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Absolutely Representing Systems of Exponentials in the Spaces of Infinitely-Differentiable Functions and Extendability in the Sense of Whitney

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Abstract

Let Q be a compactum in $\mathbb{R}^p$, $p \geqslant 1$, such that $int\mathrm{Q} \neq \varnothing$ and $Q = \overline{int\mathrm{Q}}$. Denote by $C^\infty[Q]$ the space of functions from $C^\infty(int\mathrm{Q})$ uniformly continuous in $int\mathrm{Q}$ together with all their partial derivatives. The conditions of the existence of absolutely representing systems of exponentials with purely imaginary exponents in the space $C^\infty[Q]$ and some of its subspaces of Denjoy–Carleman type are investigated. It is also proved under rather general assumptions that there is no such absolutely representing systems in the space $E(G) = \underset{\overleftarrow{Q \in \mathcal{F}_G}}{\mathrm{proj}}\ E[Q]$ where G is an arbitrary open set in $\mathbb{R}^p$, $E[Q]$ is $C^\infty[Q]$ or its subspace mentioned above and $\mathcal{F}_G$ is the totality of all non-empty compact sets $\mathcal{K}$ in G with the property $\mathcal{K} = \overline{int\mathcal{K}}$.

1.

Let Q be a set in $\mathbb{R}^p$, $p \geqslant 1$, and let $\overset{\circ}{Q}$ be its interior. A compactum Q is said to be fat if $\overset{\circ}{Q} \neq \varnothing$ and $Q = \overline{\overset{\circ}{Q}}$. Denote by $\mathcal{F}_G$ the totality of all fat compacta containing an open set G. If $G = \mathbb{R}^p$ we write $\mathcal{F}$ instead of $\mathcal{F}_{\mathbb{R}^p}$. Let $C^\infty[F]$ and $F \in \mathcal{F}$, be the Frechet space of all complex-valued functions infinitely differentiable in $\overset{\circ}{F}$ and uniformly continuous in $\overset{\circ}{\mathcal{K}}$ together with all their partial derivatives. The topology in $C^\infty[F]$ is defined by norms $\|y\|_m := \sup\{\, |y^\alpha(x)| : x \in \overset{\circ}{\mathcal{K}}\, |\alpha|_p \leqslant m\,\}$, $m = 0, 1, \ldots.$ Here $\alpha = (\alpha_1, \ldots \alpha_p) \in \mathcal{N}_0^p$, $|\alpha|_p = \sum_{k=1}^p |\alpha_k| = \sum_{k=1}^p \alpha_k$.

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If G is an arbitrary non-void open set in $\mathbb{R}^p$, then $C^\infty(G)$ is the Frechet space of all functions infinitely differentiable in G, with the topology defined by the system of norms $\|y\|_{m,F} := \sup\{\,|y^{(\alpha)}(x)| : |\alpha|_p \leqslant m,\ x \in F\,\}$, $m = 0, 1, \ldots$; $F \in \mathcal{F}_G$. It is evident that $C^\infty(G) \hookrightarrow C^\infty[\mathcal{K}], \forall \mathcal{K} \in \mathcal{F}_G$, and $C^\infty(\mathbb{R}^p) \hookrightarrow C^\infty(G)$, $C^\infty(\mathbb{R}^p) \hookrightarrow C^\infty[\mathcal{K}]$ for all open sets $G \subseteq \mathbb{R}^p$ and all $\mathcal{K}$ from $\mathcal{F}$.

Let us introduce the system

$$\mathcal{E}_\mu := \Big\{ \exp\Big(i \sum_{j=1}^{p} \mu_{j,k} x_j\Big) : \qquad k = (k_1, \ldots k_p),\ k_j = 0, \pm 1, \ldots ;\ j = 1, \ldots, p \Big\}, \qquad (1)$$

$\mu_{j,k} \in \mathbb{R}$. We are interested in finding conditions of the existence of at least one absolutely representing system of the form (1) in the spaces $C^\infty[F]$ and $C^\infty(G)$. It is worth reminding that the sequence $(x_k)_{k=1}^\infty$ of nonzero elements x_k from a complete locally convex space H is said to be an absolutely representing system (ARS) in H [4], if every element x from H can be represented in the form of a series $x = \sum_{k=1}^\infty c_k x_k$, absolutely converging in H.

An ARS X in H is said to be effective ($\mathcal{E}$ARS) [4] if for each element x the coefficients c_k of at least one series with the sum equal to x can be found constructively.

Let us say that a fat compactum $\mathcal{K}$ is a Whitney-compactum(W.-c.) if $\forall f \in C^\infty[\mathcal{K}]$ $\exists g \in C^\infty(\mathbb{R}^p) : g|_{\mathcal{K}} = f$.

Lemma 1 *For every series*

$$\sum_{|l|_p=0}^{\infty} c_l \exp\Big(i \sum_{j=1}^{p} \mu_{j,l} x_j\Big), \qquad (2)$$

the following assertions are equivalent:

1. *the series* (2) *converges absolutely in* $C^\infty[\mathcal{K}]$ *for some* $\mathcal{K} \in \mathcal{F}$*;*

2. *the series* (2) *converges absolutely in* $C^\infty[\mathcal{K}]$, $\forall \mathcal{K} \in \mathcal{F}$*;*

3. $\sum_{|l|_p=0}^\infty |c_l||\mu_l|^\alpha < \infty$, $\forall \alpha \in \mathcal{N}_0^p$, *where* $|\mu_l|^\alpha = |\mu_{1,l}|^{\alpha_1} \ldots |\mu_{p,l}|^{\alpha_p}$.

4. *the series* (2) *converges absolutely in* $C^\infty(G)$ *for some nonvoid open set* G *from* $\mathbb{R}^p$*;*

5. *the series* (2) *converges absolutely in* $C^\infty(G)$ *for all open sets* $G \subseteq \mathbb{R}^p$*;*

6. *the series* (2) *converges absolutely in* $C^\infty(\mathbb{R}^p)$*.*

The proof of Lemma 1 is very simple by virtue of the equality:

$$\left|\exp\Big(i\sum_{j=1}^{p}\mu_j x_j\Big)\right| = 1,$$

with $\forall x \in \mathbb{R}^p$, $\forall \mu = (\mu_j)_{j=1}^p \in \mathbb{R}^p$. Indeed, we have the evident implications 6)$\Rightarrow$ 5)$\Rightarrow$ 4)$\Rightarrow$ 1)$\Rightarrow$3)$\Rightarrow$6) $\Rightarrow$ 2)$\Rightarrow$ 1).

2.

Theorem 1 *Let* $\mathcal{K}$ *be a W.-c. and let* T *be an arbitrary open rectangular parallelepiped containing* $\mathcal{K}$*,* $T = \{\, x : a_j < x_j < b_j,\ j = 1, 2, ...p\,\}$*. Then the system*

$$\mathcal{E}_p^T := \left\{ \exp\Big(2\pi i\sum_{j=1}^{p}\frac{k_j x_j}{b_j - a_j}\Big) : k_s = 0, \pm 1, \ldots;\ s = 1, 2, \ldots, p \right\} \tag{3}$$

is an $\mathcal{E}ARS$ *in* $C^\infty[\mathcal{K}]$*.*

Proof. If G is an arbitrary open set in $\mathbb{R}^p$, let us denote by $C_0^\infty(G)$ the totality of all functions from $C_0^\infty(G)$ with support in G. In other words, $f \in C_0^\infty(G)$ iff $f \in C^\infty(G)$ and there exists compactum $\mathcal{K} \subset G$ such that $f \equiv 0$ in $G \setminus \mathcal{K}$. Let $y(x)$ be an arbitrary function from $C^\infty[\mathcal{K}]$ and let Y be its extension to $C^\infty(\mathbb{R}^p)$: $Y \in C^\infty(\mathbb{R}^p)$, $Y|_{\mathcal{K}} = y$. We put $d = \rho(\mathcal{K}, \partial T) = \min\{\, |x - v|_p : x \in \mathcal{K}, v \in \partial T\,\}$. A simple analysis of the proof of Theorem 1.4.1 from [3] shows that in the case $X = \mathbb{R}^p$ it is possible to determine effectively the function W from $C_0^\infty(\mathbb{R}^p)$ such that $W|_{\mathcal{K}} \equiv 1$ and $\operatorname{supp} W \subset (\mathcal{K})_{\frac{d}{2}} = \{\, x \in \mathbb{R}^p : \rho(x, \mathcal{K}) \le \frac{d}{2}\,\}$. Then $w_1 := w \cdot Y \in C_0^\infty(T)$ and $w_1|_{\mathcal{K}} \equiv y$.

Let us form the Fourier series of the function w_1 with respect to the system $\mathcal{E}_p^T$:

$$w_1 \sim \sum_{|k|_p=0}^{\infty} v_k \exp\Big\langle i2\pi k, \frac{x}{b-a}\Big\rangle, \tag{4}$$

where $\left\langle i2\pi k, \dfrac{x}{b-a}\right\rangle := 2\pi i \sum_{j=1}^{p} \dfrac{k_j x_j}{b_j - a_j}$ and

$$\prod_{j=1}^{p}(b_j - a_j)v_k = \int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p} w_1(x)\exp\left\langle -2\pi k i, \frac{x}{b-a}\right\rangle dx, \quad \forall k \in \mathcal{Z}^p \tag{5}$$

(as usual, $\mathcal{Z} = \{0, \pm 1, \pm 2, \dots\}$).

Integrating by parts the equality (5) and taking into account that $W_1^{(\gamma)}(x) \equiv 0$ near the boundary T for all $\gamma \in N_0^p$, we obtain $\forall \beta \in N_0^p$:

$$\prod_{j=1}^{p}(b_j - a_j)|v_k| \le \frac{(b-a)^\beta}{(2\pi)^{|\beta|_p}|k|^\beta}\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}|w_1^{(\beta)}(x)|\,dx,$$

where $(b-a)^\beta := \prod\limits_{j=1}^{p}(b_j - a_j)^{\beta_j}$, $|k|^\beta = |k_1|^{\beta_1}...|k_p|^{\beta_p}$; $(0)^{\beta_j} = 1, 1 \leqslant j \leqslant p$. Hence

$$(2\pi)^{|\beta|_p}|v_k| \le \frac{(b-a)^\beta}{|k|^\beta}\sup\{|w_1^{(\beta)}(x)| : x \in T\}, \quad k \in \mathcal{Z}^p, \beta \in \mathcal{N}_0^p \tag{6}$$

Further, $\forall k \in \mathcal{Z}^p, \forall m \in \mathcal{N}_0^p$ and for $F = \overline{T}$

$$\left\| v_k \exp\left\langle 2\pi k i, \frac{x}{b-a}\right\rangle \right\|_{m,F} \le$$
$$\le |v_k|(2\pi)^m \max\left\{ |k|^\gamma (b-a)^{-\gamma} : |\gamma|_p \leqslant m \right\}. \tag{7}$$

We put $\beta_j = \gamma_j + 2p$, $j = 1, 2, ..., p$, for each $\gamma \in \mathcal{N}_0^p$ such that $|\gamma|_p \leqslant m$. Then $|\beta|_p \leqslant m + 2p^2$ and $\sup\left\{|w_1^{(\beta)}(x)| : x \in T\right\} \le \|w_1\|_{m+2p^2,F}$. The relations (6), (7) imply the following inequality

$$\left\| v_k \exp\left\langle 2\pi k i, \frac{x}{b-a}\right\rangle \right\|_{m,F} \leqslant A_m \|w_1\|_{m+2p^2,F}|k|^{-2p},$$
$$k \in \mathcal{Z}^p, m \geqslant 0, F = \overline{T}.$$

Therefore the series in the right-hand side of (4) converges absolutely in $C^\infty[\overline{T}]$ moreover this series converges uniformly on $\overline{T}$. Hence

$$w_1(x) = \sum_{|k|_p=0}^{\infty} v_k \exp\langle 2\pi k i, \frac{x}{b-a}\rangle, \quad x \in \overline{T}, \tag{8}$$

and the series (8) converges absolutely in $C^\infty[\overline{T}]$. Consequently, the series at the right-hand side of (8) converges absolutely in $C^\infty[\mathcal{K}]$, and its sum is equal to $y(x)$ for all x from $\mathcal{K}$. We are done. □

Corollary Let $-\infty < a < 0 < b < +\infty$. The sequence

$$\mathcal{E}_{(\theta)} := \left\{ \exp \frac{i2kx\pi}{(b-a)}\theta \right\}_{|k|=0}^{\infty}, \qquad k \in \mathcal{Z}_0,$$

is an $\mathcal{E}$ARS in $C^\infty[a,b]$ for each $\theta \in (0,1)$.

Indeed, $\forall \theta \in (0,1)$, $\left(\frac{a}{\theta}, \frac{b}{\theta}\right) \supset [a,b]$, and we can put in Theorem 1 $p = 1$, $T = \left(\frac{a}{\theta}, \frac{b}{\theta}\right)$.

The last result is exact. To show it we remark that for each $\theta \geqslant 1$ we have $\frac{a}{\theta} \in [a,b]$, $\frac{b}{\theta} \in [a,b]$, and for every function $v(x)$ from the closure in $C^\infty[a,b]$ of linear span of $\mathcal{E}_\theta$ the equality $v\left(\frac{a}{\theta}\right) = v\left(\frac{b}{\theta}\right)$ is valid. But the last equality is not true, for example, for the function $y(x) = x$ from $C^\infty[a,b]$. Therefore the system $\mathcal{E}_{(\theta)}$ is not even complete in the space $C^\infty[a,b]$ for each $\theta \geqslant 1$. A fortiori $\mathcal{E}_\theta$ is not an ARS in $C^\infty[a,b]$, if $\theta \geqslant 1$.

3.

The following result is nearly evident.

Theorem 2 *Let $\mathcal{K}$ be an arbitrary fat compactum in $\mathbb{R}^p$. Suppose that there exists at least one ARS of the form* (1) *in $C^\infty[\mathcal{K}]$. Then $\mathcal{K}$ is a W.-c.*

Proof. If $\mathcal{E}_\mu$ (1) is an ARS in $C^\infty[\mathcal{K}]$ and if $y(x)$ is an arbitrary function from $C^\infty[\mathcal{K}]$, then there exists the series

$$\sum_{|k|_p=0}^{\infty} y_k \exp\Big(i \sum_{j=1}^{p} \mu_{j,k} x_j\Big) \tag{9}$$

converging absolutely to $y(x)$ in $C^\infty[\mathcal{K}]$. By Lemma 1 the series (9) converges absolutely in $C^\infty(\mathbb{R}^p)$. If $Y(x)$ is its sum, then $Y \in C^\infty(\mathbb{R}^p)$ and $Y|_\mathcal{K} = y$. □

Remark 2.1 *If the series* (9) *converges absolutely in* $C^\infty(\mathbb{R}^p)$*, then by the same Lemma* 1 *condition (3) is fulfilled. Hence every series*

$$\sum_{|k|_p=0}^{\infty} y_k \Big(\exp\Big(i\sum_{j=1}^{p}\mu_{j,k}x_j\Big)\Big)^{(\alpha)}, \qquad \alpha\in\mathcal{N}_0^p, \tag{10}$$

converges absolutely at each point x *from* $\mathbb{R}^p$*. If* $Y(x)$ *is the sum of the series* (9) *in* $\mathbb{R}^p$*, then* $\forall x\subset\mathbb{R}^p$ $|Y(x)|\leq\sum_{|k|_p=0}^{\infty}|y_k|<\infty$*, and for all* $\alpha\in\mathcal{N}_0^p$*.*

$$|Y^{(\alpha)}(x)|\leqslant\sum_{|k|_p=0}^{\infty}|y_k||\mu_k|^{\alpha}<\infty.$$

Denote by $BC^\infty(\mathbb{R}^p)$ the set of all functions from $C^\infty(\mathbb{R}^p)$ bounded in $\mathbb{R}^p$ together with every their derivative.

Then we can formulate some strengthening of Theorem 2.

Theorem 3 *Let all assumptions of Theorem* 2 *be fulfilled. Then for each* $y\in C^\infty[\mathcal{K}]$ $\exists Y\in BC^\infty(\mathbb{R}^p): Y|_{\mathcal{K}}=y$*.*

Now we can formulate the summarizing result.

Theorem 4 *Let* $\mathcal{K}$ *be an arbitrary fat compactum in* $\mathbb{R}^p$*. Then the following assertions are equivalent:*

1. $\mathcal{K}$ *is a W.-c.;*
2. $\forall y\in C^\infty[\mathcal{K}]\quad\exists Y\in BC^\infty(\mathbb{R}^p): Y|_{\mathcal{K}}=y$*;*
3. *there exists an ARS in* $C^\infty[\mathcal{K}]$ *of the form* (1)*;*
4. *there exists an* $\mathcal{E}ARS$ *in* $C^\infty[\mathcal{K}]$ *of the form*(1)*;*
5. *if* T *is an arbitrary rectangular open parallelepiped containing* $\mathcal{K}$*, then the corresponding system* $\mathcal{E}_p^T(3)$ *is an* $\mathcal{E}ARS$ *in* $C^\infty[\mathcal{K}]$*.*

Proof. Implications 5) $\Rightarrow$ 4) $\Rightarrow$ 3), 2) $\Rightarrow$ 1) are evident. By Theorem 3 3) $\Rightarrow$ 2). Finally Theorem 1 is equivalent to the implication 1) $\Rightarrow$ 5). □

Remark 4.1 *If any of equivalent assertions 1)–5) takes place, then each function* y *from* $C^\infty[\mathcal{K}]$ *can be extented to* $\mathbb{R}^p$ *as the sum* Y *of a certain series* (8) *absolutely converging in* $C^\infty(\mathbb{R}^p)$. *But the function* $Y(x)$ *is* $p-$*periodic:* $Y(X_1) = Y(X_2)$, *if* $(X_1)_m = (X_2)_m + (b_m - a_m)$, $m = 1, 2, \ldots, p$. *This period of the extension* $Y(x)$ *of the function* $y(x)$ *can vary in rather broad limits. Namely we can construct the required extension* $Y(x)$ *with period* $(\alpha_1, \alpha_2, \ldots, \alpha_p)$, *if there exists the point* $(a_1, \ldots, a_p)$ *such that* $\mathcal{K} \subset \{\, x : a_j < x_j < a_j + \alpha_j),\ j = 1, 2, \ldots, p\,\}$.

Remark 4.2 *According to [7] a connected fat compactum* $\mathcal{K}$ *in* $\mathbb{R}^p$ *is a W.-c., if* $\mathcal{K}$ *has the property* $(\mathcal{P})$: *there exists constants* $\mathcal{M} < \infty$ *and* $\gamma \in (0, 1]$ *such that every pair of points* $X^{(1)}$, $X^{(2)}$ *from* $\mathcal{K}$ *can be connected by a rectifiable curve* $\mathcal{L}$ *in* $\mathcal{K}$ *of length not exceeding* $\mathcal{M}\big(|X^{(1)} - X^{(2)}|_p\big)^\gamma$ *and with ends in* $X^{(1)}$ *and* $X^{(2)}$. *In particular, each convex fat compactum in* $\mathbb{R}^p$ *has the property* $(\mathcal{P})$. *According to theorem* 4 *the space* $C^\infty[\mathcal{K}]$ *has an* $\mathcal{E}ARS$ *of the form* (3) *for every connected fat compact set with the property* $(\mathcal{P})$ *and in particular for each convex fat compactum* $\mathcal{K}$.

4.

Let us investigate now the problem of the existence of an ARS of exponentials of the form (1) in the space $C^\infty(G)$, where G is an arbitrary nonempty open set in $\mathbb{R}^p$. We shall see that in this case the results will differ essentially from those obtained above for $C^\infty[\mathcal{K}]$, $\mathcal{K} \in \mathcal{F}$.

Lemma 2 *Let* G *be an arbitrary open nonempty set in* $\mathbb{R}^p$. *Suppose that* $C^\infty(G)$ *has at least one ARS of the form* (1). *Then*

$$\forall y \in C^\infty(G) \quad \exists Y \in BC^\infty(\mathbb{R}^p) : Y|_G = y.$$

Proof. Let us fix an arbitrary $y(x)$ from $C^\infty(G)$. If $\mathcal{E}_\mu(1)$ is an ARS in $C^\infty(G)$ then there exists a series

$$\sum_{|k|_p=0}^{\infty} y_k \exp\Big\langle i \sum_{j=1}^{p} \mu_{j,k} x_j \Big\rangle \tag{11}$$

converging to $y(x)$ absolutely in $C^\infty(G)$. By Lemma 1 the series (11) converges absolutely in $C^\infty(\mathbb{R}^p)$. By virtue of remark to Theorem 2 the sum $Y(x)$ of the series (11) belongs to $BC^\infty(\mathbb{R}^p)$. It is clear that $Y|_G = y$. □

Corollary If G is such as in Lemma 2 and if $C^\infty(G)$ contains at least one function unbounded in G, then $C^\infty(G)$ has no ARS of the form (1).

Theorem 5 *If G is an arbitrary nonempty open set in $\mathbb{R}^p$, then there exists no ARS of the form* (1) *in the space $C^\infty(G)$.*

Proof. If the set G is unbounded, then the function $f(x) := \sum_{j=1}^{p}(x_j)^2$ belongs to $C^\infty(G)$ but is not bounded in G. Suppose now that the set G is bounded. Then G has at least one finite boundary point $\gamma = (\gamma_1, \dots, \gamma_p)$. It is easy to see that the function $\varphi(x) = \dfrac{1}{\sum_{j=1}^{p}(x_j - \gamma_j)^2}$ belongs to $C^\infty(G)$ but is not bounded in G. It remains only to exploit the corollary of Lemma 2. □

5.

Now we apply the results obtained above to the problem of stability of an ARS under the passage to projective limit. This problem was posed in [4] and can be formulated in the following manner. Let H_n be a complete locally convex space, $\forall n \geqslant 1\, H_{n+1} \hookrightarrow H_n$. Let

$$H := \underset{\leftarrow}{\operatorname{proj}}\, H_n$$

be the space $\bigcap_{k=1}^{\infty} H_k$ with the topology of projective limit. Let $x_k \neq 0$, $x_k \in H_n$, $\forall k, n \geqslant 1$. Suppose that $X := (x_k)_{k=1}^{\infty}$ is an ARS in each H_n, $n = 1, 2, \dots$. Will X be an ARS in H? This problem has been first investigated in one special situation, when H is the Frechet space $H(G)$ of all functions analytic in the convex domain $G \subset \mathbb{C}^p$, $x_k = \exp < \lambda_k, z >$ are exponentials with complex exponents $\lambda_k \in \mathbb{C}^p$, $p \geqslant 1$, and $H_n = H(G_n)$, where $(G_n)_{n=1}^{\infty}$ is an increasing sequence of convex domains $G_n \subset G$ approximating G: $\overline{G}_n \subset G_{n+1} \subset G = \bigcup_{m=1}^{\infty} G_m$. The first results (for $p = 1$) belong to Korobeinik [4]. Later, Abanin obtained rather general but not final results for $p \geqslant 1$ ([4], [1]) as well as for the regarded special situation.

The first results concerning the general situation appeared in the paper [2] (Theorems 2.1 and 2.2). We show here only Theorem 2.1(the reader can find easily the formulation of Theorem 2.2 in [2]).

Theorem A[[2], theorem 2.1] Let H_n be a nuclear Frechet space with the topology defined by seminorms $(p_j^n)_{j=1}^{\infty}, n \geqslant 1$. Let $H_{n+1} \hookrightarrow H_n$ for all $n \geqslant 1$. Suppose that $U := (u_k)_{k=1}^{\infty}$ is the sequence of elements from H such that $\forall n \geqslant 1, \quad U$ is an ARS in H_n and

$$\lim_{k\to\infty} p_j^n(u_k)/p_{j+1}^n(u_k) = 0, \quad \forall j, n \geqslant 1. \tag{12}$$

Then U is an ARS in H.

In 1994 Abanin found an error in the proof of Theorem A on page 202 of [2]. In connection with this fact he remarked in [1], ch.1, §8, that the validity of Theorem A and of all its corollaries obtained in [2] remain to be open. A bit later, Korobeinik found a similar error in the proof of Theorem 2.2 ([2], p. 205).

Consequently the last theorem together with its Corollary 2.2 from [2] remained unproved as well. We shall show in this paragraph with the help of results obtained above that Theorem A is not true. As we shall see further, theorem 2.2 [2] is also false.

Theorem 6 *Theorem A is not true.*

Proof. Let us fix an arbitrary bounded convex domain G in $\mathbb{R}^p$, $p \geqslant 1$, and some bounded open rectangular parallelepiped T containing G. We can always construct a sequence of nonempty convex compact sets $\mathcal{K}_n$ in G such that $\forall n \geqslant 1 \quad K_n \subseteq \overset{\circ}{\mathcal{K}}_{n+1} \subset G = \bigcup_{m=1}^{\infty} \mathcal{K}_m$. Taking into account Remark 4.2, to Theorem 4 at the end of §3 we state that $\mathcal{E}_p^T$ (3) is an $\mathcal{E}$ARS in every space $C^{\infty}[\mathcal{K}_n]$, $n \geqslant 1$. Since by the same Remark 4.2 every convex compactum $\mathcal{K}_n$ is a W.-c., the space $C^{\infty}[\mathcal{K}_n]$ coincides both algebraically and topologically with the space $C_{\infty}[\mathcal{K}_n]$ of traces on $\mathcal{K}_n$ of all functions from the nuclear Frechet space $C^{\infty}(\mathbb{R}^p)$. Hence (see e.g.[6]) $C^{\infty}(\mathcal{K}_n)$ is a nuclear Frechet space. Let us put $u_k = \exp\Big\langle 2\pi i k, \dfrac{x}{b-a}\Big\rangle, k \in Z^p$, $p_j^n(y) = \max\Big\{|y^{(\alpha)}(x)| : |\alpha|_p \leqslant j, x \in \mathcal{K}_n\Big\}$. Then

$$p_j^n(u_k) = \max\Big\{\frac{(2\pi)^{|\alpha|_p}|k|^{\alpha}}{(b-a)^{|\alpha|_p}} : |\alpha|_p \leqslant j\Big\},$$

and

$$\lim_{|k|_p\to\infty} p_j^n(u_k)/p_{j+1}^n(u_k) = 0.$$

If, in particular, $p = 1$, the last equality implies the Relation (12). By Theorem A $\mathcal{E}_1^T$ is an ARS in

$$C^{\infty}(G) = \operatorname*{proj}_{\leftarrow} C^{\infty}[\mathcal{K}_n],$$

where $G=(-R,+R), 0<R<\infty, T=[a,b], -\infty<a<-R<+R<b<+\infty, \mathcal{K}_n=[-R_n,R_n], 0<R_n\uparrow R$. On the other hand, according to Theorem 5 there is no ARS $\mathcal{E}_1^T$ of the form (3) in the space $C^\infty(-R,R)$. □

6.

Assertions similar to Tsheorems 1–5 can be obtained for some subspaces of $C^\infty[\mathcal{K}]$ and $C^\infty(G)$. Let us consider as example of such a subspace the Carleman–Beurling–type space. Let $F\in\mathcal{F}$, $\mathcal{M}_0=1$, $\mathcal{M}_l>0$, $\mathcal{M}_l\to\infty$, $h\in(0,+\infty)$. We put

$$\mathcal{E}_{(\mathcal{M}_l)}[F,h]:=\Big\{\, y(x)\in C^\infty[F]: \qquad \|y\|_h:=\sup\Big[\frac{|y^{(\alpha)}(x)|}{h^{|\alpha|_p}\mathcal{M}_{|\alpha|_p}}: x\in\mathring{F}, \alpha\in\mathcal{N}_0^p\Big]<\infty\Big\}.$$

It is easy to check that $\mathcal{E}_{(\mathcal{M}_l)}[F,h]$ is a Banach space with the norm $\|y\|_h$. We put for each $d\in[0,+\infty)$:

$$\mathcal{E}_{(\mathcal{M}_l)}[F]_d:=\underset{\overleftarrow{h>d}}{\operatorname{proj}}\,\mathcal{E}_{(\mathcal{M}_l)}[F,h].$$

Let us fix $d\in[0,+\infty)$. The set of Carleman–Beurling spaces $\{\mathcal{E}_{(\mathcal{M}_l)}[F]_d\}_{F\in\mathcal{F}}$ has the following properties:

1) $\forall F\in\mathcal{F}\quad \mathcal{E}_{(M_l)}[F]_d\hookrightarrow C^\infty[F]$;

2) if $F_1\subseteq F_2$, $F_j\in\mathcal{F}$, $j=1,2$, then $\mathcal{E}_{(\mathcal{M}_l)}[F_2]_d\hookrightarrow\mathcal{E}_{(\mathcal{M}_l)}[F_1]_d$.

Suppose that the numbers $(\mathcal{M}_l)$ tend to infinity sufficiently fast that

$$\lim_{l\to\infty}(\mathcal{M}_l)^{1/l}=\infty. \tag{13}$$

Then condition (13) implies the additional property of the set $\mathcal{E}_{(\mathcal{M}_l)}[F]_d$, $F\in\mathcal{F}$: 3) $\forall\mu\in\mathbb{R}^p\,\forall F\in\mathcal{F}\quad \exp\langle i\mu,x\rangle\in\mathcal{E}_{(\mathcal{M}_l)}[F]_d$. For an arbitrary open set $G\subseteq\mathbb{R}^p$ we introduce the space

$$\mathcal{E}_{(\mathcal{M}_l)}(G)_d:=\underset{\overleftarrow{F\in\mathcal{F}_G}}{\operatorname{proj}}\,\mathcal{E}_{(\mathcal{M}_l)}[F]_d.$$

It is evident that 4) $\mathcal{E}_{(\mathcal{M}_l)}(\mathbb{R}^p)_d\hookrightarrow\mathcal{E}_{(\mathcal{M}_l)}(G)_d$ for all open sets G from $\mathbb{R}^p$; 5) $\mathcal{E}_{(\mathcal{M}_l)}(\mathbb{R}^p)_d\hookrightarrow\mathcal{E}_{(\mathcal{M}_l)}[F]_d, \forall F\in\mathcal{F}$; 6) $\mathcal{E}_{(\mathcal{M}_l)}(G)_d\hookrightarrow\mathcal{E}_{(\mathcal{M}_l)}[F]_d, \forall F\in\mathcal{F}_G$.

At last, for every series (1) the following assertions are equivalent:

a_7) the series (1) converges absolutely in $\mathcal{E}_{(\mathcal{M}_l)}[F]_d$ for some $F \in \mathcal{F}$;

b_7) the series (1) converges absolutely in $\mathcal{E}_{(\mathcal{M}_l)}(R^p)_d$;

c_7) the series (1) converges absolutely in $\mathcal{E}_{(\mathcal{M}_l)}[F]_d, \forall F \in \mathcal{F}$;

d_7) the series (1) converges absolutely in $\mathcal{E}_{(\mathcal{M}_l)}(G)_d$ for some nonempty open set G;

e_7) the series (1) converges absolutely in $\mathcal{E}_{(\mathcal{M}_l)}(G)_d$ for all nonempty open set G from $\mathbb{R}^p$.

It is easy to prove an analogue of Lemma 1 according to which each of the assertions a_7)–e_7) is equivalent to the following one:

$$f_7) \qquad \sum_{|l|_p=0}^{\infty} |c_l| \sup \left\{ \frac{|\mu_l|^\alpha}{h^{|\alpha|_p} \mathcal{M}_{|\alpha|_p}} : \alpha \in \mathcal{N} \right\} < \infty, \quad \forall h > d.$$

We shall say that a fat compactum $\mathcal{K} \subset \mathbb{R}^p$ is a Carleman–Beurling d–compactum (CBdC) if $\forall y \in \mathcal{E}_{(M_l)}[\mathcal{K}]_d \, \exists Y \in \mathcal{E}_{(M_l)}(\mathbb{R}^p)_d : Y|_{\mathcal{K}} = y$. As in the case of $C^\infty[\mathcal{K}]$ we can obtain the following results.

Theorem 7 *Let $\mathcal{K}$ be a CBdC and let T be an arbitrary open rectangular parallelepiped containg $\mathcal{K}$, $T = \{ x : a_j < b_j < d_j, j = 1, 2, ...p \}$. Suppose that the numbers $\mathcal{M}_l$ have the following properties:*

$$\sum_{l=1}^{\infty} \frac{\mathcal{M}_{l-1}}{\mathcal{M}_l} < \infty; \tag{14}$$

$$\begin{aligned} &\limsup_{l\to\infty} \left[\sum_{j=0}^{l} \frac{\mathcal{M}_{l-j}\mathcal{M}_j}{\mathcal{M}_l} \right]^{1/l} \leqslant 1, \quad \text{if} \quad 0 < d < \infty; \\ &\limsup_{l\to\infty} \left[\sum_{j=0}^{l} \frac{\mathcal{M}_{l-j}\mathcal{M}_j}{\mathcal{M}_l} \right]^{1/l} < \infty, \quad \text{if} \quad d = 0; \end{aligned} \tag{15}$$

$$\begin{aligned} &\limsup_{l\to\infty} \left(\frac{\mathcal{M}_{l+1}}{\mathcal{M}_l} \right)^{1/l} \leqslant 1, \quad \text{if} \quad 0 < d < \infty; \\ &\limsup_{l\to\infty} \left(\frac{\mathcal{M}_{l+1}}{\mathcal{M}_l} \right)^{1/l} < \infty, \quad \text{if} \quad d = 0. \end{aligned} \tag{16}$$

Then the system $\mathcal{E}_p^T$ (3) is an $\mathcal{E}ARS$ in $\mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d$.

Denote by $B\mathcal{E}_{(\mathcal{M}_l)}(\mathbb{R}^p)_d$ the set of all functions from $\mathcal{E}_{(\mathcal{M}_l)}(\mathbb{R}^p)_d$ bounded in $\mathbb{R}^p$ together with all their derivatives.

Theorem 8 *Let $\mathcal{K}$ be an arbitrary fat compactum in $\mathbb{R}$ and let the conditions* (14)–(16) *be fulfilled. Then FAAE:*

1. *$\mathcal{K}$ is a CBdS;*
2. $\forall y \in \mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d \, \exists Y \in B\mathcal{E}_{(\mathcal{M}_l)}(\mathbb{R}^p)_d : Y|_{\mathcal{K}} = y$;
3. *there exists an ARS in $\mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d$ of the form* (1)*;*
4. *there exists an $\mathcal{E}ARS$ in $\mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d$ of the form* (1)*;*
5. *if T is an arbitrary restangular parallelepiped containing $\mathcal{K}$, then the correspondent system $\mathcal{E}_p^T$* (3) *is an $\mathcal{E}ARS$ in $\mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d$.*

Theorem 9 *Let $\mathcal{M}_0 = 1$, $\mathcal{M}_l > 0$, $0 \leqslant d < \infty$, and*

$$\lim_{l\to\infty} \frac{l}{(\mathcal{M}_l)^{1/l}} = 0. \tag{17}$$

If G is an arbitrary nonempty open set in $\mathbb{R}^p$, then there is no ARS of the form (1) *in the space $\mathcal{E}_{(\mathcal{M}_l)}(G)_d$.*

Remark 9.1 *The sequence $\mathcal{M}_l = (l!)^\beta$, $\beta > 1$ satisfies the conditions* (14), (17) *and the first ones in the pairs of conditions* (15), (16)*.*

Therefore Theorems 7–9 are valid for Gevrey spaces of normal ($0 < d < \infty$) and minimal ($d = 0$) type:

$$\mathcal{E}_{((l!)^\beta)}[\mathcal{K}]_d = \Big\{ y \in C^\infty[\mathcal{K}] : \\ \forall h > d \sup\Big[\frac{|y^{(\alpha)}(x)|}{h^{|\alpha|_p}((|\alpha|_p)!)^\beta} : \alpha \in N_0^p, x \in \mathring{\mathcal{K}}\Big] < \infty \Big\};$$

$$\mathcal{E}_{((l!)^\beta)}(G)_d = \operatorname*{proj}_{\overleftarrow{\mathcal{K}\in\mathcal{F}_G}} \mathcal{E}_{((l!)^\beta)}[\mathcal{K}]_d.$$

7.

As the last example we consider Carleman–Roumieu–type space

$$\mathcal{E}_{\{\mathcal{M}_l\}}[\mathcal{K}]_d = \operatorname*{ind}_{\overrightarrow{h<d}} \mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}, h], \text{ where } 0 < d \leqslant \infty, \mathcal{M}_0 = 1, \mathcal{M}_l > 0$$

and the condition (13) is fulfilled. For an arbitrary open set G we put

$$\mathcal{E}_{\{\mathcal{M}_l\}}(G)_d = \operatorname*{proj}_{\overleftarrow{\mathcal{K}\in\mathcal{F}_G}} \mathcal{E}_{\{\mathcal{M}_l\}}[\mathcal{K}]_d.$$

With the help of approximately the same arguments as in the case of Carleman–Beurling–type spaces we can obtain exact analogues of properties 1)–6) of §6 with substitution $\{\mathcal{M}_l\}$ instead of $(\mathcal{M}_l)$. Besides, each of the assertions $a_7)$–$e_7)$ after the same substitution is equivalent to the following one:

$$f_7) \quad \exists h < d : \sum_{|l_p|=0}^{\infty} |c_l| \sup \left\{ \frac{|\mathcal{M}_l|^\alpha}{h^{|\alpha|_p} \mathcal{M}_{|\alpha|_p}} : \alpha \in N_0^p \right\} < \infty.$$

We shall say that the compactum $\mathcal{K} \in \mathcal{F}$ is a Carleman–Roumieu d–compactum (CRdC) if

$$\forall y \in \mathcal{E}_{\{\mathcal{M}_l\}}[\mathcal{K}]_d \quad \exists Y \in \mathcal{E}_{\{\mathcal{M}_l\}}(\mathbb{R}^p)_d : Y|_{\mathcal{K}} = y.$$

One can prove with the help of approximately the same arguments as in the case of the spaces $C^\infty[\mathcal{K}]$, $C^\infty(G)$, $\mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d$, $\mathcal{E}_{(\mathcal{M}_l)}(G)_d$ the exact analogues of Theorems 7–9. In order to formulate these results we only need to replace $(\mathcal{M}_l)_{l=1}^\infty$ everywhere in formulations of Theorems 7–9 by $\{\mathcal{M}_l\}_{l=1}^\infty$. In particular, such results are valid for Gevrey spaces of maximal type:

$$\mathcal{E}_{\{(l!)^\beta\}}[\mathcal{K}]_\infty = \Big\{ y \in C^\infty[\mathcal{K}] : \\ \exists h > 0 \sup\Big[\frac{|y^{(\alpha)}(x)|}{h^{|\alpha|_p}((|\alpha|_p)!)^\beta} : \alpha \in N_0^p, x \in \mathring{\mathcal{K}} \Big] < \infty \Big\};$$

$$\mathcal{E}_{\{(l!)^\beta\}}(G)_\infty = \operatorname*{proj}_{\overleftarrow{\mathcal{K}\in\mathcal{F}_G}} \mathcal{E}_{((l!)^\beta)}[\mathcal{K}]_\infty.$$

The analogues of Theorems 7–9 for Carleman–Roumieu–type spaces enable one to construct an example rejecting Theorem 2.2 from [2]. We give here a short discription of such an example. First, we put $p = 1$, fix some $R \in (0, +\infty)$ and select an arbitrary sequence $\mathcal{M}_l$, $l \geqslant 0$ such that $\mathcal{M}_0 = 1$, $\mathcal{M}_l > 0$ and conditions (17) is fulfilled. Moreover we take $(\mathcal{M}_l)$ in such a manner that the following relations are valid with $m_l = \dfrac{\mathcal{M}_{l+1}}{\mathcal{M}_l}$, $l \geqslant 0$:

$$m_0 = 1, \quad m_l \to \infty, \quad \limsup_{l\to\infty} (m_l)^{1/l} < \infty, \tag{18}$$

$$\limsup_{n\to\infty} \frac{m_n}{n} \sum_{j>n}^{\infty} \frac{1}{m_j} < \infty. \tag{19}$$

In order to satisfy all these requirements we can put in particular $\mathcal{M}_0 = 1$, $\mathcal{M}_l = (l!)^\gamma$, $l \geqslant 1$ with an arbitrarily fixed $\gamma > 1$. Let $(R_n)_{n=1}^\infty$ be an arbitrary sequence of numbers such that $0 < R_n \uparrow R$. According to Remark 1 to Theorem 5.4 of the paper [5], if the condition (18)–(19) are fulfilled then the system $U = (u_k)_{k=0}^\infty$ where $u_{2k} = \exp \dfrac{ik\pi x}{R}$, $u_{2k+1} = \exp\left(-\dfrac{ik\pi x}{R}\right)$, $k = 0, 1, \ldots$ is an ARS in

$$\mathcal{E}_{\{\mathcal{M}_l\}}[-R_n, R_n]_\infty = \operatorname*{ind}_{\overrightarrow{\gamma}} B^n_\gamma,$$

where $n \geqslant 1$ and

$$B^n_\gamma = \{f \in C^\infty[-R_n, R_n] : ||f||_{(\gamma!)^s} < \infty\}, \quad \gamma = 1, 2, \ldots;$$

and s is a fixed sufficiently large natural number. One can check without special difficulties that all suppositions of Theorem 2.2 from [2] are fulfilled in the regarded situation. By this heorem U is an ARS in $\mathcal{E}_{(\mathcal{M}_l)}(-R, R)$. At the same time according to the analogue of Theorem 9 for the space $\mathcal{E}_{\{\mathcal{M}_l\}}(G)_d$ for the case $d = \infty$, $G = (-R, R)$ there is no ARS of exponentials with imaginary exponents in the space $\mathcal{E}_{\{\mathcal{M}_l\}}(-R, R)_\infty$.

8.

As was shown above, Whitney-compact sets, Carleman–Beurling d–compact sets and Carleman–Roumieu d–compact sets can be characterized by existence in the corresponding spaces $C^\infty[\mathcal{K}], \mathcal{E}_{(\mathcal{M}_l)}[\mathcal{K}]_d$ and $\mathcal{E}_{\{\mathcal{M}_l\}}[\mathcal{K}]_d$ an ARS of exponentials with imaginary exponents. It will be very interesting to characterize such compacta in different manner namely, in terms of geometrical properties of $\mathcal{K}$ for W.-c. and in terms of properties of numbers $\mathcal{M}_l$ for CBdC and CRdC.

References

[1] A.V. Abanin, On continuation and stability of weakly sufficient sets, Izvestija Vuzov, Mathematica, 1987, $\mathcal{N}$4, p. 3 - 10 (Russian); English transl. in USSR Izvestija Vuzov

[2] Chan-Por, Les systemes de representation absolue dans les espaces des fonctions holomorphes, Studia Mathem., 1989, v. XCIV, p. 193 - 212

[3] L. Hörmander, The Analysis of Linear Partial Differential Operators. I. Distribution theory and Fourier analysis, Springer, 1983

[4] Yu.F. Korobeinik, Representing Systems, Russian Mathematical Surveys, 1981, v.36, $\mathcal{N}$1, p. 75 - 137

[5] Yu.F. Korobeinik, Effectively representing ϑ-trigonometrical systems and their applications. I, Manuscript dep. by VINITI 29.06.1999 $\mathcal{N}$2132 - B99, Rostov on Don, 1999, 35 pp (Russian).

[6] A. Pietsch, Nukleare Localkonvexe Räume, Academic - Verlag, Berlin, 1965

[7] H. Whitney, Functions differentiable on the boundaries of regions, Annals of Mathem., 1934, vol. 33, $\mathcal{N}$3, p. 482 - 485

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