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Combinatorial patchworking of real pseudo-holomorphic curves

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Abstract

The Viro method of construction of real algebraic varieties with prescribed topology uses convex subdivisions of Newton polyhedra. We show that in the case of arbitrary (not necessarily convex) subdivisions of polygons corresponding to $\mathbb{C}P^2$ and rational ruled surfaces Σ_a, $a \geq 0$ the Viro method produces *pseudo-holomorphic* curves. The version of the Viro method discussed in the paper also gives a possibility to construct singular pseudo-holomorphic curves by gluing singular algebraic curves whose collections of singularities do not permit to glue these curves in the framework of the standard Viro method. As an application, we construct a series of singular real pseudo-holomorphic curves in $\mathbb{C}P^2$ whose collections of singular points do not occur on known algebraic curves of the same degree.

1. Introduction

The Viro method of gluing of polynomials appeared to be the most powerful construction of real algebraic varieties with prescribed topology [23, 24, 27, 28] (see also [7], 11.5, [12, 17]). It provides a nice interaction of real algebraic geometry, toric geometry and combinatorics, and gives rise to various generalizations and applications.

Consider an example of the Viro construction. Let $T_d \subset \mathbb{R}^2$, $d \in \mathbb{N}$, be the triangle with vertices $(0,0)$, $(0,d)$, $(d,0)$,

$$\tau : \quad T_d = \Delta_1 \cup ... \cup \Delta_N$$

a triangulation with the set of vertices $V \subset \mathbb{Z}^2$, and $\sigma : V \to \{\pm 1\}$ any function. Out of this combinatorial data we construct piecewise-linear plane curves. Denote by $T_d^{(1)}$, $T_d^{(2)}$ and $T_d^{(3)}$ the copies of T_d under the reflections with respect to the coordinate axes and the origin. Take in $T_d^{(1)}$, $T_d^{(2)}$ and $T_d^{(3)}$ the triangulations symmetric to τ, and define σ at the vertices of new triangulations by

$$\sigma(\varepsilon_1 i, \varepsilon_2 j) = \varepsilon_1^i \varepsilon_2^j \sigma(i,j), \quad (i,j) \in V, \quad \varepsilon_1, \varepsilon_2 = \pm 1.$$

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Now in each triangle of the triangulation of $T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)}$, having vertices with different values of σ, we draw the midline which separates the vertices with different signs. The union $C(\tau, \sigma)$ of all these midlines is a broken line homeomorphic to a disjoint union of circles and segments. Introduce natural maps:

$$\Phi : T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)} \to \mathbb{R}P^2, \quad \Psi : \mathrm{Int}(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)}) \to \mathbb{R}^2,$$

where Φ is continuous onto, identifying antipodal points on $\partial(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)})$, and Ψ is a homeomorphism. The curve $\Phi(C(\tau, \sigma)) \subset \mathbb{R}P^2$ (resp., $\Psi(C(\tau, \sigma)) \subset \mathbb{R}^2$) is called *projective* (resp., *affine*) *T-curve of degree d*.

The Viro theorem states that a projective (resp., affine) T-curve of degree d is isotopic in $\mathbb{R}P^2$ (resp., in $\mathbb{R}^2$) to a nonsingular algebraic projective (resp., affine) curve of degree d, providing that the triangulation τ is *convex*. A subdivision $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ of a convex polygon Δ into convex polygons $\Delta_1, \ldots, \Delta_N$ is called *convex*, if there exists a convex piecewise-linear function $\nu : \Delta \to \mathbb{R}$, whose linearity domains are $\Delta_1, \ldots, \Delta_N$ (sometimes such subdivisions are called *regular* or *coherent*; see [31, 7]). The Viro theorem, in fact, endows the combinatorial broken line $C(\tau, \sigma)$ with a rich structure, which implies a number of restrictions to the topology of $C(\tau, \sigma)$ (see an account of known results in [18, 25, 26, 30]).

In a more general situation, the initial data of the Viro construction is a convex subdivision $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ of a convex polygon Δ into convex polygons $\Delta_1, \ldots, \Delta_N$, and a collection $A_{i,j}$ of real or complex numbers indexed by the integer points (i, j) of Δ. The polynomials

$$F_i(x, y) = \sum_{(i,j) \in \Delta_i} A_{i,j} x^i y^j, \quad i = 1, \ldots, N$$

are often supposed to be non-degenerate (see Section 2 for the definition). The result of the construction is an algebraic curve with Newton polygon Δ in the toric surface $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ associated with Δ.

The convexity of the subdivision in the Viro method is very important. However, one can try to perform the Viro construction using *non-convex* subdivisions. The result of such a construction is not necessarily algebraic. In the present paper we restrict ourselves to the case of curves in $\mathbb{C}P^2$ and curves in rational ruled surfaces Σ_a, $a \geq 0$, and show that the Viro construction applied to arbitrary (*i.e.*, not necessarily convex) subdivisions of corresponding polygons produces *pseudo-holomorphic* curves. A Riemann surface M embedded in X (where X stands for $\mathbb{C}P^2$ or a rational ruled surface Σ_a, $a \geq 0$) is a (*nonsingular*) *real pseudo-holomorphic curve*, if it is a J-holomorphic curve in some tame almost complex structure J on X (see [6]) such that $\mathrm{Conj}_* \circ J = J^{-1} \circ \mathrm{Conj}_*$ and $\mathrm{Conj}(M) = M$, where $\mathrm{Conj} : X \to X$ is the standard real structure on X (this is the real structure of X as a real toric surface associated with a polygon). More generally, by a (singular) real pseudo-holomorphic curve (cf. [22]) we mean the image $\varphi(M) \subset X$ of M, where $\varphi : M \to X$ is a Conj-invariant map, which embeds the complement of a finite set $K \subset M$, the (non-compact) smooth surface $\varphi(M \backslash K)$ is tangent to some tame almost complex structure J on X such that $\mathrm{Conj}_* \circ J = J^{-1} \circ \mathrm{Conj}_*$, and for any point

$z \in \varphi(K)$ the germ of $\varphi(M)$ at z is a bouquet of finitely many discs whose tangent cones at z are J-invariant 2-planes. The latter holds, for instance, if the germ of $\varphi(M)$ at z is the image of a singular algebraic curve germ by a diffeomorphism of a small ball in $\mathbb{C}^2$ onto a neighborhood of z in X. Denote by $\mathbb{R}X$ the fixed point set of Conj. The fixed point set $\mathbb{R}M \subset \mathbb{R}X$ of Conj restricted to a real pseudo-holomorphic curve M is called the *real point set* of M.

Fix an integer homology class $h \in H_2(X)$, and denote by $\mathbb{R}\mathcal{M}_h$ the set of nonsingular real pseudo-holomorphic curves in X realizing the class h. On can ask for a classification of the real point sets of curves $M \in \mathbb{R}\mathcal{M}_h$ up to isotopy in $\mathbb{R}X$. This question is similar to the first part of the Hilbert 16-th problem. It is interesting to compare two classifications. In the case $X = \mathbb{C}P^2$ the classifications coincide for the curves of degree ≤ 7 (we say that a pseudo-holomorphic curve in $\mathbb{C}P^2$ is of degree d if the curve realizes the class $d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$). Recently. S. Orevkov [15] obtained a classification of the real point sets of maximal real pseudo-holomorphic curves of degree 8 up to isotopy in $\mathbb{R}P^2$ (a nonsingular real pseudo-holomorphic curve is called *maximal* if its real point set has $g+1$ connected components, where g is the genus of the curve). The corresponding classification of algebraic curves is still not completed. A real pseudo-holomorphic curve M in X is called *algebraically unrealizable* if the class of $\mathbb{R}M$ up to isotopy (up to fiberwise isotopy if X is a rational ruled surface) in $\mathbb{R}X$ cannot be represented by the real point set of an algebraic curve realizing the same homology class $[M] \in H_2(X)$. S. Orevkov and E. Shustin [14], showed that there exist algebraically unrealizable nonsingular real pseudo-holomorphic curves in the rational ruled surface Σ_2. Nonsingular real pseudo-holomorphic curves form a subclass of *flexible* curves. Flexible curves in $\mathbb{C}P^2$ were introduced by O. Viro [26]. Viro analyzed the properties of real algebraic curves used in topological proofs of restrictions, and defined flexible curves as topological surfaces in $\mathbb{C}P^2$ satisfying these properties (see Section 3 for the definition). J.-Y. Welschinger [29] constructed a series of algebraically unrealizable (in the same sense as above) flexible curves in rational ruled surfaces.

There exist algebraically unrealizable reducible real pseudo-holomorphic curves in $\mathbb{C}P^2$ with two irreducible components, one of degree 1 and the other of degree 6 (see [4, 14]). For the moment, no example of algebraically unrealizable *nonsingular* real pseudo-holomorphic curve in $\mathbb{C}P^2$ is known.

The version of the Viro method discussed in the present paper produces the curves which we call *C-curves*. These curves are pseudo-holomorphic but not necessarily algebraic. Thus, this version can be a source of examples of algebraically unrealizable real pseudo-holomorphic curves. The first possibility to produce in this way a real pseudo-holomorphic curve which has a chance to be algebraically unrealizable was already mentioned above: one can use *non-convex* subdivisions in the construction. Another possibility is to modify slightly the construction and glue *singular* polynomials whose collections of singularities do not permit to glue these polynomials in the framework of the standard Viro method. This possibility is illustrated in Section 5, where we construct a series of singular real pseudo-holomorphic curves in $\mathbb{C}P^2$ whose collections of singular points do not occur on *known* algebraic curves of the same degree.

The material is organized as follows. In Section 2 we present the construction of complex and real C-curves. This construction is a particular case of the construction of C-hypersurfaces described in [11]. We also prove that any (nonsingular) complex C-curve in a toric surface is isotopic to a complex algebraic curve with the same Newton polygon. The material of Section 2 is basically contained in [23]. However, our setting is a little bit different. Section 3 is devoted to topology of real C-curves. We indicate a proof of the fact that (nonsingular) real C-curves in $\mathbb{C}P^2$ are *flexible curves* in the sense of [26], and thus verify all known topological restrictions on nonsingular real algebraic curves in $\mathbb{C}P^2$ (see also [11]). In Section 4 we consider C-curves in $\mathbb{C}P^2$ and in the rational ruled surfaces Σ_a, $a \geq 0$, introduce an appropriate pencil of pseudo-lines, and show that any real C-curve in $\mathbb{C}P^2$ or in a rational ruled surface is pseudo-holomorphic. Section 5 contains a construction of a series of singular real pseudo-holomorphic curves in $\mathbb{C}P^2$.

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2. C-curves in toric surfaces

In this section we describe the construction of C-curves. All the details and the proofs can be found in [11], where the construction of C-hypersurfaces of any dimension is given.

2.1. Notations and definitions

Put $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$, $\mathbb{R}_+^* = \{x \in \mathbb{R}, x > 0\}$ and $\mathbb{C}^* = \{z \in \mathbb{C}, z \neq 0\}$. Further on the term *polygon* means a convex (possibly degenerate) polygon in the nonnegative quadrant $\mathbb{R}_+^2$ of $\mathbb{R}^2$. If all the vertices of a polygon have integer coordinates, then the polygon is called *integer*.

Given a complex polynomial $F(z,w) = \sum_{i,j} A_{i,j} z^i w^j$ in two variables, by $\Delta(F)$ we denote its Newton polygon, *i.e.*, the convex hull of the set $\{(i,j) \in \mathbb{R}^2 \ : \ A_{i,j} \neq 0\}$. The truncation of F on a face δ of $\Delta(F)$ is the polynomial $F^\delta(i,j) = \sum_{(i,j)\in\delta} A_{i,j} z^i w^j$. A polynomial $F \in \mathbb{C}[z,w]$ is called *non-degenerate*, if F and any truncation F^δ on a proper face δ of $\Delta(F)$ has a nonsingular zero set in $(\mathbb{C}^*)^2$ (cf. [23]).

2.2. Extension of the moment map

Let Δ be a polygon, V_Δ the set of vertices of Δ, and $\mu_\Delta : (\mathbb{R}_+^*)^2 \to I(\Delta)$ the moment map (see [1, 2, 5], [7], 6.1), where $I(\Delta)$ is the complement in Δ of the union of all its proper faces :

$$\mu_\Delta(x,y) = \frac{\sum_{(i,j)\in V_\Delta} x^i y^j \cdot (i,j)}{\sum_{(i,j)\in V_\Delta} x^i y^j} \ . \tag{1}$$

Split the complex torus $(\mathbb{C}^*)^2$ in the product $(\mathbb{R}_+^*)^2 \times (S^1)^2$:

$$(z,w) \in (\mathbb{C}^*)^2 \ \mapsto \ (|z|,|w|) \in (\mathbb{R}_+^*)^2, \ \left(\frac{z}{|z|}, \frac{w}{|w|}\right) \in (S^1)^2 \ .$$

Note that the inverse map $(\mathbb{R}_+^*)^2 \times (S^1)^2 \to (\mathbb{C}^*)^2$ naturally extends to a surjection $\theta : \mathbb{R}_+^2 \times (S^1)^2 \to \mathbb{C}^2$. Put

$$\mathbb{C}I(\Delta) = \theta(I(\Delta) \times (S^1)^2) \subset \mathbb{C}^2, \quad \mathbb{C}\Delta = \theta(\Delta \times (S^1)^2) \subset \mathbb{C}^2 \ ,$$

and call $\mathbb{C}\Delta$ the *complexification of* Δ.

Proposition 2.1. *The complexification $\mathbb{C}\Delta$ of Δ is a (possibly singular) PL-manifold with boundary. If the dimension of Δ is less than 2, then $\mathbb{C}\Delta$ is not singular. If the dimension of Δ is 2, then the singular set of $\mathbb{C}\Delta$ is the union of $\mathbb{C}v$ over all vertices v of Δ which are intersections of Δ with coordinates axes. The real part $\mathbb{R}\Delta$ of $\mathbb{C}\Delta$ is the union of Δ with all its symmetric copies with respect to the coordinate axes.*

Proof. Straightforward. $\square$

Define *the extended moment map* $\mathbb{C}\mu_\Delta : (\mathbb{C}^*)^2 \to \mathbb{C}I(\Delta)$ by

$$\mathbb{C}\mu_\Delta(xu, yv) = \theta(\mu_\Delta(x,y)\ , (u,v)), \text{where } (x,y) \in (\mathbb{R}_+^*)^2,\ (u,v) \in (S^1)^2\ ,$$
$$\text{and } \theta(\mu_\Delta(x,y), (u,v)) \in \theta(I(\Delta) \times (S^1)^2) = \mathbb{C}I(\Delta)\ .$$

As an easy consequence of classical results we obtain the following statement.

Proposition 2.2. *The map $\mathbb{C}\mu_\Delta$ is surjective and commutes with the complex conjugation* Conj*. It is a diffeomorphism when the dimension of Δ is equal to 2. The real part of $\mathbb{C}I(\Delta)$ is the image of $(\mathbb{R}^*)^2$.*

2.3. Real and complex chart of a polynomial

Let $F \in \mathbb{C}[z,w]$ be a polynomial and Δ a polygon. The closure $\mathbb{C}Ch_\Delta(F) \subset \mathbb{C}\Delta$ of the set $\mathbb{C}\mu_\Delta(\{F=0\} \cap (\mathbb{C}^*)^2)$ is called the *complex chart* of the polynomial F in Δ. If F is real then $\mathbb{R}Ch_\Delta(F) = \mathbb{C}Ch_\Delta(F) \cap \mathbb{R}\Delta$ is called the *real chart* of F in Δ. If Δ coincides with the Newton polygon of F, we denote the complex and the real charts of F in Δ by $\mathbb{C}Ch(F)$ and $\mathbb{R}Ch(F)$, respectively.

This definition is a key ingredient of the Viro construction (see [23, 17], cf. [20, 11]).

Proposition 2.3. *Let $F \in \mathbb{C}[z,w]$ be a non-degenerate polynomial. Suppose that the Newton polygon Δ of F has dimension 2 and $\Delta \subset (\mathbb{R}_+^*)^2$. Then the set $\mathbb{C}Ch(F)$ is a smooth surface with boundary $\partial\mathbb{C}Ch(F) = \mathbb{C}Ch(F) \cap \partial\mathbb{C}\Delta$. For any edge δ of Δ,*

$$\mathbb{C}Ch(F) \cap \mathbb{C}\delta = \mathbb{C}Ch(F^\delta)\ . \qquad (2)$$

If F is real, then $\mathbb{C}Ch(F)$ is invariant with respect to the involution of complex conjugation Conj *in $\mathbb{C}^2$, and $\mathbb{R}Ch(F)$ is a smooth curve in $\mathbb{R}\Delta$ with boundary $\partial\mathbb{R}Ch(F) = \mathbb{R}Ch(F) \cap \partial\mathbb{R}\Delta$.*

Remark 2.1. If Δ intersects with coordinate axes, then the statement of Proposition 2.3 holds true when substituting $\mathbb{C}\Delta \backslash \mathrm{Sing}(\mathbb{C}\Delta)$ and $\mathbb{R}\Delta \backslash \mathrm{Sing}(\mathbb{C}\Delta)$ for $\mathbb{C}\Delta$ and $\mathbb{R}\Delta$, respectively, but we will not use this below.

2.4. Gluing of charts

Let $\mathcal{S}$ be a subdivision of a 2-dimensional integer polygon $\Delta \subset \mathbb{R}^2_+$ into 2-dimensional integer polygons : $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ (*i.e.*, $\Delta_i \cap \Delta_j$ is empty or a common vertex or a common edge), and let $\mathcal{A} = \{A_{i,j},\ (i,j) \in \Delta \cap \mathbb{Z}^2\}$ be a collection of complex numbers such that $A_{i,j} \neq 0$ if (i,j) is a vertex of some Δ_k, $1 \leq k \leq N$. Further on, speaking on subdivisions of polygons and corresponding collections of numbers, we always assume the above properties.

Assume that the polynomials

$$F_k(z,w) = \sum_{(i,j)\in\Delta_k} A_{i,j} z^i w^j \ , \quad k = 1, ..., N,$$

are non-degenerate. The union $\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \bigcup_{k=1}^N \mathbb{C}Ch(F_k)$ of the complex charts of the polynomials $F_1, ..., F_N$ is called a (*nonsingular*) *C-curve in* $\mathbb{C}\Delta$. If all the numbers $A_{i,j}$ are real, the union $\mathbb{R}Ch(\mathcal{S},\mathcal{A}) = \bigcup_{k=1}^N \mathbb{R}Ch(F_k)$ of the real charts of $F_1, ..., F_N$ is called a (*nonsingular*) *C-curve in* $\mathbb{R}\Delta$.

Proposition 2.4. *A (nonsingular) C-curve $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is a piecewise-smooth surface in $\mathbb{C}\Delta$ with boundary $\partial\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \partial\mathbb{C}\Delta \cap \mathbb{C}Ch(\mathcal{S},\mathcal{A})$. If all the numbers $A_{i,j}$ are real, then $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is invariant with respect to* Conj*, and the set $\mathbb{R}Ch(\mathcal{S},\mathcal{A})$ is a piecewise-smooth curve in $\mathbb{R}\Delta$ with boundary $\partial\mathbb{R}Ch(\mathcal{S},\mathcal{A}) = \partial\mathbb{R}\Delta \cap \mathbb{R}Ch(\mathcal{S},\mathcal{A})$.*

Definition 2.1. A homeomorphism of (resp., an isotopy in) $\mathbb{C}\Delta$ is called *tame* if for any face δ of Δ the restriction of this homeomorphism (resp., isotopy) to $\mathbb{C}\delta$ is a homeomorphism of (resp., an isotopy in) $\mathbb{C}\delta$. In addition, we call such objects *equivariant* if they commute with Conj.

Remark 2.2. For given Δ and $\mathcal{S}$, and different $\mathcal{A}, \mathcal{A}' : \Delta \cap \mathbb{Z}^2 \to \mathbb{C}$, the corresponding C-curves $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ and $\mathbb{C}Ch(\mathcal{S},\mathcal{A}')$ are tame isotopic in $\mathbb{C}\Delta$ (not equivariantly, in general). Indeed, one can connect $\mathcal{A}$ and $\mathcal{A}'$ by a family $\mathcal{A}_t$, $t \in [0,1]$ such that the polynomials $F_{k,t}$ are non-degenerate for all $k = 1, ..., N$, $t \in [0,1]$.

2.5. C-curves in toric surfaces and Viro theorem

Let $\Delta \subset \mathbb{R}^2_+$ be an integer polygon, and F be a non-degenerate polynomial with Newton polygon Δ. Denote by $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ the toric surface corresponding to Δ, and by $Z(F)$ the algebraic curve in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ defined by F.

Represent $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ as the closure of the surface

$$\{(z^i w^j)_{(i,j)\in\Delta\cap\mathbb{Z}^2} \ : \ (z,w) \in (\mathbb{C}^*)^2\} \subset \mathbb{C}P^{n-1} \ ,$$

where $n = \#(\Delta \cap \mathbb{Z}^2)$ (see, for example [5]). Define a map $\nu_\Delta : \mathbb{C}\Delta \to \mathrm{Tor}_{\mathbb{C}}(\Delta)$ in the following way. First, put $\nu_\Delta\big|_{\mathbb{C}I(\Delta)} = (\mathbb{C}\mu_\Delta)^{-1}$. Note that $\nu_\Delta\big|_{\mathbb{C}I(\Delta)}$ is a diffeomorphism of $\mathbb{C}I(\Delta)$ and $(\mathbb{C}^*)^2 \subset \mathrm{Tor}_{\mathbb{C}}(\Delta)$. Then extend $\nu_\Delta\big|_{\mathbb{C}I(\Delta)}$ to $\partial\mathbb{C}\Delta$. Namely, given an edge or a vertex δ of Δ and a point $\mathbb{C}\mu_\delta(z,w) \in \mathbb{C}I(\delta)$, where $(z,w) \in (\mathbb{C}^*)^2$, we put

$\nu_\Delta(\mathbb{C}\mu_\delta(z,w)) = (a_{i,j})_{(i,j)\in\Delta\cap\mathbb{Z}^2}$, where

$$a_{i,j} = z^i w^j,\ (i,j)\in\delta, \quad a_{i,j}=0,\ (i,j)\in\Delta\backslash\delta\ .$$

Proposition 2.5. *The map ν_Δ is equivariant, continuous and surjective, and one has $\nu_\Delta(\mathbb{C}Ch(F)) = Z(F)$.*

Definition 2.2. In the notation of Section 2.4, given a subdivision $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ and a collection of complex numbers $\mathcal{A} = \{A_{i,j}:\ (i,j)\in\Delta\cap\mathbb{Z}^2\}$, define a (*nonsingular*) *C-curve with Newton polygon* Δ in $\mathrm{Tor}_\mathbb{C}(\Delta)$ as

$$T\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \nu_\Delta(\mathbb{C}Ch(\mathcal{S},\mathcal{A})) \subset \mathrm{Tor}_\mathbb{C}(\Delta)\ .$$

If all the numbers $A_{i,j}$ are real, the corresponding C-curve is called *real* and its real part $\nu_\Delta(\mathbb{R}Ch(\mathcal{S},\mathcal{A}))$ is denoted by $T\mathbb{R}Ch(\mathcal{S},\mathcal{A})$.

Proposition 2.6. *A (nonsingular) C-curve $T\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ with Newton polygon Δ is a closed piecewise-smooth surface in* $\mathrm{Tor}_\mathbb{C}(\Delta)$*. This surface is invariant with respect to the involution of complex conjugation* Conj *in* $\mathrm{Tor}_\mathbb{C}(\Delta)$ *if all the numbers in $\mathcal{A}$ are real.*

In [11] the statement corresponding to Proposition 2.6 is formulated and proved only in the case of projective spaces (and not for arbitrary projective toric varieties). However, the proof given in [11] automatically extends to the case of projective toric surfaces.

Definition 2.3. A homeomorphism of (resp., an isotopy in) $\mathrm{Tor}_\mathbb{C}(\Delta)$ is called *tame* if its restriction to $\mathrm{Tor}_\mathbb{C}(\delta)$ is a homeomorphism of (resp., an isotopy in) of $\mathrm{Tor}_\mathbb{C}(\delta)$ for any edge or vertex δ of Δ. In addition, we call homeomorphisms of (resp., isotopies in) $\mathrm{Tor}_\mathbb{C}(\Delta)$ *equivariant* if they commute with the involution of complex conjugation Conj of $\mathrm{Tor}_\mathbb{C}(\Delta)$.

Remark 2.3. Similarly to Remark 2.2, two C-curves in $\mathrm{Tor}_\mathbb{C}(\Delta)$ with the same Newton polygon Δ and with the same subdivision $\mathcal{S}$ of Δ are tame isotopic in $\mathrm{Tor}_\mathbb{C}(\Delta)$ (not equivariantly, in general).

Definition 2.4. A subdivision $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ of a polygon Δ is called *convex* if there exists a piecewise-linear convex function $\mu : \Delta \to \mathbb{R}$ whose domains of linearity are the polygons $\Delta_1, \ldots, \Delta_N$.

Theorem 2.7 (Complex Viro theorem, see [23])**.** *In the notation of Section 2.4, let $\mathcal{S}$ be a convex subdivision of an integer polygon Δ into 2-dimensional integer polygons, and $\mu : \Delta \to \mathbb{R}$ a convex piecewise-linear function certifying the convexity of $\mathcal{S}$. Then, for sufficiently small positive t, the curve in* $\mathrm{Tor}_\mathbb{C}(\Delta)$ *given by the polynomial*

$$F_t(z,w) = \sum_{(i,j)\in\Delta} A_{i,j} t^{\mu(i,j)} z^i w^j,$$

is nonsingular, and $T\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is tame isotopic in $\mathrm{Tor}_\mathbb{C}(\Delta)$ *(equivariantly, if all the numbers in $\mathcal{A}$ are real) to $T\mathbb{C}Ch(F_t)$.*

2.6. Topology of C-curves

Proposition 2.8 (see [11]). *Any C-curve M in* $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ *is isotopic to a close smooth surface M_{sm} in* $\mathrm{Tor}_{\mathbb{C}}(\Delta)$. *If M is real, the surface M_{sm} can be chosen* Conj*-invariant, and the isotopy can be made equivariant. The tangent bundle of a* Conj*-invariant smoothing M_{sm} of a real C-curve M is equivariantly isotopic to a* Conj*-invariant bundle of complex lines by an isotopy preserving the tangent bundle to the real part of M_{sm}.*

Proposition 2.9 (see [11]). *Let two subdivisions $\mathcal{S} = \{\Delta_k,\ k = 1,...,N\}$ and $\mathcal{S}' = \{\Delta_{kl},\ l = 1,...,r_k,\ k = 1,...,N\}$ of an integer polygon Δ into 2-dimensional integer polygons satisfy*

$$\Delta_k = \bigcup_{l=1}^{r_k} \Delta_{kl}, \quad k = 1,...,N,$$

so that the subdivision $\mathcal{S}'$ is given by piecewise-linear function $\mu : \Delta \to \mathbb{R}$, whose restrictions $\mu_k = \mu\big|_{\Delta_k}$, $k = 1,...,N$, are convex. Then the C-curves $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ and $\mathbb{C}Ch(\mathcal{S}',\mathcal{A}')$ are tame isotopic, provided $\mathcal{A},\mathcal{A}' : \Delta \to \mathbb{C}$ define non-degenerate polynomials F_k, F_{kl}, $l = 1,...,r_k$, $k = 1,...,N$. Similarly, $T\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ and $T\mathbb{C}Ch(\mathcal{S}',\mathcal{A}')$ are tame isotopic in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$.

A subdivision $\Delta = \Delta_1 \cup ... \cup \Delta_N$ of an integer polygon Δ into integer polygons is called *maximal* if it cannot be refined to another subdivision of Δ into integer polygons. In this case all the integral points in Δ are vertices of $\Delta_1,...,\Delta_N$, and these polygons are triangles of area $1/2$.

Corollary 2.10. *Given an integer polygon Δ, any C-curve in $\mathbb{C}\Delta$ is tame isotopic to a C-curve constructed out of a maximal subdivision of Δ. The same is true for C-curves in* $\mathrm{Tor}_{\mathbb{C}}(\Delta)$.

Proof. Let $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ be a subdivision of Δ into integer polygons, and $f : \Delta \to \mathbb{R}$ be a smooth convex function. Define a piecewise-linear function $\mu : \Delta \to \mathbb{R}$ as follows: put $\mu(i,j) = f(i,j)$, where $(i,j) \in \Delta \cap \mathbb{Z}^2$, then define the graph of $\mu\big|_{\Delta_k}$ to be the "lower" part of the convex hull of $\{(i,j,\mu(i,j)\ :\ (i,j) \in \Delta_k\}$. The function μ defines a maximal subdivision of Δ inscribed into the initial subdivision and satisfying the conditions of Proposition 2.9, which completes the proof. ☐

Corollary 2.11 (cf. [11]). *Given an integer polygon Δ, any C-curve in $\mathbb{C}\Delta$ is tame isotopic (not equivariantly, in general) to an algebraic curve in $\mathbb{C}\Delta$ with Newton polygon Δ.*

Proof. By Corollary 2.10 we can assume that a C-curve in $\mathbb{C}\Delta$ is constructed out of a maximal triangulation of Δ. We will transform any given triangulation into a convex triangulation, so that in each transformation step the conditions of Proposition 2.9 hold true.

Let $\mathcal{S}$ be a triangulation of Δ, and O a vertex of Δ. Denote by $O(\mathcal{S})$ the star of O in $\mathcal{S}$. We construct a triangulation $\mathcal{S}'$ of Δ such that $O(\mathcal{S}') \supset O(\mathcal{S})$ and $O(\mathcal{S}') \neq O(\mathcal{S})$. Then, in finitely many steps we come to $O(\mathcal{S}) = \Delta$, which corresponds to a convex triangulation, and the required statement will follow from Theorem 2.7 and Proposition 2.9.

Let $O, P_1, ..., P_r$ be all the vertices of $O(\mathcal{S})$ numbered successively clockwise along $\partial O(\mathcal{S})$. Any segment $[P_i, P_{i+1}]$ either lies on $\partial\Delta$, or is an edge of a unique triangle $T_i \in \mathcal{S}$, $T_i \not\subset O(\mathcal{S})$.

(i) Assume that, for some $i = 1, ..., r$, the vertex $Q \ne P_i, P_{i+1}$ of T_i lies in the sector between the rays $[OP_i)$ and $[OP_{i+1})$, and $Q \notin [OP_i) \cup [OP_{i+1})$ (see Figure 1a). Then we change the triangulation of Δ as shown in Figure 1b,c. These changes satisfy the conditions of Proposition 2.9 and lead to a triangulation with a strongly greater star of O.

(ii) Assume that, for some $i = 1, ..., r-1$, $T_i = T_{i+1}$, *i.e.*, the vertices of the latest triangle are P_i, P_{i+1}, P_{i+2} (see Figure 1d). Then we perform the transformation shown in Figure 1e, once again increasing the star of O.

(iii) Assume that there are no triangles T_i as in (i), (ii). Then any triangle T_i is either "left", *i.e.*, the vertex Q lies in the sector between $[OP_1)$ and $[OP_i)$, or "right", *i.e.*, Q lies in the sector between $[OP_{i+1})$ and $[OP_r)$. If there exist "left" triangles, consider the "left" triangle T_i with the minimal i. If $i = 1$, we have the situation shown in Figure 1f. Then we change the triangulation as shown in Figure 1g, increasing the star of O. If $i > 1$, then the triangle T_{i-1} must be "right", which means that we have a situation shown in Figure 1h. Then we change the triangulation as shown in Figure 1i, increasing the star of O. $\square$

2.7. Singular C-curves

The construction of C-curves described in Sections 2.1 - 2.5 can be performed even if the polynomials involved in the construction are not non-degenerate. Introduce the following definition. A polynomial $F \in \mathbb{C}[z, w]$ is called *non-degenerate on the boundary*, if any truncation F^δ on a proper face δ of the Newton polygon $\Delta(F)$ of F has a nonsingular zero set in $(\mathbb{C}^*)^2$ (cf. with the definition of a *non-degenerate* polynomial in Section 2.1). One can define a complex chart in Δ_F of a polynomial which is non-degenerate on the boundary (and its real chart in Δ_F if the polynomial is real) exactly in the same way as in Section 2.3. The charts of polynomials which are non-degenerated on the boundary can be glued in the same way as in Section 2.4. Namely, let $\mathcal{S}$ be a subdivision of a 2-dimensional integer polygon $\Delta \subset \mathbb{R}^2_+$ into 2-dimensional integer polygons : $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$, and let $\mathcal{A} = \{A_{i,j},\ (i,j) \in \Delta \cap \mathbb{Z}^2\}$ be a collection of complex numbers such that $A_{i,j} \ne 0$ if (i,j) is a vertex of some Δ_k, $1 \le k \le N$. Assume that the polynomials

$$F_k(z,w) = \sum_{(i,j)\in\Delta_k} A_{i,j} z^i w^j \ , \quad k = 1, ..., N,$$

are non-degenerate on the boundary, and that at least one of these polynomials is not non-degenerate. The union $\mathbb{C}Ch(\mathcal{S}, \mathcal{A}) = \bigcup_{k=1}^N \mathbb{C}Ch(F_k)$ of the complex charts of the polynomials $F_1, ..., F_N$ is called a *singular C-curve in* $\mathbb{C}\Delta$. If all the numbers $A_{i,j}$ are real, the union $\mathbb{R}Ch(\mathcal{S}, \mathcal{A}) = \bigcup_{k=1}^N \mathbb{R}Ch(F_k)$ of the real charts of $F_1, ..., F_N$ is called a

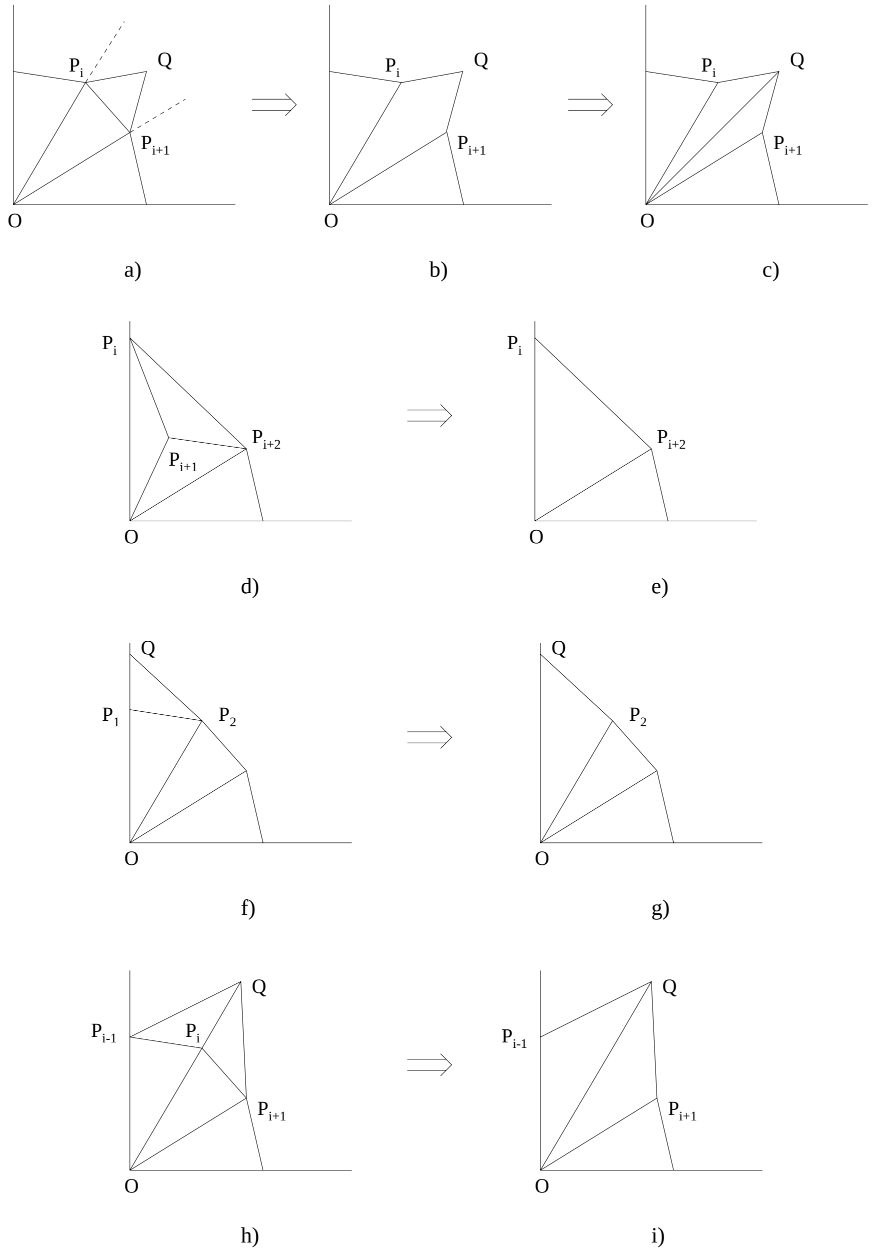

Figure 1. Transformation of a triangulation

singular C-curve in $\mathbb{R}\Delta$. We define a *singular C-curve with Newton polygon* Δ in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ as $T\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \nu_\Delta(\mathbb{C}Ch(\mathcal{S},\mathcal{A})) \subset \mathrm{Tor}_C(\Delta)$.

We have two propositions which are completely similar to Propositions 2.4 and 2.6.

Proposition 2.12. *A singular C-curve $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is a singular surface in $\mathbb{C}\Delta$ with boundary $\partial\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \partial\mathbb{C}\Delta \cap \mathbb{C}Ch(\mathcal{S},\mathcal{A})$. The only singular points of $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ are those of $\mathbb{C}Ch(F_k)$, $k = 1,\ldots,N$. If all the numbers in $\mathcal{A}$ are real, then $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is invariant with respect to* Conj*, and the set $\mathbb{R}Ch(\mathcal{S},\mathcal{A})$ is a (possibly singular) curve in $\mathbb{R}\Delta$ with boundary $\partial\mathbb{R}Ch(\mathcal{S},\mathcal{A}) = \partial\mathbb{R}\Delta \cap \mathbb{R}Ch(\mathcal{S},\mathcal{A})$.*

Proposition 2.13. *A singular C-curve $T\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is a singular surface in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$. The only singular points of $T\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ are the images under ν_Δ of the singular points of $\mathbb{C}Ch(F_k)$, $k = 1,\ldots,N$. This surface is invariant with respect to the involution of complex conjugation* Conj *in* $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ *if all the numbers in $\mathcal{A}$ are real.*

Further on, we will continue to use the term *C-curve* for nonsingular C-curves, and will use the term *singular C-curve* for singular ones.

3. Topology of real C-curves

It follows from Corollary 2.11 that the genus of a C-curve with Newton polygon Δ in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ coincides with the genus $g(\Delta)$ of a nonsingular algebraic curve with Newton polygon Δ in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ (the number $g(\Delta)$ is equal to the number of integer points lying strongly inside of Δ). Thus, we have the following statement.

Proposition 3.1 (Harnack inequality for C-curves)**.** *For a real C-curve M with Newton polygon Δ in* $\mathrm{Tor}_{\mathbb{C}}(\Delta)$, *one has the inequality $b_0(\mathbb{R}M) \le g(\Delta) + 1$, where $b_0(\mathbb{R}M)$ is the number of connected components of $\mathbb{R}M$.*

We formulate other topological properties of real C-curves only in the case when $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ is the projective plane. If Δ is the triangle with vertices $(0,0)$, $(d,0)$ and $(0,d)$, then a C-curve with Newton polygon Δ in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ is called *a C-curve of degree d in $\mathbb{C}P^2$*.

Let M be an oriented smooth connected closed surface in $\mathbb{C}P^2$. Then M is called a *flexible curve of degree d* (see [26]) if

- it realizes $d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$,
- the genus of M is equal to $(d-1)(d-2)/2$,
- M is invariant under the complex conjugation,
- the field of tangent planes to M on $M \cap \mathbb{R}P^2$ can be equivariantly deformed to the field of lines in $\mathbb{C}P^2$ tangent to $M \cap \mathbb{R}P^2$.

According to Proposition 2.8 and Corollary 2.11 a Conj-invariant smoothing of a real C-curve of degree d in $\mathbb{C}P^2$ is a flexible curve of degree d. Thus, all the restrictions on the topology of flexible curves are applicable to real C-curves. We formulate here in the framework of real C-curves the principal known restrictions on flexible curves. An extensive list of restrictions to the topology of flexible curves can be found in [26].

Let us start with definitions. The standard definitions applicable to real algebraic curves can be naturally extended to real C-curves. A real C-curve A of degree d in $\mathbb{C}P^2$ is called an *M-curve* or *maximal* if $\mathbb{R}A$ has $(d-1)(d-2)/2+1$ connected components. A real C-curve A of degree d in $\mathbb{C}P^2$ is called an *$(M-i)$-curve* if $\mathbb{R}A$ has $(d-1)(d-2)/2+1-i$ connected components. A connected component of the real part of a real C-curve of degree d in $\mathbb{C}P^2$ is called an *oval* if it divides $\mathbb{R}P^2$ into two parts. The part homeomorphic to a disk is called the *interior* of the oval. All the connected components of the real part of a real C-curve of an even degree in $\mathbb{C}P^2$ are ovals. Exactly one connected component of the real part of a real C-curve of an odd degree in $\mathbb{C}P^2$ is not an oval. This component is called *nontrivial*. An oval is *even* (resp., *odd*) if it lies inside of an even (resp., odd) number of other ovals of the curve. The numbers of even and odd ovals of a curve are denoted by p and n, respectively. The Euler characteristic of a connected component of the complement in $\mathbb{R}P^2$ of the real part of a real C-curve is called the *characteristic* of an oval bounding the component from outside. A component of the complement in $\mathbb{R}P^2$ of the real part of a real C-curve is said to be *even* if each of its inner bounding ovals contains inside an odd number of ovals.

Theorem 3.2.

- Harnack inequality. *The number of connected components of the real part of a real C-curve of degree d in $\mathbb{C}P^2$ is at most $(d-1)(d-2)/2+1$.*
- Gudkov-Rokhlin congruence. *For a maximal real C-curve of degree $2k$ in $\mathbb{C}P^2$, one has*
$$p - n \equiv k^2 \mod 8.$$
- Gudkov-Krakhnov-Kharlamov congruence. *Let A be a real C-curve of degree $2k$ in $\mathbb{C}P^2$. If A is an $(M-1)$-curve, then*
$$p - n \equiv k^2 \pm 1 \mod 8.$$
- Strengthened Petrovsky inequalities. *For a real C-curve A of degree $2k$ in $\mathbb{C}P^2$, one has*
$$p - n^- \le \frac{3k(k-1)}{2} + 1, \quad n - p^- \le 3k(k-1),$$
where p^- (resp., n^-) is the number of even (resp., odd) ovals of $\mathbb{R}A$ with negative characteristic.
- Strengthened Arnold inequalities. *For a real C-curve A of degree $2k$ in $\mathbb{C}P^2$, one has*
$$p^- + p^0 \le \frac{k^2 - 3k + 3 + (-1)^k}{2}, \quad n^- + n^0 \le \frac{k^2 - 3k + 2}{2},$$
where p^0 (resp., n^0) is the number of even (resp., odd) ovals of $\mathbb{R}A$ with characteristic 0.
- Extremal properties of strengthened Arnold inequalities. *For a real C-curve of degree $2k$ in $\mathbb{C}P^2$, one has*
  $p^- = p^+ = 0$, if k is even and $p^- + p^0 = (k^2 - 3k + 4)/2$,
  $n^- = n^+ = 0$, if k is odd and $n^- + n^0 = (k^2 - 3k + 2)/2$.

A real C-curve A in $\mathbb{C}P^2$ is said to be of *type I* if its real part $\mathbb{R}A$ divides A into two parts; otherwise, the curve is of *type II*. For a curve A of type I, the orientations of two halves of $A \setminus \mathbb{R}A$ induce on $\mathbb{R}A$ two opposite orientations which are called *complex orientations*. Note that a real C-curve A is of type I if and only if all the algebraic curves used in the construction of A are of type I and complex orientations on the real parts of these curves can be chosen in such a way that they induce an orientation of $\mathbb{R}A$.

A pair of ovals of the real part of a real C-curve in $\mathbb{C}P^2$ is *injective* if one of these ovals is inside of the other one. A collection of ovals is called a *nest* if any two of them form an injective pair. An injective pair of ovals of a real C-curve is *positive* (resp., *negative*) if the complex orientations of the ovals can be induced (resp., cannot be induced) from some orientation of the annulus bounded by the ovals. Take an oval of a real C-curve of type I and of an odd degree in $\mathbb{C}P^2$, and consider the Möbius band which is the complement in $\mathbb{R}P^2$ of the interior of the oval. The oval is called *positive* (resp., *negative*) if the integer homology class realized in the Möbius band by the oval equipped with a complex orientation differs in sign (resp., coincides) with the class of the doubled nontrivial component equipped with the complex orientation.

Theorem 3.3.

- Klein congruence. *Let A be a real C-curve of type I in $\mathbb{C}P^2$. If A is an $(M-i)$-curve, then $i \equiv 0 \mod 2$.*
- Arnold congruence. *For a real C-curve of type I and of degree $2k$, one has*
$$p - n \equiv k^2 \mod 4 .$$
- Rokhlin-Mishachev formulae. *For a real C-curve A of type I and of degree $2k$ in $\mathbb{C}P^2$, one has*
$$2(\Pi^+ - \Pi^-) = l - k^2,$$
where l is the number of ovals of $\mathbb{R}A$, and Π^+ and Π^- are the numbers of positive and negative injective pairs, respectively. For a real C-curve A of type I and of degree $2k+1$ in $\mathbb{R}P^2$, one has
$$2(\Pi^+ - \Pi^-) + \Lambda^+ - \Lambda^- = l - k(k+1),$$
where Λ^+ and Λ^- are the numbers of positive and negative ovals, respectively.
- Kharlamov- Marin congruence. *Let A be a real C-curve of degree $2k$ in $\mathbb{C}P^2$. If A is an $(M-2)$-curve and $p - n \equiv k^2 + 4 \mod 8$, then A is of type I.*
- Rokhlin inequalities. *Let A be a real C-curve of type I and of degree $2k$ in $\mathbb{C}P^2$. If k is even, then $4\nu + p - n \le 2k^2 - 6k + 8$, where ν is the number of odd nonempty exterior bounding ovals of even components of $\mathbb{R}P^2 \setminus \mathbb{R}A$.*

  If k is odd, then $4\pi + n - p \le 2k^2 - 6k + 7$, where π is the number of even nonempty exterior bounding ovals of even components of $\mathbb{R}P^2 \setminus \mathbb{R}A$.
- Extremal properties of strengthened Arnold inequalities. *Let A be a real C-curve of degree $2k$ in $\mathbb{C}P^2$.*

  If k is even and $p^- + p^0 = (k^2 - 3k + 4)/2$, then A is of type I.

  If k is odd and $n^- + n^0 = (k^2 - 3k + 2)/2$, then A is of type I.

The Harnack inequality in the case of real C-curves constructed using a maximal triangulation was, first, proved in [10] and then in a different way in [8]. Rokhlin-Mishachev formulae in the case of real C-curves constructed using a maximal triangulation was proved in [16].

It is interesting that one restriction on real C-curves is not proved in the case of flexible curves:

if A is a real C-curve of degree d in $\mathbb{C}P^2$, then the sum of the depths of any two nests of $\mathbb{R}A$ is at most $d/2$.

In the case of real algebraic curves this statement is known as Hilbert's theorem and is an immediate corollary of the Bézout theorem. For real C-curves constructed out of trinomials, the restriction was proved in [3]. In general, for real pseudo-holomorphic curves, the statement follows from the results of [6].

There are examples of C-curves beyond the range of known algebraic curves: using non-convex triangulations F. Santos [19] constructed T-curves in $\mathbb{R}P^2$ whose number of even ovals is greater than in the best known algebraic examples of the same degree. Corollary 4.5 in the next section shows that, as any real C-curve in the projective plane, the curves constructed in [19] are pseudo-holomorphic.

4. Pencils of lines and pseudo-holomorphic structure on C-curves

4.1. Refinement

Let Δ be an integer polygon, and $[a, b]$ its projection on the horizontal coordinate axis. A vertex of Δ is called h-extremal if it is the only preimage of a or the only preimage of b in the above projection.

Let $\mathcal{T}: \ \Delta = \Delta_1 \cup ... \cup \Delta_N$ be a subdivision of Δ into integer polygons. We define a new subdivision $\mathcal{T}^{ref}$ of Δ which we call the *horizontal refinement* of $\mathcal{T}$. Denote by V the set of vertices of the polygons $\Delta_1, ..., \Delta_N$, by E_0 the set of vertical edges of $\Delta_1, ..., \Delta_N$ and by E the set of the other edges of polygons $\Delta_1, ..., \Delta_N$. Consider continuous piecewise-linear functions defined on $[a, b]$ whose graphs are unions of edges belonging to E, and denote by $\mathcal{P}$ the set of graphs of these functions. For any $v \in V$ and $e \in E$, put

$$\mathcal{P}(v) = \#\{\Gamma \in \mathcal{P} \ : \ v \in \Gamma\}, \quad \mathcal{P}(e) = \#\{\Gamma \in \mathcal{P} \ : \ e \subset \Gamma\} \ .$$

Fix a sufficiently small positive number ε and replace each edge $e \in E$ by the union $\widetilde{e}$ of $2\mathcal{P}(e)$ parallel translates of e in the following way:

- if $e \subset \partial\Delta$ then
$$\widetilde{e} = \{e, \ tr_\varepsilon(e), \ \ldots \ , \ tr_\varepsilon^{\mathcal{P}(e)-1}(e)\} \ ,$$
or
$$\widetilde{e} = \{e, \ tr_{-\varepsilon}(e), \ \ldots \ , \ tr_{-\varepsilon}^{\mathcal{P}(e)-1}(e)\} \ ,$$
where tr_δ is the translation by the vector $(0, \delta)$, and the sign is chosen so that all new edges intersect Δ;
- if e does not belong to the boundary $\partial\Delta$ of Δ then
$$\widetilde{e} = \{tr_{-\varepsilon/2}^{2\mathcal{P}(e)-1}(e), \ tr_{-\varepsilon/2}^{2\mathcal{P}(e)-3}(e), \ \ldots \ , \ tr_{-\varepsilon/2}(e), \ tr_{\varepsilon/2}(e), \ \ldots \ , \ tr_{\varepsilon/2}^{2\mathcal{P}(e)-1}(e)\} \ .$$

All the endpoints of the edges belonging to $\widetilde{E} = \bigcup_{e\in E} \widetilde{e}$ and their intersection points lie in small neighborhoods of the vertices of $\mathcal{T}$. For any vertex v in V, we choose such a closed neighborhood in the form of a rectangle R_v^1 with vertical and horizontal sides. Choosing ε smaller if necessary, we can suppose that no edge which belongs to $\widetilde{E}$ intersects the horizontal sides of R_v^1 (see Figure 2(a)). Put $R_v^2 = R_v^1 \cap \Delta$.

The intersection points of the edges belonging to $\widetilde{E}$ with the vertical sides of R_v^2 is called *marked points*. If $v = (i, j)$, $a < i < b$, then on the vertical sides of R_v^2 we take the minimal segments s_1, s_2 containing all the marked points, and put R_v^3 to be the convex hull of the union of s_1, s_2 and the intersection points of R_v^2 with vertical edges of $\mathcal{T}$ adjacent to v (see Figure 2(b)). Note that the number of marked points on s_1 and s_2 is the same and equal to $2\mathcal{P}(v)$. Then we subdivide R_v^3 by non-intersecting lines joining the respective marked points on s_1 and s_2 (see Figure 2(c)). If the obtained subdivision of R_v^3 contains triangles, then we remove the sides of these triangles lying inside of R_v^3 (see Figure 2(d)).

If R_v^2 has a vertical edge $s_1 \subset \partial\Delta$ (see Figure 3(a)), we take the minimal segment s_2 on the other vertical side of R_v^2, which contains all the intersection points with the edges belonging to $\widetilde{E}$. Then put R_v^3 to be the convex hull of s_1 and s_2 (see Figure 3(b)), and subdivide R_v^3 by horizontal lines through marked points on s_2 (see Figure 3(c)). If the obtained subdivision of R_v^3 contains triangles, then we remove the sides of these triangles lying inside of R_v^3 (see Figure 3(d)).

If R_v^2 is a triangle, then v is its vertex and is an h-extremal vertex of Δ. In this case put $R_v^3 = R_v^2$ and subdivide it into triangles by joining marked points with v (see Figure 4).

Now define the *horizontal refinement* $\mathcal{T}^{ref}$ of Δ into the following convex polygons:

- the polygons of the subdivisions of R_v^3, $v \in V$, introduced above;
- the (closures of the) complements in $\Delta\backslash \bigcup_{v\in V} R_v^3$ to the edges from $E_0 \cup \widetilde{E}$.

An example of horizontal refinement is shown in Figure 5.

The following statement accumulates the key properties of the horizontal refinement.

Lemma 4.1. *(i) For any $k = 1, ..., N$ there is exactly one element Δ_k^{ref} of $\mathcal{T}^{ref}$ such that $\Delta_k^{ref} \subset \Delta_k$ and $\Delta_k \backslash \Delta_k^{ref}$ lies in a small neighborhood of $\partial\Delta_k$. Furthermore, for any vertex v of Δ_k, the polygon Δ_k^{ref} has exactly two vertices in a small neighborhood of v.*

(ii) For any edge $e \in E$ there are exactly $2\mathcal{P}(e)-1$ elements of $\mathcal{T}^{ref}$ which are congruent parallelograms each one with a pair of sides parallel and close to e.

(iii) The elements of $\mathcal{T}^{ref}$ which are not mentioned in (i) and (ii) lie in $\bigcup_{v\in V} R_v^3$.

(iv) Each polygon in $\mathcal{T}^{ref}$, which does not contain an h-extremal vertex of Δ, has exactly two vertical edges; each polygon in $\mathcal{T}^{ref}$, which contains an h-extremal vertex of Δ, is a triangle with one vertical edge.

The proof is straightforward from the construction (see Figures 2, 3 and 4).

The subdivisions of Δ into convex (not necessary integer) polygons which satisfy the property (iv) in Lemma 4.1 will be called *h-subdivisions*.

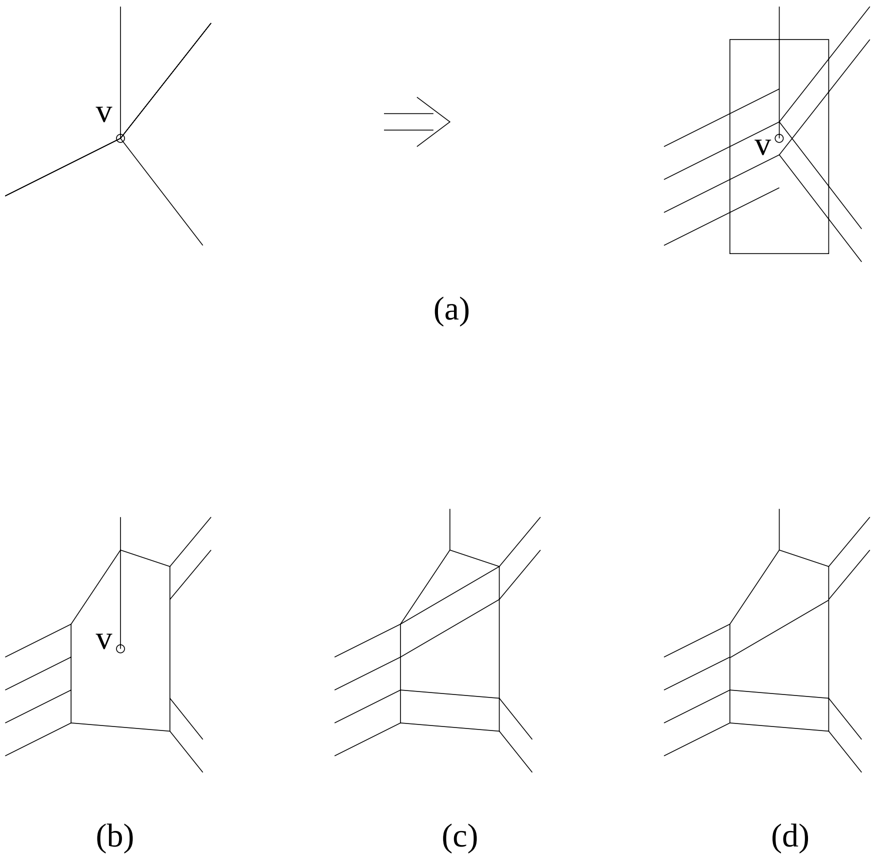

Figure 2. Refinement of a subdivision in a neighborhood of a vertex

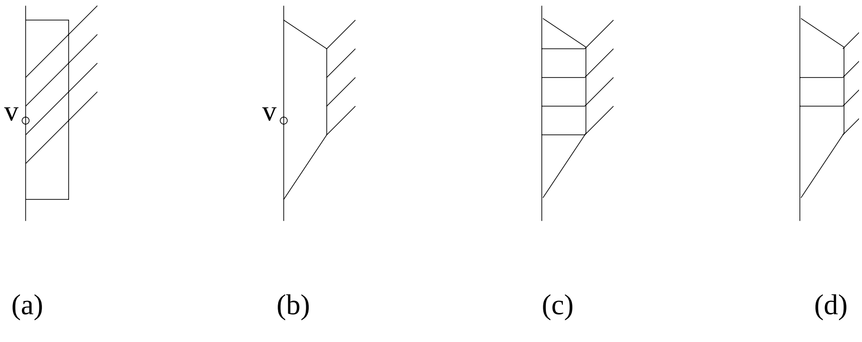

Figure 3. Case when v belongs to a vertical edge of Δ

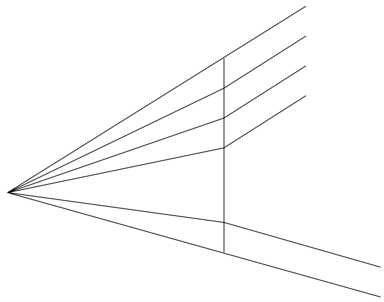

Figure 4. Case when v is h-extremal

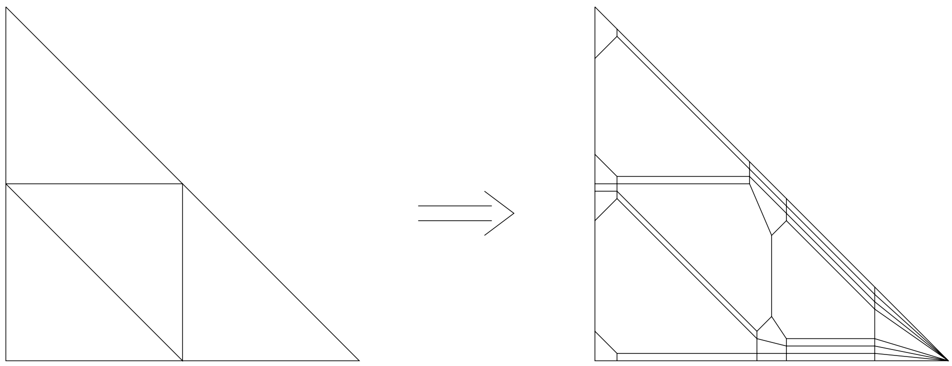

Figure 5. Example of horizontal refinement

4.2. S^2-bundles over $\mathbb{C}P^1$ associated with h-subdivisions

Now we impose an extra condition on Δ. We assume that $\Delta \subset (\mathbb{R}^*_+)^2$ and that Δ is

- either an integer quadrangle which has two vertical sides, one horizontal side and one side with a nonnegative integer slope m,
- or an integer triangle with one horizontal side of length d and one vertical side of length d.

Let $\mathcal{S}$ be an h-subdivision of Δ, and Λ the (toric part of the) pencil of straight lines $\{y = \text{const}\}$ in $(\mathbb{C}^*)^2$.

Lemma 4.2. *(1) Let δ be a polygon of the subdivision $\mathcal{S}$ which has vertical edges σ_1 and σ_2. Then the charts $\mathbb{C}Ch_\delta(L)$ of lines $L \in \Lambda$ are disjoint and lie in $\mathbb{C}I(\delta) \cup \mathbb{C}I(\sigma_1) \cup \mathbb{C}I(\sigma_2)$. Any chart $\mathbb{C}Ch_\delta(L)$ is homeomorphic to a cylinder whose boundary consists of one circle in $\mathbb{C}I(\sigma_1)$ and one circle in $\mathbb{C}I(\sigma_2)$.*

(2) Let σ be a non-vertical edge of $\mathcal{S}$ with vertices v_1 and v_2. Then the charts $\mathbb{C}Ch_\sigma(L)$, $L \in \Lambda$, are homeomorphic to cylinders whose boundary consists of one circle in $\mathbb{C}v_1$ and one circle in $\mathbb{C}v_2$. The charts of lines $\{y = c_1\}$, $\{y = c_2\}$ coincide if $c_1/c_2 \in \mathbb{R}$, and are disjoint otherwise.

(3) Let δ' and δ'' be two polygons of $\mathcal{S}$ with a common vertical side s. Then, for any line $L \in \Lambda$, the charts $\mathbb{C}Ch_{\delta'}(L)$ and $\mathbb{C}Ch_{\delta''}(L)$ intersect along their common boundary component in $\mathbb{C}I(s)$. Similarly, for a pair of non-vertical edges δ' and δ'' of $\mathcal{S}$ with a common vertex v and any line $L \in \Lambda$, the charts $\mathbb{C}Ch_{\delta'}(L)$ and $\mathbb{C}Ch_{\delta''}(L)$ intersect along their common boundary component in $\mathbb{C}I(v)$.

(4) Let Δ be a triangle with h-extremal vertex v, and $\delta \in \mathcal{S}$ be a triangle with vertex v and vertical edge σ. Then the charts $\mathbb{C}Ch_\delta(L)$, $L \in \Lambda$, are homeomorphic to cylinders whose boundary consists of one circle in $\mathbb{C}\sigma$ and one circle in $\mathbb{C}v$. The charts of lines $\{y = c_1\}$, $\{y = c_2\}$ intersect along the common boundary circle in $\mathbb{C}v$ if $c_1/c_2 \in \mathbb{R}$, and are disjoint otherwise.

The proof Lemma 4.2 is straightforward from the definitions and statements of Section 2.

If Δ is a quadrangle, denote by σ_1 and σ_2 its vertical sides in such a way that σ_1 is not shorter than σ_2. If Δ is a triangle, denote by σ_1 its vertical side. Let σ_1' be the projection of σ_1 to the vertical coordinate axis. This projection extends to the projection $\pi_1 : \mathbb{C}\sigma_1 \to \mathbb{C}\sigma_1'$, defining, in fact, a splitting $\mathbb{C}\sigma_1 = \mathbb{C}\sigma_1' \times S^1$.

Lemma 4.3. *If Δ is a quadrangle, there exists a surjective piecewise-smooth map $\pi : \mathbb{C}\Delta \to \mathbb{C}\sigma_1'$ such that*

(i) $\pi\big|_{\sigma_1} = \pi_1$,

(ii) all fibers of π are unions of charts of lines $L \in \Lambda$ in the complexifications of polygons of $\mathcal{S}$, and are homeomorphic to cylinders with boundary circles in $\mathbb{C}\sigma_1, \mathbb{C}\sigma_2$.

If Δ is a triangle with an h-extremal vertex v, there exists a surjective piecewise-smooth map $\pi : (\mathbb{C}\Delta \backslash \mathbb{C}v) \to \mathbb{C}\sigma_1'$ such that

(i) $\pi\big|_{\sigma_1} = \pi_1$,

(ii) any fiber of π is a union of the complements to $\mathbb{C}v$ of the charts of some line $L \in \Lambda$ in the complexifications of polygons of $\mathcal{S}$, and is homeomorphic to cylinder $S^1 \times [0, \infty)$ with boundary circle in $\mathbb{C}\sigma_1$; the closure of a fiber of π has one more boundary circle in $\mathbb{C}v$.

Proof. Assume that Δ is a quadrangle. Let $s \subset \sigma_1$ be an edge of the subdivision $\mathcal{S}$. Then, there exists a unique sequence of polygons $\delta_1, ..., \delta_r \in \mathcal{S}$ such that

- s is a side of δ_1,
- for any $i = 1, ..., r-1$, the polygons δ_i and δ_{i+1} have a common vertical side,
- one of the vertical sides of δ_r belongs to σ_2.

Pick a point p in $\mathbb{C}I(s)$, and consider the point $p' = \pi_1(p)$ in $\mathbb{C}I(\sigma_1')$. According to Lemma 4.2, there exists a uniquely defined line $L \in \Lambda$ such that $p \in \mathbb{C}Ch_{\delta_1}(L)$. Note that p' does not depend on the choice of p in $\mathbb{C}Ch_{\delta_1}(L) \cap \mathbb{C}\sigma_1$. Again according to

Lemma 4.2, the union $\bigcup_{i=1}^r \mathbb{C}Ch_{\delta_i}(L)$ is a piecewise-smooth cylinder. We denote it by Π_p. Put $\pi(\Pi_p) = p$.

Let $v \in \sigma_1$ be a vertex of $\mathcal{S}$. Then, there exists a unique sequence of non-vertical edges $\delta_1, ..., \delta_r \in \mathcal{T}$ such that

- v is a vertex of δ_1,
- for any $i = 1, ..., r-1$, the edges δ_i and δ_{i+1} have a common vertex,
- one of the vertices of δ_r belongs to σ_2.

Similarly, pick a point p in $\mathbb{C}v$, and consider the point $p' = \pi_1(p)$ and the line $L \in \Lambda$ such that $p \in \mathbb{C}Ch_{\delta_1}(L)$. Once again p' does not depend on the choice of p in $\mathbb{C}Ch_{\delta_1}(L) \cap \mathbb{C}\sigma_1$. According to Lemma 4.2, the union $\bigcup_{i=1}^r \mathbb{C}Ch_{\delta_i}(L)$ is a piecewise-smooth cylinder. Again we denote it by Π_p and put $\pi(\Pi_p) = p$.

The case of triangular Δ can be treated similarly. $\square$

For the case of quadrangular Δ, we have constructed a trivial fibration $\mathbb{C}\Delta \to \mathbb{C}\sigma_1'$ whose fibers are cylinders. The map $\nu_\Delta : \mathbb{C}\Delta \to \mathrm{Tor}_\mathbb{C}(\Delta)$ factors $\partial\mathbb{C}\Delta$ by an S^1-action. The toric surface $\mathrm{Tor}_\mathbb{C}(\Delta)$ is isomorphic to the rational ruled surface Σ_m, i.e., to the space of a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ with self-intersection $-m$ of the base section. Now we note that ν_Δ takes $\mathbb{C}\sigma_1$ to $\mathrm{Tor}_\mathbb{C}(\sigma_1) = \mathrm{Tor}_\mathbb{C}(\sigma_1') = \mathbb{C}P^1$, and takes each fiber Π_p into a sphere S^2, contracting the boundary components into points. At last, we observe that the fibers Π_p, $p \in \partial\mathbb{C}\sigma_1'$ are identified by ν_Δ so that the induced fibration $\pi_\mathcal{S} : \mathrm{Tor}_\mathbb{C}(\Delta) = \Sigma_m \to \mathrm{Tor}_\mathbb{C}(\sigma_1') = \mathbb{C}P^1$ defines an S^2-bundle with self-intersection $-m$ of the base section $E \overset{\text{def}}{=} \mathrm{Tor}_\mathbb{C}(\sigma_1) \subset \mathrm{Tor}_\mathbb{C}(\Delta)$. In addition, this fibration commutes with the complex conjugation. Hence there exists an equivariant piecewise-smooth homeomorphism $\mathrm{Tor}_\mathbb{C}(\Delta) \to \Sigma_m$ which takes the fibration $\pi_\mathcal{S}$ to the standard one.

For the case of triangular Δ, we have a trivial fibration $\mathbb{C}\Delta\backslash\mathbb{C}v \to \mathbb{C}\sigma_1'$ with fibers whose closures in $\mathbb{C}\Delta$ are cylinders. The map $\nu_\Delta : \mathbb{C}\Delta \to \mathrm{Tor}_\mathbb{C}(\Delta) = \mathbb{C}P^2$ factors $\partial\mathbb{C}\Delta$ by an S^1-action. Then ν_Δ takes $\mathbb{C}\sigma_1$ to $\mathrm{Tor}_\mathbb{C}(\sigma_1) = \mathrm{Tor}_\mathbb{C}(\sigma_1') = \mathbb{C}P^1$, and takes the closures of all fibers Π_p into spheres S^2, which pass through the point $z_0 = \mathrm{Tor}_\mathbb{C}(v)$ and are disjoint in $\mathbb{C}P^2\backslash\{z_0\}$. One can easily see that these spheres represent a generator of $H_2(\mathbb{C}P^2)$, and thus, intersect with multiplicity 1 at z. Hence there exists an equivariant piecewise-smooth homeomorphism $\mathrm{Tor}_\mathbb{C}(\Delta) \to \mathbb{C}P^2$ which takes the pencil of the above spheres to the pencil of straight lines through z_0.

4.3. Horizontal refinement of a C-curve and pseudo-holomorphic structure

Let Δ be a polygon as in section 4.2, $X = \mathrm{Tor}_\mathbb{C}(\Delta)$, and $\mathcal{T} :\ \Delta = \Delta_1 \cup \ldots \cup \Delta_N$ a subdivision of Δ into integer polygons. Choose a collection $\{A_{i,j}\ :\ (i,j) \in \Delta \cap \mathbb{Z}^2\}$ of real numbers such that $A_{i,j} \neq 0$ if (i,j) is a vertex of one of the polygons $\Delta_1, \ldots, \Delta_N$, and the polynomials

$$F_k(x,y) = \sum_{(i,j)\in\Delta_k\cap\mathbb{Z}^2} A_{i,j} x^i y^j, \quad k = 1, ..., N,$$

are non-degenerate on the boundary. This data defines a (may be, singular) C-curve M in the surface X.

Consider the horizontal refinement $\mathcal{T}^{ref}$ of $\mathcal{T}$. We define a new surface $M^{ref} \subset X$ as follows. For any $k = 1, ..., N$, we take the chart $\mathbb{C}Ch_{\Delta_k^{ref}}(F_k)$, and for any non-vertical edge σ of Δ_k, $k = 1, ..., N$, and any parallelogram $\delta \in \mathcal{T}^{ref}$ with sides parallel and close to σ, we take the chart $\mathbb{C}Ch_\delta(F_k^\sigma)$. At last, we define

$$M_{ref} = \nu_\Delta \left(\bigcup_{k=1}^{N} \mathbb{C}Ch_{\Delta_k^{ref}}(F_k) \cup \bigcup_{\delta} \mathbb{C}Ch_\delta(F_k^\delta) \right) \subset X \ .$$

Denote by $\Pi(X)$ the family of (the closures of) the fibers of the fibration on X, defined by the subdivision $\mathcal{T}_{ref}$ as in Section 4.2.

Lemma 4.4. *(1) The set M_{ref} is a surface with finitely many singular points in X and is equivariantly isotopic to M.*

(2) There is a finite set $K \subset \Pi(X)$ such that, any fiber $\Pi_p \in \Pi(X) \backslash K$ intersects with M_{ref} exactly at d points, where d is the length of the projection of Δ on the horizontal coordinate axis. Moreover, all these intersection points are transverse and positive with respect to the naturally induced orientations of Π_p and M_{ref}.

(3) If Δ is triangular, then M_{ref} does not pass through z_0. If Δ is quadrangular, then M_{ref} intersects with E exactly at l points, where l is the length of σ_2. Moreover, all these intersections are transverse and positive with respect to the natural orientations of E and M_{ref}.

Proof. (1) By construction, M_{ref} is the union of surfaces with boundary. The surfaces which correspond to adjacent polygons of the subdivision $\mathcal{T}^{ref}$ are glued along their boundary components. Singular points of M_{ref} correspond to singular points of F_k, $k = 1, \dots, N$.

The construction of the refinement $\mathcal{T}^{ref}$ depends on the small parameter $\varepsilon > 0$ in the following way. If ε tends to 0, then

- each polygon Δ_k^{ref} tends to Δ_k, $k = 1, ..., N$, so that, for any vertex $v \in \Delta_k$ the pair of the vertices of Δ_k^{ref} in a small neighborhood of v tends to v;
- each parallelogram σ with a pair of edges parallel and close to a non-vertical edge e of the subdivision $\mathcal{T}$ tends to e so that the vertices of σ tend pairwise to the vertices of e.

Hence, for any $k = 1, ..., N$, the moment map $\mathbb{C}\mu_{\Delta_k^{ref}} : (\mathbb{C}^*)^2 \to \mathbb{C}\Delta_k^{ref} \hookrightarrow \mathbb{C}^2$ uniformly converges to $\mathbb{C}\mu_{\Delta_k} : (\mathbb{C}^*)^2 \to \mathbb{C}\Delta_k \hookrightarrow \mathbb{C}^2$. Furthermore, for any non-vertical edge e of Δ_k, $1 \le k \le N$, and a parallelogram σ with a pair of edges s_1, s_2 parallel and close to e, the moment maps $\mathbb{C}\mu_{s_i} : (\mathbb{C}^*)^2 \to \mathbb{C}s_i \hookrightarrow \mathbb{C}^2$, $i = 1, 2$, uniformly converge to $\mathbb{C}\mu_e : (\mathbb{C}^*)^2 \to \mathbb{C}e \hookrightarrow \mathbb{C}^2$, and the chart $\mathbb{C}Ch_{s_i}(F_k^e)$ is the product of $\mathbb{C}Ch_e(F_k^e)$ with a segment, which contracts to a point as $\varepsilon \to 0$. This altogether gives an isotopy of M_{ref} and M in Σ_m.

(2) Denote by K the finite subset of $\Pi(X)$ corresponding to

- the projections of the vertices of the subdivision $\mathcal{T}^{ref}$ in the side σ_1,
- the projections of the boundary components in $\mathbb{C}\sigma_1$ of the charts (with respect to the subdivision $\mathcal{T}^{ref}$) of all non-generic lines.

The following observations immediately imply the second statement of Lemma. Any generic line $L \in \Lambda$ intersects with the curve $\{F_k = 0\} \subset (\mathbb{C}^*)^2$ at $l_h(\Delta_k)$ points, where l_h denotes the length of the projection of Δ_k on the horizontal coordinate axis, and all the intersections are transversal. Similarly, for any non-vertical edge $e \subset \Delta_k$, any line $L \in \Lambda$ intersects with the curve $\{F_k^e = 0\}$ at $l_h(e)$ points, and all the intersections are transversal.

(3) The last statement of Lemma immediately follows from the construction. □

Corollary 4.5. *There exists a complex structure on X and an equivariant isotopy of X, preserving the lines of $\Pi(X)$, which deforms M into a real pseudo-holomorphic curve.*

Proof. First, we deform M into M_{ref}, then choose on X a complex structure such that the fibers of $\Pi(X)$ are complex straight lines. Then we transform the complex structure on X and M_{ref} as follows.

If $X = \mathbb{C}P^2$, let U, V, $\overline{V} \subset U$ be small open tubular neighborhoods of some straight line $L \in \Pi(X)$. If $X = \Sigma_m$, let U, V, $\overline{V} \subset U$ be small open tubular neighborhoods of the union of E and one of the lines $L \in \Pi(X)$. Since the intersections of M_{ref} with L and E are transverse and positive, one can equivariantly deform the complex structure of X in U, preserving the complex lines $L \in \Pi(X)$, so that $M_{ref} \cap V$ will become holomorphic.

In neighborhoods of singular points of M_{ref} we deform the complex structure of X into that defined by the corresponding (extended) moment maps.

Notice that in the (finitely many) non-singular points of M_{ref}, where the intersection with lines of $\Pi(X)$ is not transverse, the surface M_{ref} has corners. Furthermore, these intersections are of multiplicity 2. We smooth up M_{ref} in neighborhoods of these points and deform locally the complex structure of X to make M_{ref} holomorphic in these neighborhoods.

In $X \backslash (L \cup E) \simeq \mathbb{C}^2$, the surface M_{ref} is represented as a graph of a multivalued function $f = f(z, \overline{z}) : \mathbb{C} \to \mathbb{C}$ with finitely many poles, ramification and singular points (cf. [13], section 5.3, [4], section 4.1). Then the argument of [4], section 4.1, shows how to construct a tame almost complex structure in X in which M_{ref} becomes pseudo-holomorphic. Indeed, the key observation in this argument is the uniform boundedness of $|\partial f/\partial \overline{z}|$, which immediately follows from the fact that, by construction, M_{ref} is holomorphic in a neighborhood of infinity and in neighborhoods of the singular and ramification points. □

Remark 4.1. Corollary 4.5 remains true if we remove the condition $\Delta \in (\mathbb{R}_+^*)^2$ in the description of Δ given in Section 4.2 (i.e., if $\Delta \in (\mathbb{R}_+)^2$).

5. Example

We construct an infinite series of singular pseudo-holomorphic curves in the projective plane with collections of singular points which do not occur on known algebraic curves of the same degree.

Proposition 5.1. *For any integer $d \geq 3$ and any positive integer k such that $k \leq N(d) = [\frac{d^2-2d-3}{5}]$, there exists an irreducible singular real pseudo-holomorphic curve of degree d in $\mathbb{C}P^2$ with k real singular points of type A_3 as only singularities.*

Remark 5.1. (1) It is not known if there exist irreducible algebraic curves of degree d in the plane with $N(d)$ singularities A_3, but we would like to point out that $N(d)$ singular points of type A_3 on an algebraic curve of degree d would be dependent since each singular point of type A_3 imposes 3 conditions and the dimension of the space of curves of degree d is $d(d+3)/2$.

(2) The Hirano construction [9] produces a series of real algebraic curves of degrees $d = 2^{2r+1}$ with $4(2^{4r}-1)/5 = (d^2-4)/5$ singular points of type A_3. This is even greater than in our examples. However, these curves are reducible (at least four components), and almost all their singular points are imaginary. One can smooth out some of these singularities and obtain irreducible symplectic curves of these special degrees with $d^2/5+O(d)$ singular points of type A_3.

Proof. Given $d \geq 3$, we take a subdivision of the triangle T_d with vertices $(0,0)$, $(d,0)$, $(0,d)$ into convex lattice polygons $\Delta_1, \ldots, \Delta_N$ and take polynomials $F_1, \ldots, F_N \in \mathbb{R}[x,y]$ with Newton polygons $\Delta, ..., \Delta_N$, respectively, such that

- each polynomial F_k, $k = 1, \ldots, N$ is non-degenerate on the boundary,
- $F_i^\sigma = F_j^\sigma$ for any edge $\sigma = \Delta_i \cap \Delta_j$, $i \neq j$,
- each curve $\{F_k = 0\}$, $k = 1, \ldots, N$ is either non-singular in $(\mathbb{C}^*)^2$, or has one singular point of type A_3 in $(\mathbb{R}^*)^2$.

This data defines a singular real C-curve with singular points of type A_3, which is pseudo-holomorphic according to Corollary 4.5.

First, we naturally cover the plane by parallel translates of the polygons δ_1, δ_2 shown in Figure 6(a,b), then take the part of this tilling consisting of polygons lying entirely inside T_d, and, finally, complete the subdivision of T_d by complementary convex polygons (see Figure 6(c)).

Observe that the polynomial

$$F(x,y) = (4x - y^2)(x + 1 - y) = 4x^2 + 4x - 4xy - xy^2 - y^2 + y^3$$

has Newton polygon δ_1, is non-degenerate on the boundary, and defines a curve $\{F = 0\}$ with one singular point of type A_3 in $(\mathbb{R}^*)^2$ and without other singular points in $(\mathbb{C}^*)^2$. Respectively, the polynomial $G(x,y) = x^2y^3F(x^{-1}, y^{-1})$ has Newton polygon δ_2, is non-degenerate on the boundary, and defines a curve $\{G = 0\}$ with one singular point of type A_3 in $(\mathbb{R}^*)^2$ and without other singular points in $(\mathbb{C}^*)^2$.

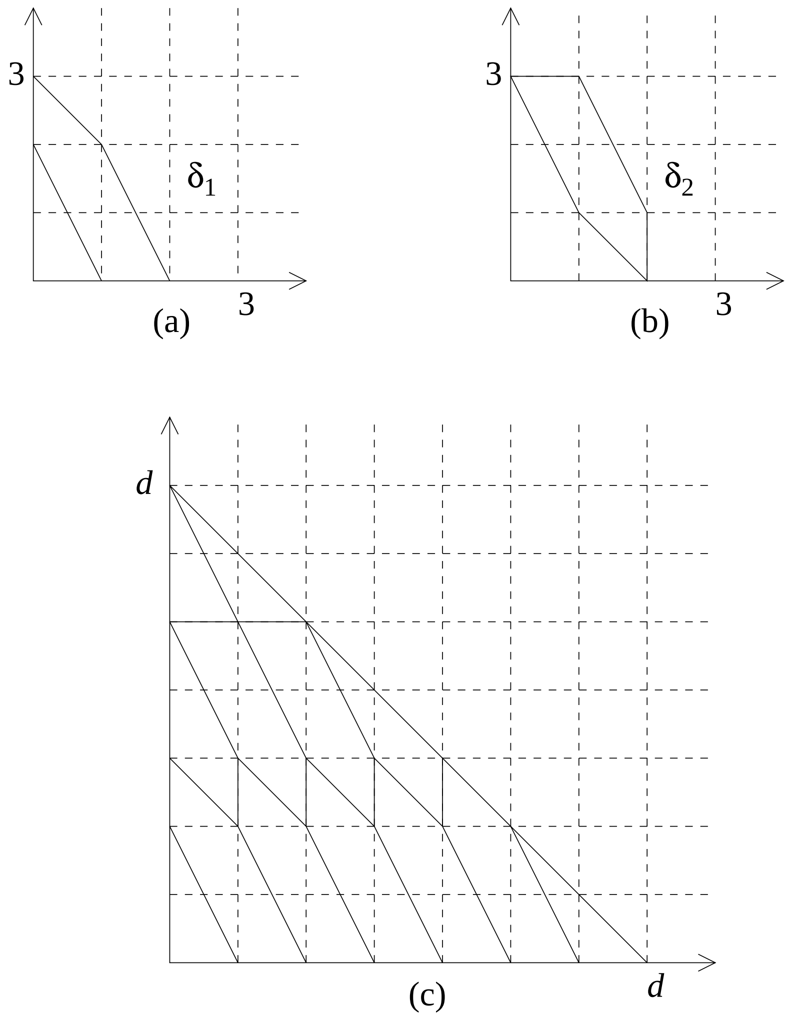

Figure 6. Construction of singular C-curve with many tacnodes

If Δ_k is a parallel translate of δ_1, then put $F_k(x,y) = (-1)^q x^p y^{5q} F(x,y)$ choosing p and q in such a way that Δ_k would be the Newton polygon of F_k. Similarly, if Δ_k is a parallel translate of δ_2, then put $F_k(x,y) = (-1)^{q+1} x^p y^{2+5q} G(x,y)$ choosing p and q in such a way that Δ_k would be the Newton polygon of F_k. Finally, consider the triangles and quadrangles of the subdivision one by one, and for each of them choose a non-degenerate polynomial which has given triangle or quadrangle as Newton polygon and which is coherent with all already chosen polynomials (including the polynomials whose Newton polygons are parallel translates of δ_1 and δ_2). One can easily check that $F_i^\sigma = F_j^\sigma$ for any edge $\sigma = \Delta_i \cap \Delta_j$, $i \neq j$.

Notice that the singular C-curve obtained has two (complex) components. Indeed, the curve $\{F = 0\}$ (and similarly, $\{G = 0\}$) consists of two components, a line and a conic. The chart of the line intersects with the complexification of the edges $[(0,2),(0,3)]$, $[(0,3),(1,2)]$, and $[(1,0),(2,0)]$, and the chart of the conic intersects with the complexification of the edges $[(0,2),(1,0)]$, $[(1,2),(2,0)]$. Since the curves corresponding to the polygons of the subdivision, which are not translates of δ_1, δ_2, are nonsingular in $(\mathbb{C}^*)^2$, we conclude that one component of our singular C-curve contains the charts of all the lines and all the conics, except for the conics corresponding to the translates of δ_1 along the horizontal axis. To make the singular C-curve constructed irreducible we replace the polynomial, corresponding to δ_1 as an element of the subdivision, by $F'(x,y) = F(x,y) + \lambda xy$ with generic λ which defines an irreducible curve.

Finally, note that the number of singular points of type A_3 on our singular C-curve is $N(d)$. In a similar way one can construct for any positive integer $k \leq N(d)$ an irreducible singular pseudo-holomorphic curve of degree d in $\mathbb{C}P^n$ with k real points of type A_3 as only singularities. $\square$

Remark 5.2. The subdivision of T_d used in the proof of Proposition 5.1 is convex. However, we do not automatically get an algebraic curve which is isotopic to the curve constructed, since the version of the Viro theorem adapted to construction of singular algebraic curves (see [21]) requires the independence of conditions imposed by all singular points.

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