Ix_0 = fx_0 = Jy_0 = gy_0.$$

Since I and f are compatible (resp.compatible of type (A), compatible of type (P)) and $Ix_0 = fx_0$, by Lemma 1 $Ifx_0 = fIx_0$. By (8)

$$f^2x_0 = fIx_0 = Ifx_0 = I^2x_0.$$

If $I^2x_0 \neq Ix_0$ then $Ifx_0 \neq Jy_0$ and by (b) we have successively
$F(d(f^2x_0, gy_0), d(Ifx_0, Jy_0), d(Ifx_0, f^2x_0), d(Jy_0, gy_0), d(Ifx_0, gy_0), d(Jy_0, Ifx_0)) < 0$,
$F(d(f^2x_0, Ix_0), d(I^2x_0, Ix_0), 0, 0, d(I^2x_0, Ix_0), d(I^2x_0, Ix_0)) < 0$
a contradiction of (F_3^*). Therefore, $Ix_0 = I^2x_0$. Hence

$$(9) fIx_0 = Ix_0 = I^2x_0.$$

Similary, we have

$$(10) gJy_0 = Jy_0 = J^2y_0.$$

Let $u = Ix_0 = Jy_0$. Then $fu = fIx_0 = Ifx_0 = I^2x_0 = Iu$, which implies $fu = Iu$. Similary, $gu = Ju$. Since $u = Ix_0 = I^2x_0$,then $Iu = u$. Similary, $Ju = u$. Therefore,

$$(11) f(u) = u = Iu = Ju = gu$$

and u is a common fixed point of f,g,I and J.
Suppose that g and J have another common fixed point $v \neq u$, then $d(u, v) \neq 0$ and by (b) we have successively
$F(d(fu, gv), d(Iu, Jv), d(Iu, fu), d(Jv, gv), d(Iu, gv), d(Jv, fu)) < 0$
$F(d(u, v), d(u, v), 0, 0, d(u, v), d(u, v)) < 0$, a contradiction of (F_3^*).
Thus $u = v$. Similarly, u is unique common fixed point of f and I.

Corollary 1. Let f,g,I,J be self maps of a compact metric space (X,d) such that
$a) f(X) \subset J(X)$ and $g(X) \subset I(X)$,

$(b')(1+pd(Ix,Jy))d(fx,gy) < pmax\{d(Ix,fx)d(Jy,gy),d(Ix,gy)d(Jy,fx)\}$
$+ max\{d(Ix,Jy),d(Ix,fx),d(Jy,gy),\frac{1}{2}(d(Ix,gy)+d(Jy,fx))\}$
for all x,y in X for which the right hand side of (b') is positive, where $p > 0$.

c) the pair $\{f,I\}$ is compatible (resp. compatible of type (A), compatible of type (P)) and the pair $\{g,J\}$ is weakly compatible,

d) f and I are continuous,

then f,g,I and J have a unique common fixed point.

Proof. Follows from Theorem 5 and Ex.3.

Remark. If $I = J = id$, by Corollary 1 , Theorem 1 follows.

Corollary 2. Theorem 2.

Proof. Follows from Theorem 5 and Ex.1.

Corollary 3. Theorem 3.

Proof. Follows from Theorem 5 and Ex.2 for $c_1 = c, c_2 = \frac{1}{2}(1-c), c_3 = 1-c$

Corollary 4. Theorem 4.

Proof. Follows from Theorem 5 and Ex.4 if $f = S, g = T$ and $I = J = id$.

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