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Some Applications of the Lattice Finite Representability in Spaces of Measurable Functions

P. Gómez Palacio, J.A. López Molina and M. J. Rivera

Abstract

We study the lattice finite representability of the Bochner space $L_p(\mu_1, L_q(\mu_2))$ in $\ell_p\{\ell_q\}$, $1 \leq p, q < \infty$, and then we characterize the ideal of the operators which factor through a lattice homomorphism between $L_\infty(\mu)$ and $L_p(\mu_1, L_q(\mu_2))$.

Key Words: Integral operators. Ultraproducts of spaces and maps.

1. Introduction

The classical works of Dacunha-Castelle and Krivine [1], Heinrich [5] and Haydon and Levy and Raynaud [4] show that ultraproducts and finite representability are closely connected. The so called "local theory" in Banach spaces, i. e. the research in terms of finite dimensional subspaces, has much enriched our understanding of Banach operator ideals. The ultraproducts technique allows the study some operators in terms of its finite dimensional parts. The representation of ultraproducts of classical sequence spaces by spaces of measurable functions lies at the heart of the operators theory.

The factorization theorems of some operator ideals, for example the p-nuclear, p-integral, (p,q)-factorable or (p,q)-dominated operators (see [2]), have been an important

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tool in the study of many of their topological and metric properties, as well as in the study of some topological and metric properties of tensor products of Lapresté. The aim of this paper is the study of some operator ideals characterized by the fact that they factor through a lattice homomorphism between a $L_\infty(\mu)$ and a sublattice of a Bochner space $L_p(\mu_1, L_q(\mu_2))$.

The notation is standard. All the spaces considered are Banach spaces over the real field, since we shall use results in the theory of Banach lattices.

2. Finite and lattice finite representability in spaces of measurable functions.

Definition 1 *A Banach space (lattice) X is said to be (lattice) finitely representable in a family of Banach spaces (lattices) $\{X_i, i \in I\}$ if, for every finite dimensional subspace (sublattice) M of X and for every $\varepsilon > 0$, there are an index $i \in I$, a finite dimensional subspace (sublattice) N of X_i and an isomorphism (lattice isomorphism) $J : M \to N$ so that $\|J\|\|J^{-1}\| \leq 1 + \varepsilon$.*

If for every $i \in I$, the space (lattice) X_i is a subspace (sublattice) of a Banach space (lattice) Y, then X is said to be (lattice) finitely representable in Y.

A Banach space E is said to be a $\mathcal{L}_{p,\lambda}$-space, $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$, if for every finite dimensional subspace P of E there is a finite dimensional subspace Q of E containing P such that the Banach-Mazur distance $d(Q, \ell_p^{dim(Q)}) < \lambda$.

If E is a $\mathcal{L}_{p,\lambda}$-space for every $\lambda > 1$, then it is finitely representable in ℓ_p. In particular, the space $L_p(\mu)$ is finitely representable in ℓ_p. Moreover, from [7] chapter 6 §17, theorem 7, a Banach space is finitely representable in ℓ_p if and only if it is isometric to a subspace of some $L_p(\mu)$.

Let D be an index set and $\mathcal{D}$ a non-trivial ultrafilter on D. Given $\{A^d, d \in D\}$ a family of Banach spaces $(A_d)_\mathcal{D}$ denotes the corresponding ultraproduct (ultrapower if $A_d = A$ for every $d \in D$). If every A^d is a Banach lattice, $(A^d)_\mathcal{D}$ has a canonical order which makes it a Banach lattice. If we have another family of Banach spaces $\{B_d, d \in D\}$ and a family of operators $\{T^d \in \mathcal{L}(A^d, B^d), d \in D\}$ such that $\sup_{d\in D} \|T^d\| < \infty$, $(T^d)_\mathcal{D} \in \mathcal{L}((A^d)_\mathcal{D}, (B^d)_\mathcal{D})$ denotes the canonical ultraproduct operator. The main ideas on ultraproducts of Banach spaces used in this paper are stated in [5] and [4].

Ultraproducts and finite representability are closely connected. In fact a Banach space X is (lattice) finitely representable in a family of Banach spaces (lattices) $\{X_i, i \in I\}$ if and only if there is a subset $J \subset I$ and an ultrafilter $\mathcal{U}$ on J such that X is isometric to a subspace (sublattice) of $(X_j)_{\mathcal{U}}$, see [4].

From the principle of local reflexivity, the bidual X'' of a Banach space X is finitely representable in X. The lattice version of Conroy and Moore proves that the bidual E'' of a Banach lattice E is lattice finitely representable in E.

The aim of the following theorem is the study of finite representability in some spaces of measurable functions.

Theorem 2 *Given a measure space (Ω, Σ, μ), let X be a Banach space (lattice) of measurable functions such that the set of simple functions with finite measure support $\mathcal{T}$ is a dense subset of X. Let $\{G_\delta, \delta \in \Lambda\}$ be the set of all subspaces G_δ of simple functions generated for a finite and pairwise disjoint set of characteristic functions of $\mathcal{T}$. Then, there is an ultrafilter $\mathcal{U}$ in Λ such that X is isometric (lattice isometric) to a subspace (sublattice) of $(G_\delta)_{\mathcal{U}}$.*

Proof. We consider the order in Λ, $\delta_1 \leq \delta_2$ if and only if $G_{\delta_1} \subset G_{\delta_2}$. Fix a $\delta_0 \in \Lambda$, we put $R(\delta_0) := \{\delta \in \Lambda : \delta_0 \leq \delta\}$. Then $\mathcal{R} := \{R(\delta), \delta \in \Lambda\}$ is a filter basis of subsets of Λ. According to the lemma of Zorn, let $\mathcal{U}$ be an ultrafilter containing $\mathcal{R}$.

We define $T : \mathcal{T} \to (G_\delta)_{\mathcal{U}}$ such that for every $x \in \mathcal{T}$, $T(x) = (x_\delta)_{\mathcal{U}}$ with $x_\delta = x$ if $x \in G_\delta$, and $x_\delta = 0$ if $x \notin G_\delta$. T is a positive isometric map, which can be extended to X. Then X is isometric (lattice isometric) to a subspace (sublattice) of $(G_\delta)_{\mathcal{U}}$. □

Remark 3 *With the hypothesis of the Theorem 2, X is (lattice) finitely representable in $\{G_\delta, \delta \in \Lambda\}$.*

In some cases there is a Banach sequence space (lattice) λ_X such that every G_δ is isometric (lattice isometric) to some λ_X^m, and then X is (lattice) finitely representable in λ_X, and even in the family λ_X^m where m varies over all natural numbers. For example if $X = L_p(\mu)$, then $\lambda_X = \ell_p$. Hence it is easy to see the lattice version of [7] chapter 6 §17, theorem 7,

Corollary 4 *A Banach lattice is lattice-finitely representable in ℓ_p if and only if it is isometric to a sublattice of some $L_p(\mu)$.*

The definitions and results in the theory of tensor norms and operator ideals involved in this paper are exposed in [2]. Given a pair of Banach spaces E and F and a tensor norm α, $E \otimes_\alpha F$ represents the space $E \otimes F$ endowed with the α-normed topology. The completion of $E \otimes_\alpha F$ is denoted by $E\hat{\otimes}_\alpha F$, and the norm of z in $E\hat{\otimes}_\alpha F$ by $\alpha(z; E \otimes F)$. If there is no risk of mistake, we write $\alpha(z)$ instead of $\alpha(z; E \otimes F)$. Recalled from the metric mapping property that if $A_i \in \mathcal{L}(E_{i1}, E_{i2})$, $i = 1, 2$, then $A_1 \otimes A_2 \in \mathcal{L}(E_{11} \otimes_\alpha E_{21}, E_{12} \otimes_\alpha E_{22})$ with $\|A_1 \otimes A_2\| \leq \|A_1\|\|A_2\|$.

The next proposition, involving $\mathcal{L}_{\infty\lambda}$ spaces, is an extension of a well known result of Hollstein's [6] proposition 2,2. For every Banach space F and for every closed subspace F_0 of F, $K_{F_0} : F \to F/F_0$ represents the canonical quotient map, and the open unit ball in F is denoted by $\overset{\circ}{B}_F$.

Proposition 5 *([9], proposition 10) Let E be a $\mathcal{L}_{\infty,\lambda}$-space. Then, for every finitely generated tensor norm α, $E \otimes_\alpha \;\cdot\;$ isomorphically respects quotients. More precisely, for every Banach space F and for every closed subspace F_0 of F,*

$$\overset{\circ}{B}_{E\otimes_\alpha F/F_0} \subset \lambda(id_E \otimes K_{F_0})(\overset{\circ}{B}_{E\otimes_\alpha F}).$$

It is known (see [2], chapter I section 7) that for every Banach space E and for every $1 \leq p \leq \infty$, there is a norm Δ_p in $L_p(\mu) \otimes E$, called the p-natural norm, such that $L_p(\mu)\hat{\otimes}_{\Delta_p}E$ is isometric to the Bochner space $L_p(\mu, E)$ using the canonical mapping $f \otimes x \to f(.)x$. In general, Δ_p is not a tensor norm because it doesn't satisfy the metric mapping property. Moreover, if $1 \leq p < \infty$, the image of $L_p(\mu) \otimes E$ is dense in $L_p(\mu, E)$. In this case we can identify $L_p(\mu, E)$ and $L_p(\mu)\hat{\otimes}_{\Delta_p}E$ as Banach spaces. If E is a Banach lattice this isometric map is positive, hence we can identify $L_p(\mu, E)$ and $L_p(\mu)\hat{\otimes}_{\Delta_p}E$ as Banach lattices. As the set of simple functions with finite measure support is dense $L_p(\mu)$ for every $1 \leq p < \infty$, then it is easy to see that the set of simple functions of finite measure support $S = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\chi_{A_i} \otimes \chi_{B_j}$ is dense in $L_p(\mu)\hat{\otimes}_{\Delta_p}L_q(\mu')$. Moreover, if f is positive it can be approximated in $L_p(\mu)\hat{\otimes}_{\Delta_p}L_q(\mu')$ by a sequence of positive simple functions.

For $1 \leq p, q \leq \infty$ define the Bochner Banach sequence space

$$\ell_p\{\ell_q\} := \{a = (a_{ij})_{i,j=1}^{\infty} \ \mid \ \Pi_{p\{q\}}(a) := (\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty}|a_{ij}|^q)^{\frac{p}{q}})^{\frac{1}{p}} < \infty\},$$

with the usual modifications if p or q are infinite. If $1 \leq p, q < \infty$, $\ell_p\{\ell_q\}$ is an order continuous Banach sequence lattice.

For every finite set $\delta = \{A_i \otimes B_j, i = 1, .., n, j = 1, .., m\}$, where $\{A_i\}_{i=1}^n$ and $\{B_j\}_{j=1}^m$ are families of pairwise disjoint sets in Σ_1 and Σ_2, respectively, of finite and non zero measure, we denote G_δ as the linear span of $\{\chi_{A_i} \otimes \chi_{B_j}, i = 1, .., n, j = 1, .., m\}$, which is a finite dimensional sublattice of $L_p(\mu_1, L_q(\mu_2))$. Then we have the map $P_\delta : G_\delta \to \ell_p^n\{\ell_q^m\}$ so that

$$P_\delta(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\chi_{A_i} \otimes \chi_{B_j}) = (a_{ij}\mu_1(A_i)^{\frac{1}{p}}\mu_2(B_j)^{\frac{1}{q}})$$

is a positive isometric map. Then from Theorem 2, $L_p(\mu_1, L_q(\mu_2))$ is lattice isometric to a sublattice of some $(\ell_p\{\ell_q\})_{\mathcal{U}}$, and hence from [4] it is (lattice) finitely-representable in $\ell_p(\ell_q\}$. Conversely, every ultrapower of $\ell_p\{\ell_q\}$ is lattice isometric to a sublattice of some $L_p(\mu_1, L_q(\mu_2))$ (see [8]) and then

Corollary 6 *A Banach space (lattice) X is (lattice) finitely representable in $\ell_p\{\ell_q\}$ if and only if it is isometric to a subspace (sublattice) of some $L_p(\mu_1, L_q(\mu_2))$.*

3. Applications.

Given a Banach space E, a sequence of sequences $x = (x_{ij})_{i,j=1}^{\infty} \subset E$ is strongly $p\{q\}$-summing if $\pi_{p\{q\}}((x_{ij})) := \Pi_{p\{q\}}((\|x_{ij}\|)) < \infty$ and it is weakly $p\{q\}$-summing if $\varepsilon_{p\{q\}}((x_{ij})) := \sup_{\|x'\| \leq 1} \Pi_{p\{q\}}(| < x_{ij}, x' > |) < \infty$.

Definition 7 *Let E and F be Banach spaces and $1 \leq p, q < \infty$. For every $z \in E \otimes F$ we define*

$$g_{p\{q\}}(z) := \inf\{\pi_{p\{q\}}((x_{ij}))\varepsilon_{p'(q')}((y_{ij})) : z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \otimes y_{ij}\}.$$

The functional $g_{p\{q\}}$ is a tensor norm for $E \otimes F$. A suitable representation of the elements of a completed tensor product is a basic tool in the study of the operator ideals involved. The reader is referred to [3] to prove that if $z \in E\hat{\otimes}_{g_{p\{q\}}} F$, there are $\{(x_{ij})_{j=1}^{\infty}, i \in \mathbb{N}\} \subset E^{\mathbb{N}}$ and $\{(y_{ij})_{j=1}^{\infty}, i \in \mathbb{N}\} \subset F^{\mathbb{N}}$ such that $\pi_{p\{q\}}((x_{ij}))\varepsilon_{p'(q')}((y_{ij})) < \infty$, $z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}$. Moreover, $g_{pq}(z) = \pi_{p\{q\}}((x_{ij}))\varepsilon_{p'(q')}((y_{ij}))$, where the infimum is over all such representations of z.

Every representation of $z \in E'\hat{\otimes}_{g_{p\{q\}}} F$,

$$z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x'_{ij} \otimes y_{ij}$$

with $\pi_{p\{q\}}((x'_{ij}))\varepsilon_{p'(q')}((y_{ij})) < \infty$, defines a $T_z \in \mathcal{L}(E, F)$ such that $\forall x \in E$,

$$T_z(x) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < x'_{ij}, x > y_{ij}.$$

We remark that all the representations of the same z define the same map T_z. Let $\Phi_{EF} : E'\hat{\otimes}_{p\{q\}} F \to \mathcal{L}(E, F)$, $\Phi_{EF}(z) := T_z$. We set:

Definition 8 *Let E, F be Banach spaces. An operator $T : E \to F$ is said to be $p\{q\}$-nuclear if $T = \Phi_{EF}(z)$, for some $z \in E'\hat{\otimes}_{g_{p\{q\}}} F$.*

$\mathcal{N}_{p\{q\}}(E, F)$ denotes the space of the $p\{q\}$-nuclear operators $T : E \to F$ endowed with the topology of the norm

$$\mathbf{N}_{p\{q\}}(T) := \inf\{\pi_{p\{q\}}((x'_{ij}))\varepsilon_{p'(q')}((y_{ij})) : \Phi_{EF}(z) = T, z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x'_{ij} \otimes y_{ij}\}.$$

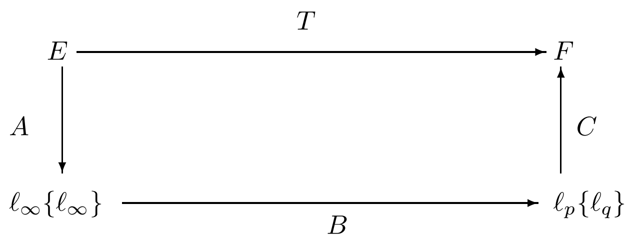
For every pair of Banach spaces E and F $(\mathcal{N}_{p\{q\}}(E, F), \mathbf{N}_{p\{q\}})$ is a component of the minimal operators ideal $(\mathcal{N}_{p\{q\}}, \mathbf{N}_{p\{q\}})$ associated to $g_{p\{q\}}$ and called the ideal of the $p\{q\}$-nuclear operators.

We have the following characterization of the $p\{q\}$-nuclear operators.

Theorem 9 *For every pair of Banach spaces E and F and for every operator T in $\mathcal{L}(E,F)$ the following assertions are equivalent:*

1) T is $p\{q\}$-nuclear.

2) T factors as follows:



where B is a positive diagonal multiplication operator defined by a positive sequence $((b_{ij})) \in \ell_p\{\ell_q\}$.

Furthermore, $\boldsymbol{N}_{p\{q\}}(T) = inf\{\|D\|\|B\|\|A\|\}$, taking the infimum over all such factors.

Proof: 1) $\Rightarrow$ 2): If T is $p\{q\}$-nuclear, given $\varepsilon > 0$, $T(.) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < x'_{ij}, . > y_{ij}$ such that $\mathbf{N}_{p\{q\}}(T) + \varepsilon \geq \pi_{p\{q\}}((x'_{ij}))\varepsilon_{p'(q')}((y_{ij}))$. Even we can suppose that $\varepsilon_{p\{q\}}((y_{ij})) = 1$, and $\mathbf{N}_{p\{q\}}(T) + \varepsilon \geq \pi_{p\{q\}}((x'_{ij}))$.

Let $A : E \to \ell_\infty\{\ell_\infty\}$ be such that $A(x) := ((\frac{<x'_{ij},x>}{\|x'_{ij}\|})_{j=1}^{\infty})_{i=1}^{\infty}$ which is linear and continuous with $\|A\| \leq 1$.

Let $B : \ell_\infty\{\ell_\infty\} \to \ell_p\{\ell_q\}$ be such that $B(((\lambda_{ij}))) := ((\lambda_{ij}\|x'_{ij}\|)_{j=1}^{\infty})_{i=1}^{\infty}$. Then

$$\|(B(((\lambda_{ij})))\|_{\ell_p\{\ell_q\}} \leq \|((\lambda_{ij}))\|_{\ell_\infty\{\ell_\infty\}}\|(((\|x'_{ij}\|)))\|_{\ell_p\{\ell_q\}} =$$

$$= \|((\lambda_{ij}))\|_{\ell_\infty\{\ell_\infty\}}\pi_{p\{q\}}((x'_{ij})))$$

and then B is linear and continuous with $\|B\| \leq \pi_{p\{q\}}((x'_{ij}))$.

Finally, let $C : \ell_p\{\ell_q\} \to F$ be such that $C((\beta_{ij})) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ij}y_{ij}$, which is linear and continuous with

$$\|C(((\beta_{ij})))\| = \sup_{\|y'\|\leq 1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < \beta_{ij}y_{ij}, y' >\leq \|(((\beta_{ij})))\|_{\ell_p\{\ell_q\}}$$

and then $\|C\| \leq 1$. Moreover, it is easy to see that $T = C \cdot B \cdot A$.

2) $\Rightarrow$ 1): $A' : (\ell_\infty\{\ell_\infty\})' \to E'$, and we put $x'_{ij} := A'(e_{ij})$. Then $< A(x), e_{ij} >=< A'(e_{ij}), x >=< x'_{ij}, x >$, and $A(x) = ((< x'_{ij}, x >)_{j=1}^\infty)_{i=1}^\infty$.

If $U = ((u_{ij}))$ such that $u_{ij} = 1$, $\forall i, j \in \mathbb{N}$, and $B(U) = ((b_{ij})_{j=1}^\infty)_{i=1}^\infty$, then B is the multiplication operator for $((b_{ij})_{j=1}^\infty)_{i=1}^\infty \in \ell_p\{\ell_q\}$ with $\|B\| = \|(((b_{ij})))\|_{\ell_p\{\ell_q\}}$.

If $C(e_{ij}) = y_{ij}$ then $C(((\beta_{ij}))) = \sum_{i=1}^\infty \sum_{j=1}^\infty \beta_{ij} y_{ij}$, $\forall ((\beta_{ij})) \in \ell_p\{\ell_q\}$, hence

$$\|C((\beta_{ij}))\|_F = \sup_{\|y'\|_{F'} \leq 1} \sum_{i=1}^\infty \sum_{j=1}^\infty |\beta_{ij} < y_{ij}, y' > | \leq \|C\| \|((\beta_{ij}))\|_{\ell_p\{\ell_q\}}.$$

Then for every y' in the unit ball of F', $((< y_{ij}, y' >)) \in \ell_{p'}\{\ell_{q'}\}$, with

$$\sup_{\|y'\|_{F'} \leq 1} \pi_{p'(q')}((< y_{ij}, y' >)) \leq \|C\|.$$

If $T = C.B.A$, then $T = \Phi_{EF}(z)$ with $z = \sum_{i=1}^\infty \sum_{j=1}^\infty b_{ij} x'_{ij} \otimes y_{ij} \in E' \hat{\otimes}_{g_{p\{q\}}} F$. □

Theorem 10 *Let (Ω, Σ, μ) be a measure space and $1 \leq p, q < \infty$. Then every positive operator $S : L_\infty(\mu) \to \ell_p\{\ell_q\}$ is $p\{q\}$-nuclear.*

Proof. As S is positive and $\ell_p\{\ell_q\}$ is order complete, we denote

$$h := \sup_{\|f\|_{L_\infty(\mu)} \leq 1} S(f)$$

and it is clear that h is positive. Let $A : L_\infty(\mu) \to \ell_\infty\{\ell_\infty\}$ be such that $\forall h \in L_\infty(\mu)$,

$$< A(h), e_{ij} >:= \frac{< S(h), e_{ij} >}{< h, e_{ij} >}$$

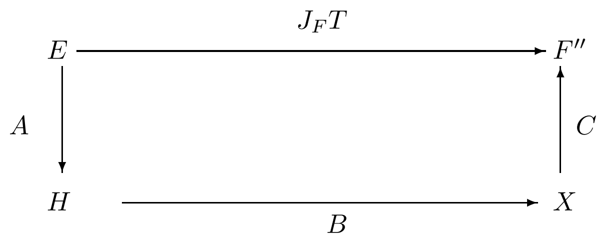
and the positive diagonal operator $D_h : \ell_\infty\{\ell_\infty\} \to \ell_p\{\ell_q\}$ such that $\forall a \in \ell_\infty$,

$$< D_h(a), e_{ij} >=< a, e_{ij} >< h, e_{ij} > .$$

It is easy to see that A and D_h are linear and continuous with $\|A\| \leq 1$, and that $S = D_h A$. Then from the characterization of the $p\{q\}$-nuclear operators, D_h is nuclear with $\mathbf{N}_{p\{q\}}(D_h) = \|D_h\| = \|h\|_{\ell_p\{\ell_q\}}$. And so, S is $p\{q\}$-nuclear with $\mathbf{N}_{p\{q\}}(S) \leq \|h\|_{\ell_p\{\ell_q\}}$. □

The $p\{q\}$-integral operators ideal $(\mathcal{I}_{p\{q\}}, \mathbb{I}_{p\{q\}})$ is the maximal operator ideal associated to that of the $p\{q\}$-nuclear operators. According to [2], for every pair of Banach spaces E and F , an operator $T : E \to F$ is $p\{q\}$-integral, if and only if, $J_F T \in (E \otimes_{(g_{p\{q\}})'} F')'$, where $J_F : F \to F''$ is the canonical isometric map. The aim of the final part of the paper is to obtain the characterization of the $p\{q\}$-integral operators by means of a factorization theorem.

Proposition 11 *For every pair of Banach spaces E, F, if $T \in \mathcal{I}_{p\{q\}}(E, F)$ then $J_F T$ factors in the following way:*



where H is an abstract M-space, B is a lattice homomorphism and X is a band of some $L_p(\mu_1, L_q(\mu_2))$.

Proof. We define

$$D := \{(M, N) : M \in FIN(E), N \in FIN(F')\},$$

where $FIN(Y)$ is the set of finite dimensional subspaces of a Banach space Y, endowed with the inclusion order

$$(M_1, N_1) \leq (M_2, N_2) \leftrightarrow M_1 \subset M_2, N_1 \subset N_2.$$

For every $(M_0, N_0) \in D$, $R(M_0.N_0) := \{(M, N) \in D : (M_0, N_0) \subset (M, N)\}$ and $\mathcal{R} = \{R(M, N), (M, N) \in D\}$. $\mathcal{R}$ is filter basis in D, and let $\mathcal{D}$ be an ultrafilter on D containing $\mathcal{R}$. If $d \in D$, M_d and N_d denote the finite dimensional subspaces of E and F' respectively such that $d = (M_d, N_d)$. For every $d \in D$, if $z \in M_d \otimes N_d$, $J_F T_{|M_d \otimes N_d} \in (M_d \otimes_{(g_H^c)'} N_d)' = M_d' \otimes_{g_H^c} N_d' = \mathcal{N}_{H^c}(M_d, N_d')$. Therefore, from Theorem 9, $J_F T_{|M_d \otimes N_d}$ factors as follows:

$$\begin{array}{ccc} M_d & \xrightarrow{J_F T_{|M_d \otimes N_d}} & N'_d \\ {\scriptstyle A^a}\downarrow & & \uparrow{\scriptstyle C^d} \\ \ell_\infty\{\ell_\infty(\Gamma^d)\} & \xrightarrow[B^d]{} & \ell_p\{\ell_q(\Gamma^d)\} \end{array}$$

where $\Gamma^d \subset \mathbb{N}$ is a finite set and B^d is a positive diagonal with $I_{p\{q\}}(T_{|M_d \otimes N_d}) \leq I_{p\{q\}}(T)$. Without loss of generality, we can assume that $\|A^d\| = \|C^d\| = 1$ and $\|C^d\| \leq I_{p\{q\}}(T)$. We define $W_E : E \rightarrow (M_d)_{\mathcal{D}}$ such that $W_E(x) = (x^d)_{\mathcal{D}}$, $x^d = x$ if $x \in M_d$ and $x^d = 0$ if $x \notin M_d$. We define $W_{F'} : F' \rightarrow (N_d)_{\mathcal{D}}$ such that $W_{F'}(a) = (a^d)_{\mathcal{D}}$ so that $a^d = a$ if $a \in N_d$ and $a^d = 0$ if $a \notin N_d$. Then we obtain the following diagram:

$$\begin{array}{ccccc} E & & \xrightarrow{J_F T} & & F'' \\ {\scriptstyle W_E}\downarrow & & & & \uparrow{\scriptstyle W'_{F'}} \\ (M_d)_{\mathcal{D}} & \xrightarrow[J_F T_{|M_d \otimes N_d}]{} & (N'_d)_{\mathcal{D}} & \xrightarrow[I]{} & ((N_d)_{\mathcal{D}})'. \end{array}$$

It follows that $A = (A^d)\mathcal{D}$, $B = (B^d)\mathcal{D}$, $C = (C^d)\mathcal{D}$, and $H = (\ell_\infty\{\ell_\infty(\Gamma^d)\})_{\mathcal{D}}$ and $X = (N'_d)_{\mathcal{D}}$, from which we obtain:

$$\begin{array}{ccc} E & \xrightarrow{J_F T} & F'' \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle C} \\ H & \xrightarrow[B]{} & X. \end{array}$$

But according to [8], every ultrapower of $\ell_p\{\ell_q\}$ is lattice isometric to a band of some Bochner space $L_p(\mu_1, L_q(\mu_2))$. $\square$

Now we have to prove the converse of the the former result, i.e., we will see every operator $T : E \to F$ having a decomposition as in the Proposition 11 is $p\{q\}$-integral. Prior to that, we need the following lemma.

Lemma 12 *([4], lemma 4.4) Let G be an order complete Banach lattice and X a finite dimensional Banach subspace of G. Then for every $\varepsilon > 0$, there is a finite dimensional Banach sublattice Y of G and an operator $A : X \longrightarrow Y$ such that*

$$\forall\, x \in X \quad \|A(x) - x\| \leq \varepsilon\, \|x\|.$$

Remark 13 *$\|A\| \leq 1 + \varepsilon$ and $\|A - id_X\| \leq \varepsilon$ are easily demonstrated.*

Theorem 14 *Let H and X be Banach lattices such that H is an abstract M-space and X is isometric to a sublattice of some $L_p(\mu_1, L_q(\mu_2))$. Then every lattice homomorphism $T : H \to X$ is $p\{q\}$-integral.*

Proof. H is an abstract M-space, hence G'' is lattice isomorphic to some $L_\infty(\mu)$ and $B : H \to L_\infty(\mu)$ denotes the corresponding positive isometric map. Then $J_X T = T''B$, where $J_X : X \to X''$ is the canonical map, and in consequence we only have to see that $T'' \in (L_\infty(\mu) \otimes_{(g_{p\{q\}})'} X')'$. For every $z \in L_\infty(\mu) \otimes X'$, given $\varepsilon > 0$, let $M \subset L_\infty(\mu)$ and $N \subset X'$ be finite dimensional subspaces and let $z = \sum_{i=1}^n f_i \otimes x_i'$ be a fix representation of z, $f_i \in M$ and $x_i' \in N$, $i = 1, 2, .., n$ such that $(g_{p\{q\}})'(z; M \otimes N) \leq (g_{p\{q\}})'(z; L_\infty(\mu) \otimes X') + \varepsilon$. Let M_1 be a finite dimensional sublattice of $L_\infty(\mu)$ and an operator $A : M \to M_1$ so that for every $f \in M$, $\|A(f) - f\| \leq \varepsilon \|f\|$. Then,

$$| < T'', z > | = | \sum_{i=1}^n < T''(f_i), x_i' > | \leq$$

$$| \sum_{i=1}^n < T''(id_{L_\infty(\mu)} - A)(f_i), x_i' > | + | \sum_{i=1}^n < T''(A(f_i)), x_i' > |$$

$$\leq \varepsilon \|T\| \sum_{i=1}^n \|f_i\| \|x_i'\| + | \sum_{i=1}^n < T''(A(f_i)), x_i' > |.$$

But from Ando (see theorem 1.4.19 of [10]), as T is a lattice homomorphism, T'' has the same property and $X_1 := T''(M_1)$ is a finite dimensional sublattice of X'', there is a finite dimensional sublattice X_2 of X and a lattice isomorphism $C : X_1 \to X_2$ such that $\|C\|\|C^{-1}\| \leq 1+\varepsilon$. But X is lattice finitely representable in $\ell_p\{\ell_q\}$, so there is a finite dimensional sublattice Z of $\ell_p\{\ell_q\}$ and a lattice isomorphism $D : X_2 \to Z$ such that $\|D\|\|D^{-1}\| \leq 1+\varepsilon$. If $R : M_1 \to Z$ denotes the map $R := DCT''$, and I_Z denotes the inclusion of Z in $\ell_p\{\ell_q\}$, we have

$$\sum_{i=1}^{n} < T''(A(f_i)), x_i' >= \sum_{i=1}^{n} < ((DC)^{-1}(DC)T'')(A(f_i)), x_i' >=$$

$$\sum_{i=1}^{n} < ((DC)^{-1})(R(A(f_i))), x_i' >= \sum_{i=1}^{n} < R(A(f_i)), ((DC)^{-1})'(x_i') >=$$

$$< R, \sum_{i=1}^{n} A(f_i) \otimes ((DC)^{-1})'(x_i') >$$

with $\sum_{i=1}^{n} A(f_i) \otimes ((DC)^{-1})'(x_i') \in M_1 \otimes Z'$.

Remark that the map $I_Z' : (\ell_p\{\ell_q\})' \to Z'$ is a canonical quotient map and M_1 is a $\mathcal{L}_{\infty,1+\varepsilon}$ space, consequentely, after Proposition 5, there is $u \in M_1 \otimes (\ell_p\{\ell_q\})'$ with a representation $u = \sum_{j=1}^{m} g_j \otimes a_j$ so that $(id_{M_1} \otimes I_Z')(u) = \sum_{i=1}^{n} A(f_i) \otimes ((DC)^{-1})'(x_i')$ and $(g_{p\{q\}})'(u; M_1 \otimes (\ell_p\{\ell_q\})') \leq (1+\varepsilon)(g_{p\{q\}})'(\sum_{i=1}^{n} A(f_i) \otimes ((DC)^{-1})'(x_i'); M_1 \otimes Z')$. We have

$$< R, \sum_{i=1}^{n} A(f_i) \otimes ((DC)^{-1})'(x_i') >=< R, (id_{M_1} \otimes I_Z')(u) >=$$

$$\sum_{j=1}^{m} < R(g_j), I_Z'(a_j) >= \sum_{j=1}^{m} < (I_Z R)(g_j), a_j >=$$

$$< I_Z R, u > .$$

But $I_Z R : M_1 \to \ell_p\{\ell_q\}$ is a positive map. Then from Theorem 10, $I_Z R$ is $p\{q\}$-integral. Accordingly,

$$| < I_Z R, u > | \leq \|I_Z R\| (g_{p\{q\}})'(u; M_1 \otimes (\ell_p\{\ell_q\})') \leq$$

$$(1+\varepsilon)\|R\|(g_{p\{q\}})'(\sum_{i=1}^{n} A(f_i) \otimes ((DC)^{-1})'(x_i'); M_1 \otimes Z') \leq$$

$$(1+\varepsilon)\|D\|\|C\|\|T''\|\|(DC)^{-1}\|(g_{p\{q\}})'(\sum_{i=1}^{n} A(f_i)\otimes x_i'; M_1\otimes N)\leq$$

$$(1+\varepsilon)^3\|T\|\|A\|(g_{p\{q\}})'(z; M\otimes N)\leq$$

$$(1+\varepsilon)^4\|T\|(g_{p\{q\}})'(z; M\otimes N)\leq(1+\varepsilon)^4\|T\|((g_{p\{q\}})'(z; L_\infty(\mu)\otimes X')+\varepsilon).$$

In consequence

$$|<T'', z>|\leq\varepsilon\|T\|\sum_{i=1}^{n}\|f_i\|\|x_i'\|+(1+\varepsilon)^4\|T\|((g_{p\{q\}})'(z; L_\infty(\mu)\otimes X')+\varepsilon).$$

Then as ε is arbitrary,

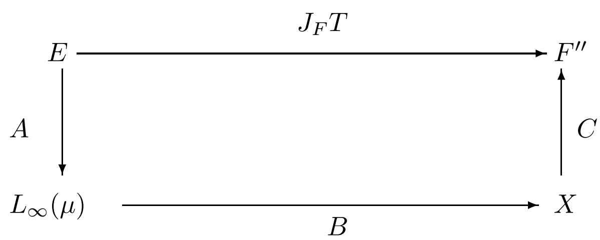
$$|<T'', z>|\leq\|T\|(g_{p\{q\}})'(z; L_\infty(\mu)\otimes X'),$$

hence $T''\in(L_\infty(\mu)\otimes_{(g_{p\{q\}})'}X')'$ with $\|T''\|_{(L_\infty(\mu)\otimes_{(g_{p\{q\}})'}X')'}\leq\|T\|=\|T''\|$. Then T'' is $p\{q\}$-integral, hence T is also $p\{q\}$-integral with $p\{q\}$-integral's norm being less or equal to $\|T\|$. □

Theorem 15 *For every pair of Banach spaces E, F, the following statements are equivalent*

1) $T\in\mathcal{I}_{p\{q\}}(E,F)$.

2) J_FT factors in the following way:



such that

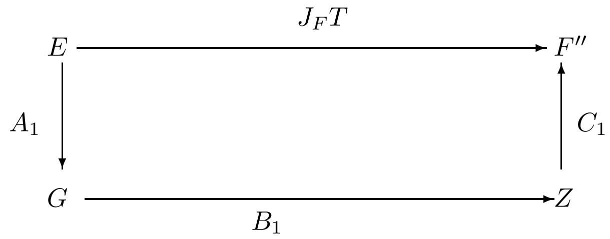
a) If $1<p,q<\infty$, $X=L_p(\mu_1, L_q(\mu_2))$, $\mu=\mu_1\times\mu_2$ and B is a multiplication operator for a function $g\in L_p(\mu_1, L_q(\mu_2))$.

b) If $p = 1$ or $q = 1$, X is isometric to a sublattice of some $L_p(\mu_1, L_q(\mu_2))$ and B is a lattice homomorphism.

Furthermore, $\boldsymbol{I}_{p\{q\}}(T) = inf\{\|D\|\|B\|\|A\|\}$, taking it over to all such factors.

Proof. 2) $\to$ 1): This is evident from the preceding theorem.

1) $\to$ 2): Using Proposition 11, $J_F T$ factors as follows:



where G is an abstract M-space, Z is a band of some $L_p(\mu_1, L_q(\mu_2))$ and B_1 is a lattice homomorphism. However, G'' is lattice isometric to some $L_\infty(\mu)$ by a positive isometric map $S : L_\infty(\mu) \to G''$.

If $1 < p, q < \infty$, then $B_1'' : G'' \to Z$. As Z is a band in $L_p(\mu_1, L_q(\mu_2))$, it is a projection band. The map $I : X \to L_p(\mu_1, L_q(\mu_2))$ denotes the inclusion map and $P : L_p(\mu_1, L_q(\mu_2)) \to Z$ the corresponding projection. Since $L_p(\mu_1, L_q(\mu_2))$ is order complete, the function of $L_p(\mu_1, L_q(\mu_2))$ exists such that

$$g := \sup_{\|h\|_{L_\infty(\mu)} \leq 1} B_1'' S(h).$$

The diagonal maps $D_{1/g} : L_\infty(\mu) \to L_\infty(\mu_1 \times \mu_2)$ such that $D_{1/g}(h) = B_1'' S(h)/g$ and $D_g : L_\infty(\mu_1 \times \mu_2) \to L_p(\mu_1, L_q(\mu_2))$ such that $D_g(f) = fg$, are well defined, linear and continuous with $B_1'' S = D_g D_{1/g}$. Then the result follows taking $A = D_{1/g} S^{-1} J_G A_1$, $B = I D_g$, $C = C_1 P$ and $X = L_p(\mu_1, L_q(\mu_2))$.

If $p = 1$ or $q = 1$, as F'' is complemented in F'''', and if $P : F'''' \to F''$ is the projection then $\|P\| \leq 1$ and $P J_{F''} J_F T = J_F T$. According to this,

$$\begin{array}{ccc} E & \xrightarrow{\quad J_F T \quad} & F'' \\ {\scriptstyle S^{-1}J_G A_1}\Big\downarrow & & \Big\uparrow{\scriptstyle PC_1''} \\ L_\infty(\mu) & \xrightarrow[\quad B_1''S \quad]{} & Z''\,. \end{array}$$

But Z is lattice finitely representable in $\ell_p\{\ell_q\}$, hence Z'' is also lattice finitely representable in $\ell_p\{\ell_q\}$, and consequently it is isometric to a sublattice of a band of some $L_p(\mu_1', L_q(\mu_2'))$. □

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P. GÓMEZ PALACIO
Universidad EAFIT
Departamento de Ciencias Básicas
Carrera 49 n- 7 sur-50 Medellín. Colombia.
e-mail: pagomez@eafit.edu.co
J.A. LÓPEZ MOLINA and M.J. RIVERA
Universidad Politécnica de Valencia
E.T.S. Ingenieros Agrónomos
Camino de Vera 46072 Valencia. Spain.
e-mails: jalopez@mat.upv.es, mjrivera@mat.upv.es

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