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Diagonal Lift in the Cotangent Bundle and its Applications

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Abstract

The purpose of this paper is to define a diagonal lift ${}^{\mathcal{D}}g$ of a Riemannian metric g of a manifold M_n to the cotangent bundle $T^*(M_n)$ of M_n, to associate with ${}^{\mathcal{D}}g$ an Levi-Civita connection of $T^*(M_n)$ in a natural way and to investigate applications of the diagonal lifts.

Key words and phrases: Cotangent bundle, Riemannian metric, diagonal lift, Levi-Civita connection, B-manifold, Killing vector field, geodesic.

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $T^*(M_n)$ the cotangent bundle over M_n. If x^i are local coordinates in a neighborhood U of a point $x \in M_n$, then a covector p at x which is an element of $T^*(M_n)$ is expressible in the form (x^i, p_i), where p_i are components of P with respect to the natural frame ∂_i. We may consider (x^i, p_i)=$(x^i, x^{\bar{i}}) = (x^J)$, $i = 1, \ldots, n$; $\bar{i} = n+1, \ldots, 2n$; $J = 1, \ldots, 2n$ as local coordinates in a neigborhood $\pi^{-1}(U)$ (π is the natural projection $T^*(M_n)$ onto M_n).

Let now M_n be a Riemannian manifold with nondegenarate metric g whose components in a coordinate neighborhood U are g_{ij} and denote by Γ^h_{ji} the Christoffel symbols formed with g_{ji}.

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We denote by $\mathcal{T}^r_s(M_n)$ the module over $F(M_n)$ ($F(M_n)$ is the ring of C^∞ functions in M_n) all tensor fields of class C^∞ and of type (r,s) in M_n. Let $X \in \mathcal{T}^1_0(M_n)$ and $w \in \mathcal{T}^0_1(M_n)$. Then ${}^CX$ (complete lift), ${}^HX$ (horizontal lift) and ${}^Vw$ (vertical lift) have, respectively, components [5]

$$ {}^CX = \begin{pmatrix} X^h \\ -p_m\partial_h X^m \end{pmatrix}, \quad {}^HX = \begin{pmatrix} X^h \\ p_m\Gamma^m_{hi}X^i \end{pmatrix}, \quad {}^Vw = \begin{pmatrix} 0 \\ w_h \end{pmatrix} \tag{1} $$

with respect to the coordinates $(x^h, x^{\overline{h}})$ in $T^*(M_n)$, where X^h and w_h are respectively local components of X and w.

In each coordinate neighborhood $U(x^h)$ of M_n, we put

$$ X_{(j)} = \frac{\partial}{\partial x^j}\ , \qquad w^{(j)} = dx^j. $$

Taking account of (1), we easily see that the components of ${}^HX_{(j)}$ and ${}^Vw^{(j)}$ are respectively given by

$$ {}^HX_{(j)} = (A_j{}^H) = \begin{pmatrix} \delta^h_j \\ p_m\Gamma^m_{jh} \end{pmatrix}, \quad {}^Vw^{(j)} = (A_{\overline{j}}{}^H) = \begin{pmatrix} 0 \\ \delta^j_h \end{pmatrix} $$

with respect to the coordinates $(x^h, x^{\overline{h}})$. We call the set $\{{}^HX_{(j)}, {}^Vw^{(j)}\}$ the frame adapted to the Riemannian connection Γ in $\pi^{-1}(U) \subset T^*(M_n)$. On putting

$$ A_{(j)} = {}^HX_{(j)}, \qquad A_{(\overline{j})} = {}^Vw^{(j)} $$

we write the adapted frame as $\{A_{(\beta)}\} = \{A_{(j)}, A_{(\overline{j})}\}$.

It is easily verified that $2n$ local 1-forms

$$ \tilde{A}^{(i)} = \left(\tilde{A}^{(i)}{}_H\right) = \left(\tilde{A}^{(i)}{}_h, \tilde{A}^{(i)}{}_{\overline{h}}\right) = \left(\delta^i_h, 0\right) = dx^i \quad i = 1, \ldots, n, $$

$$ \tilde{A}^{(i)} = \left(\tilde{A}^{(i)}{}_H\right) = \left(\tilde{A}^{(i)}{}_h, \tilde{A}^{(i)}{}_{\overline{h}}\right) = (-p_m\Gamma^m_{hi}\,\delta_{hi}) = dp_i - p_m\Gamma^m_{hi}dx^h = \delta p_i \quad \overline{i} = n+1, \ldots, 2n \tag{2} $$

form a coframe $\{\tilde{A}^\alpha\} = \{\tilde{A}^{(i)}, \tilde{A}^{(\overline{i})}\}$ dual to the adapted frame $\{A_{(\beta)}\}$, i.e. $\tilde{A}^{(\alpha)}{}_H A_{(\beta)}{}^H = \delta^\alpha_\beta$.

2. Lift ${}^{\mathcal{D}}g$ of a Riemannian g to $T^*(M_n)$

On putting locally

$$ {}^{\mathcal{D}}g = g_{ji}\tilde{A}^{(j)} \otimes \tilde{A}^{(i)} + \sum_{j,i=1}^{n} g^{ji}\tilde{A}^{(\bar{j})} \otimes \tilde{A}^{(\bar{i})} \tag{3} $$

in $T^*(M_n)$, we see that ${}^{\mathcal{D}}g$ defines a tensor field of type $(0, 2)$ in $T^*(M_n)$, which called the diagonal lift of the tensor field g to $T^*(M_n)$ with respect to Γ. From (2) and (3) we prove that ${}^{\mathcal{D}}g$ has components of the form

$$ ({}^{\mathcal{D}}g_{\beta\alpha}) = \left(\begin{array}{c|c} g_{ji} & 0 \\ \hline 0 & g^{ji} \end{array} \right) \tag{4} $$

with respect to the coframe $\{\tilde{A}^{(\alpha)}\}$ (or with respect to the adapted frame $\{A_{(\beta)}\}$) in $T^*(M_n)$ and components

$$ {}^{\mathcal{D}}g = \left(\begin{array}{c|c} g_{ji} + g^{ks}p_m p_\ell \Gamma^m_{jk}\Gamma^\ell_{is} & -g^{is}p_\ell\Gamma^\ell_{js} \\ \hline -g^{js}p_\ell\Gamma^\ell_{is} & g^{ji} \end{array} \right) \tag{5} $$

with respect to the local coordinates $(x^j, x^{\bar{j}})$, where g^{ji} denote contravariant components of g.

From (4) it easily follows that if g is a Riemannian metric in M_n, then ${}^{\mathcal{D}}g$ is a Riemannian metric in $T^*(M_n)$. The metric ${}^{\mathcal{D}}g$ is similar to that of the Riemannian extension studied by S. Sasaki in the tangent bundle [4] (for the frame bundle, see [2]).

From (1) and (5) we have

$$ {}^{\mathcal{D}}g\,({}^{H}X, {}^{H}Y) = g(X, Y). $$

We hence have

Theorem 1. Let X, $Y \in \mathcal{T}^1_0(M_n)$. Then the inner product of the horizontal lifts ${}^{H}X$ and ${}^{H}Y$ to $T^*(M_n)$with the metric ${}^{\mathcal{D}}g$ is equal to the vertical lift of the inner product of X and Y in M_n.

From (1) and (5) we have also

$$ {}^{\mathcal{D}}g({}^{V}w, {}^{V}\theta) = {}^{V}(g(w, \theta)), \quad \forall w, \theta \in \mathcal{T}^0_1(M_n), $$

$$
\begin{aligned}
{}^{\mathcal{D}}g({}^{V}w, {}^{C}X) &= -g^{js}w_j p_\ell(\partial_s X^\ell + \Gamma^\ell_{si}X^i) \\
&= -g^{js}w_j(\iota(\nabla X))_s \\
&= -{}^{V}(g(w, \iota(\nabla X)),
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathcal{D}}g({}^{C}X, {}^{C}Y) &= g_{ji}x^j y^i + g^{ji}p_k p_\ell(\nabla_j X^k)(\nabla_i Y^\ell) \\
&= g_{ji}x^j y^i + g^{ji}(\iota(\nabla X))_j(\iota(\nabla Y))_i \\
&= {}^{V}(g(X,Y)) + {}^{V}(g(\iota(\nabla X), \iota(\nabla Y))), \ \forall X, Y \in \mathcal{T}^1_0(M_n), \forall w \in \mathcal{T}^0_1(M_n),
\end{aligned} \tag{6}
$$

where $(\iota(\nabla X)$ is a 1-form with local expression:

$$\iota(\nabla X) = p_\ell \nabla_s X^\ell dx^s.$$

We recall that any element $t \in \mathcal{T}^0_r(T^*(M_n))$ is completely determined by its action on lifts of the type ${}^{C}X_1, {}^{C}X_2, \cdots, {}^{C}X_r$, where X_i, $i = 1, \ldots, r$ are arbitrary vector fields in M_n [5,p. 237]. Then ${}^{\mathcal{D}}g$ is completely determined by (6).

3. Levi-Civita Connection of ${}^{\mathcal{D}}g$

The components of the adapted frame $\{A_{(\beta)}\}$ are given by

$$(A_\beta{}^H) = (A_j{}^H, A_{\overline{j}}{}^H) = \left(\begin{array}{c|c} \delta^h_j & 0 \\ \hline p_m\Gamma^m_{jh} & \delta^j_h \end{array}\right). \tag{7}$$

The indices $\alpha, \beta, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the adapted frame. The inverse of the matrix (7) is given by

$$(\tilde{A}^\beta{}_H) = \begin{pmatrix} \tilde{A}^j{}_H \\ \tilde{A}^{\overline{j}}{}_H \end{pmatrix}, \tag{7'}$$

$\tilde{A}^j{}_H$ and $\tilde{A}^{\overline{j}}{}_H$ being defined by

$$\tilde{A}^j{}_H = (\delta^j_h, 0), \ \ \tilde{A}^{\overline{j}}{}_H = (-p_m\Gamma^m_{hj}, \delta^h_j).$$

We now consider local vector fields A_β defined in $\pi^{-1}(U)$ by

$$A_\beta = A_\beta{}^H \partial_H,$$

which will give for various types of indices

$$A_j = \partial_j + \sum_{h=1}^{n} p_m \Gamma^m_{jh} \partial_{\overline{h}}, \;\; A_{\overline{j}} = \partial_{\overline{j}}. \tag{8}$$

If $\Omega_{\gamma\beta}{}^\alpha$ denote the non-holonomic object with respect to the vector fields A_β, then we have

$$[A_\gamma, A_\beta] = \Omega_{\gamma\beta}{}^\alpha A_\alpha.$$

According to (7), (7') and (8), the components of the non-holonomic object are given by

$$\Omega_{\gamma\beta}{}^\alpha = (A_\gamma A_\beta{}^C - A_\beta A_\gamma{}^C) A^\alpha{}_C.$$

The only non-vanishing components of $\Omega_{\gamma\beta}{}^\alpha$ are

$$\begin{aligned} \Omega_{ji}{}^{\overline{h}} &= p_m R_{jih}{}^m, \\ \Omega_{ij}{}^{\overline{h}} &= -p_m R_{jih}{}^m, \\ \Omega_{j\,\overline{i}}{}^{\overline{h}} &= -\Gamma^i_{jh}, \\ \Omega_{\overline{i}j}{}^{\overline{h}} &= \Gamma^i_{jh}, \end{aligned} \tag{9}$$

where $R_{jih}{}^m$ are components of the curvature tensor of Γ with metric g_{ij}.

Components of the Riemannian connection determined by the metric ${}^{\mathcal{D}}g$ are given by

$${}^{\mathcal{D}}\Gamma^\alpha_{\gamma\beta} = \frac{1}{2}\,{}^{\mathcal{D}}g^{\alpha\epsilon}(\,A_\gamma\,{}^{\mathcal{D}}g_{\epsilon\beta} + A_\beta\,{}^{\mathcal{D}}g_{\gamma\epsilon} - A_\epsilon\,{}^{\mathcal{D}}g_{\gamma\beta}) + \frac{1}{2}(\Omega_{\gamma\beta}{}^\alpha + \Omega^\alpha{}_{\gamma\beta} + \Omega^\alpha{}_{\beta\gamma}), \tag{10}$$

where $\Omega^\alpha{}_{\gamma\beta} = {}^{\mathcal{D}}g^{\alpha\epsilon}\,{}^{\mathcal{D}}g_{\delta\beta}\Omega_{\epsilon\gamma}{}^\delta$, ${}^{\mathcal{D}}g^{\alpha\epsilon}$ are the contravariant components of the metric ${}^{\mathcal{D}}g$ with respect to the adapted frame:

$$\left({}^{\mathcal{D}}g^{\beta\alpha}\right) = \left(\begin{array}{c|c} g^{ji} & 0 \\ \hline 0 & g_{ji} \end{array}\right) \tag{11}$$

Thus, according to (8), (9), (10) and (11), the components ${}^{\mathcal{D}}\Gamma^{\alpha}_{\gamma\beta}$ with respect to adapted frame are given by

$$
\begin{aligned}
&{}^{\mathcal{D}}\Gamma^{h}_{ji} = \Gamma^{h}_{ji}, \quad {}^{\mathcal{D}}\Gamma^{h}_{j\bar{i}} = \frac{1}{2} p_m R^{h\ \ im}_{\ j}, \\
&{}^{\mathcal{D}}\Gamma^{h}_{\bar{j}i} = \frac{1}{2} p_m R^{h\ \ jm}_{\ i}, \quad \Gamma^{h}_{\bar{j}\,\bar{i}} = 0, \\
&{}^{\mathcal{D}}\Gamma^{\overline{h}}_{ji} = \frac{1}{2} p_m R_{jih}{}^{m}, \quad {}^{\mathcal{D}}\Gamma^{\overline{h}}_{j\bar{i}} = -\Gamma^{i}_{jh}, \quad {}^{\mathcal{D}}\Gamma^{\overline{h}}_{\bar{j}i} = 0, \\
&\Gamma^{\overline{h}}_{\bar{j}\,\bar{i}} = 0,
\end{aligned}
\tag{12}
$$

where $R^{h\ \ im}_{\ j} = g^{hl} g^{ki} R_{ljk}{}^{m}$.

The covariant derivative of the diagonal lift ${}^{\mathcal{D}}\varphi\,(\varphi \in \mathcal{T}^1_1(M_n))$ has components

$$
{}^{\mathcal{D}}\nabla_{\delta}\,{}^{\mathcal{D}}\varphi^{\alpha}_{\beta} = A_{\delta}\,{}^{\mathcal{D}}\varphi^{\alpha}_{\beta} + {}^{\mathcal{D}}\,\Gamma^{\alpha}_{\delta\epsilon}\,{}^{\mathcal{D}}\varphi^{\epsilon}_{\beta} - {}^{\mathcal{D}}\,\Gamma^{\epsilon}_{\delta\beta}\,{}^{\mathcal{D}}\varphi^{\alpha}_{\epsilon}
$$

with respect to the adapted frame, where the components of ${}^{\mathcal{D}}\varphi$ are given by [5, p. 291]

$$
{}^{\mathcal{D}}\varphi^{\alpha}_{\beta} = \left(\begin{array}{c|c} \varphi^{i}_{j} & 0 \\ \hline 0 & -\varphi^{j}_{i} \end{array}\right) \tag{13}
$$

with respect to the adapted frame.

Let us consider a 2n-dimensional Riemannian manifold M_{2n} with the almost complex structure φ. If tensor of Riemann metric g_{ij} satisfies

$$
g_{mj}\varphi^{m}_{i} = g_{im}\varphi^{m}_{j},
$$

then we call this Riemann metric a pure metric in an almost complex manifold M_{2n} and call an almost $\mathcal{B}$-manifold an almost complex space with a pure metric.

Now, if a pure metric satisfies

$$
\nabla_j F^{h}_{i} = 0 \quad \text{or} \quad \phi_k g_{ij} = 0,
$$

where ϕ is the Tachibana operator, then we call this manifold a $\mathcal{B}$-manifold (see [1], [3]) (∇_k denotes the covariant differention with respect to the Christoffel symbols formed with g_{ij}). Taking account of (8) and (12), we find that ${}^{\mathcal{D}}\nabla\,{}^{\mathcal{D}}\varphi$ has components given by

$$\begin{aligned}
{}^{\mathcal{D}}\nabla_k\, {}^{\mathcal{D}}\varphi_j^i &= \nabla_k \varphi_j^i, \\
{}^{\mathcal{D}}\nabla_k\, {}^{\mathcal{D}}\varphi_{\overline{j}}^{\overline{i}} &= -\nabla_k \varphi_i^j, \\
{}^{\mathcal{D}}\nabla_{\overline{k}}\, {}^{\mathcal{D}}\varphi_j^i &= \frac{1}{2} p_\ell \left(R^i{}_m{}^{k\ell} \varphi_j^m - R^m{}_j{}^{k\ell} \varphi_m^i \right), \\
{}^{\mathcal{D}}\nabla_k\, {}^{\mathcal{D}}\varphi_{\overline{j}}^i &= -\frac{1}{2} p_\ell \left(R^i{}_k{}^{m\ell} \varphi_m^j + R^m{}_k{}^{j\ell} \varphi_m^i \right), \\
{}^{\mathcal{D}}\nabla_k\, {}^{\mathcal{D}}\varphi_j^{\overline{i}} &= \frac{1}{2} p_\ell \left(R_{kmi}{}^\ell \varphi_j^m + R_{kjm}{}^\ell \varphi_i^m \right).
\end{aligned}$$

all the others being zero, with respect to the adapted frame.

From (11) and (13) we easily find that ${}^{\mathcal{D}}g$ is pure with respect to the structure ${}^{\mathcal{D}}\varphi$. Thus we have

Theorem 2. The cotangent bundle of $\mathcal{B}$-manifold is $\mathcal{B}$-manifold with respect to the metric ${}^{\mathcal{D}}g$ and the structure ${}^{\mathcal{D}}\varphi$ if and only if the Riemannian manifold is locally flat.

4. Killing vector fields

A vector field $X \in T_0^1(M_n)$ is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold with metric g, if $\mathcal{L}_X\, g = 0$ [5, p78]. In terms of components g_{ji} of g, X is a Killing vector field if and only if

$$\mathcal{L}_X\, g_{ji} = X^\alpha \nabla_\alpha g_{ji} + g_{\alpha i} \nabla_j X^\alpha + g_{j\alpha} \nabla_i X^\alpha = \nabla_j X_i + \nabla_i X_j = 0,$$

X^α being components of X, where ∇ is the Riemannian connection of the metric g.

Let $\tilde{X}$ be a covector field in $T^*(M_n)$ and

$$\left(\tilde{X}_\alpha\right) = \left(\tilde{X}_h, \tilde{X}_{\overline{h}}\right)$$

its components with respect to the adapted frame. Then the covariant derivative ${}^{\mathcal{D}}\nabla\tilde{X}$ has components

$$ {}^{\mathcal{D}}\nabla_\beta \tilde{X}_\alpha = A_\beta \tilde{X}_\alpha + {}^{\mathcal{D}}\Gamma_{\beta\alpha}^\gamma \tilde{X}_\gamma, \tag{14}$$

${}^{\mathcal{D}}\Gamma_{\beta\alpha}$ being given by (12), with respect to the adapted frame. From (1) we see that components of ${}^{C}X$, ${}^{H}X$ and ${}^{V}w$

$$ {}^{C}X^{\alpha} = A^{\alpha}{}_{A}\,{}^{C}X^{A}, \;\; {}^{H}X^{\alpha} = A^{\alpha}{}_{A}\,{}^{H}X^{A}, \;\; {}^{V}w^{\alpha} = A^{\alpha}{}_{A}\,{}^{V}w^{A} $$

with respect to the adapted frame ${}^{C}X$, ${}^{H}X$ and ${}^{V}w$ are given respectively by

$$ \left({}^{C}X^{\alpha}\right) = \begin{pmatrix} X^{h} \\ -p_m \nabla_h X^m \end{pmatrix}, \quad \left({}^{H}X^{\alpha}\right) = \begin{pmatrix} X^{h} \\ 0 \end{pmatrix}, \quad \left({}^{V}w^{\alpha}\right) = \begin{pmatrix} 0 \\ w_h \end{pmatrix} $$

by virtue of (7').

The associated covector fields of the complete, horizontal and vertical lifts to $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$ are given respectively by

$$ \begin{aligned} \left({}^{C}X_{\beta}\right) &= \left({}^{\mathcal{D}}g_{\beta\alpha}\,{}^{C}X^{\alpha}\right) = \left(X_j, -g^{ji}p_m\nabla_i X^m\right), \\ \left({}^{H}X_{\beta}\right) &= \left({}^{\mathcal{D}}g_{\beta\alpha}\,{}^{H}X^{\alpha}\right) = \left(X_j, 0\right), \\ \left({}^{V}w_{\beta}\right) &= \left({}^{\mathcal{D}}g_{\beta\alpha}\,{}^{V}w^{\alpha}\right) = \left(0, w^j\right) \end{aligned} \tag{15} $$

with respect to the adapted frame, where $X_j = g_{ji}X^i$, $w^j = g^{ji}w_i$.

We now compute the Lie derivatives of the metric ${}^{\mathcal{D}}g$ with respect to ${}^{C}X$, ${}^{H}X$ and ${}^{V}w$, by means of (14) and (15). The Lie derivatives of ${}^{\mathcal{D}}g$ with respect to ${}^{C}X$, ${}^{H}X$ and ${}^{V}w$ have respectively components

$$ \begin{aligned} \left(\mathcal{L}_{{}^{C}X}\,{}^{\mathcal{D}}g_{\beta\alpha}\right) &= \left({}^{\mathcal{D}}\nabla_{\beta}\,{}^{C}X_{\alpha} + {}^{\mathcal{D}}\nabla_{\alpha}\,{}^{C}X_{\beta}\right) \\ &= \left(\begin{array}{c|c} \nabla_j X_i + \nabla_i X_j & -p_m g^{ki} g^{tm}\left(\nabla_j\nabla_k X_t + R_{\ell jkt}X^{\ell}\right) \\ \hline -p_m g^{kj} g^{tm}\left(\nabla_i\nabla_k X_t + R_{\ell ikt}X^{\ell}\right) & -\left(g^{is}\nabla_s X^j + g^{js}\nabla_s X^i\right) \end{array}\right), \end{aligned} \tag{16} $$

$$ \left(\mathcal{L}_{{}^{H}X}\,{}^{\mathcal{D}}g_{\beta\alpha}\right) = \left({}^{\mathcal{D}}\nabla_{\beta}\,{}^{H}X_{\alpha} + {}^{\mathcal{D}}\nabla_{\alpha}\,{}^{H}X_{\beta}\right) = \left(\begin{array}{c|c} \nabla_j X_i + \nabla_i X_j & -p_m g^{ki} R_{\ell jk}{}^{m} X^{\ell} \\ \hline -p_m g^{kj} R_{\ell ik}{}^{m} X^{\ell} & 0 \end{array}\right), $$

$$ \left(\mathcal{L}_{{}^{V}w}\,{}^{\mathcal{D}}g_{\beta\alpha}\right) = \left({}^{\mathcal{D}}\nabla_{\beta}\,{}^{V}w_{\alpha} + {}^{\mathcal{D}}\nabla_{\alpha}\,{}^{V}w_{\beta}\right) = \left(\begin{array}{c|c} 0 & g^{is}\nabla_j w_s \\ \hline g^{js}\nabla_i w_s & 0 \end{array}\right) $$

with respect to the adapted frame in $T^*(M_n)$.

Since we have

$$\nabla_i \nabla_k X_t + R_{\ell i k t} X^\ell = 0, \qquad \mathcal{L}_X\, g^{ji} = -\left(g^{js}\nabla_s X^i + g^{is}\nabla_s X^j\right) = 0$$

as a consequence of $\mathcal{L}_X\, g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$ (see [6, p.17]), we conclude by means of (16) that the complete lift ${}^C X$ is a Killing vector field in $T^*(M_n)$ if and only if X is a Killing vector field in M_n.

We next have

$$R_{\ell i k}{}^m X^\ell = 0$$

as a consequence of the vanishing of the second covariant derivative of X. Conversely, the conditions $\mathcal{L}_X\, g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$ and $R_{\ell i k}{}^m X^\ell = 0$ imply that the second covariant derivative of X vanishes. Summing up these results, we have

Theorem 3. Necessary and sufficient conditions in order that the

a) complete ${}^C X \in \mathcal{T}_0^1(T^*(M_n))$,

b) horizontal ${}^H X \in \mathcal{T}_0^1(T^*(M_n))$ and

c) vertical ${}^V w \in \mathcal{T}_0^1(T^*(M_n))$

lifts to $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$, of a vector field X and covector field w in M_n be a Killing vector field in $T^*(M_n)$ are that,

a) X is a Killing vector field in M_n,

b) X is a Killing vector field with vanishing second covariant derivative in M_n and

c) w is parallel in M_n.

5. Geodesics in $T^*(M_n)$ with metric ${}^{\mathcal{D}}g$

Let C be a curve in M_n expressed locally by $x^h = x^h(t)$ and $w_h(t)$ be a covector field along C. Then, in the cotangent bundle $T^*(M_n)$, we define a curve $\tilde{C}$ by

$$x^h = x^h(t), \;\; x^{\tilde{h}} \overset{\text{def}}{=} p_h = w_h(t) \tag{17}$$

If the curve C satisfies at all the points the relation

$$\frac{\delta w_n}{dt} = \frac{dw_n}{dt} - \Gamma_{j\ h}^{\ i} \frac{dx^j}{dt} w_i = 0, \tag{18}$$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve C in M_n. Thus, if the initial condition $w_h = w_h^0$ for $t = t_0$ is given, there exists a unique horizontal lift expressed by (17).

We now consider differential equations of the geodesics of the cotangent bundle $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$. If t is the arc lenght of a curve $x^A = x^A(t)$ in $T^*(M_n)$, equations of geodesic in $T^*(M_n)$ have the usual form

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + {}^{\mathcal{D}}\Gamma_C{}^A{}_B \frac{dx^C}{dt}\frac{dx^B}{dt} = 0 \tag{19}$$

with respect to the induced coordinates $(x^i,\, x^{\bar{i}}) = (x^i,\, \rho_i)$ in $T^*(M_n)$.

We find it more convenient to refer equations (19) to the adapted frame $\{A_{(i)},\, A_{(\bar{i})}\}$. Using (2), we now write

$$\theta^h = A^{(h)}{}_A\, dx^A = dx^h,$$

$$\theta^{\bar{h}} = A^{(\bar{h})}{}_A\, dx^A = \delta p_h,$$

and put

$$\frac{\theta^h}{dt} = A^{(h)}{}_A \frac{dx^A}{dt} = \frac{dx^h}{dt},$$

$$\frac{\theta^{\bar{h}}}{dt} = A^{(\bar{h})}{}_A \frac{dx^A}{dt} = \frac{\delta p_h}{dt} = \frac{\delta p_h}{dt} - \Gamma_j{}^i{}_h \frac{dx^j}{dt} p_i$$

along a curve $x^A = x^A(t)$, i.e., $x^h = x^h(t)$, $p_h = p_h(t)$ in $T^*(M_n)$.

If we therefore write down the form equivalent to (19), namely,

$$\frac{d}{dt}\Big(\frac{\theta^\alpha}{dt}\Big) + {}^{\mathcal{D}}\,\Gamma_\delta{}^\alpha_\beta\Big(\frac{\theta^\gamma}{dt}\Big)\Big(\frac{\theta^\beta}{dt}\Big) = 0$$

with respect to the adapted frame and take account of (12), then we have

$$\begin{cases} \frac{\delta^2 x^h}{dt^2} + p_m R^h{}_j{}^{im}\, \frac{dx^j}{dt}\frac{\delta p_i}{dt} = 0, \\ \frac{\delta^2 p_h}{dt^2} + \frac{1}{2} p_m R_{j\,i\,h}{}^m\, \frac{dx^j}{dt}\frac{dx^i}{dt} = 0 \end{cases} \tag{20}$$

Since we have

$$R_{j\,i\,h}{}^{m}\,\frac{dx^j}{dt}\frac{dx^i}{dt}=0$$

as a consequence of $R_{(j\,i)\,h}{}^{m}=0$, we conclude by means of (20) that a curve $x^i=x^i(t)$, $p_h=p_h(t)$ in $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$ is a geodesic in $T^*(M_n)$, if and only if

$$\begin{cases}\frac{\delta^2x^h}{dt^2}+p_mR^h_{\ j}{}^{\ im}\ \frac{dx^j}{dt}\frac{\delta p_i}{dt}=0, & \text{(a)}\\ \frac{\delta^2p_h}{dt^2}=0 & \text{(b)}\end{cases}\tag{21}$$

If a curve satisfying (21) lies on a fibre given by x^h =const, then (20, (b)) reduces to

$$\frac{d^2p_h}{dt^2}=0$$

so that $p_h=a_h\,t+b_h$, a_h and b_h being constant. Thus we have

Theorem 4. If geodesic $x^h=x^{h(t)}$, $p_h=p_h(t)$ lies in a fibre of $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$, the a geodesic is expressed by linear equations $x^h=c^h$, $p_h=a_h\,t+b_h$, where a_h, b_h and c^h are constant.

From (18) and (21), we have

Theorem 5. The horizontal lift of a geodesic in M_n is always geodesic in $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$.

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