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Gauge theory and Stein fillings of certain 3-manifolds

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Abstract

In the following we show that a Stein filling S of the 3-torus T^3 is homeomorphic to $D^2 \times T^2$. In the proof we also show that if S is Stein and ∂S is diffeomorphic to the Seifert fibered 3-manifold $-\Sigma(2,3,11)$ then $b_1(S) = 0$ and $Q_S = H$. Similar results are obtained for the Poincaré homology sphere $\pm\Sigma(2,3,5)$; in studying these fillings we apply recent gauge theoretic results, and prove our theorems by determining certain Seiberg-Witten invariants.

1. Introduction

Suppose that M is a closed, oriented 3-manifold and ξ is a 2-plane field on M. This ξ is called a *contact structure* if it is completely nonintegrable, i.e., for the 1-form (locally) defining ξ as $\ker \alpha$, the expression $\alpha \wedge d\alpha$ is nowhere 0. (For more about contact manifolds see [1].) A 3-manifold M in a Kähler surface X inherits a natural contact structure provided that M is *convex*, that is, there exists a vector field v on X transverse to M such that near M $\mathcal{L}_v\omega = \omega$ for the Kähler form ω ($\mathcal{L}$ stands for the Lie derivative). In this case the complex lines in TM form a 2-plane field ξ satisfying the definition of a contact structure. For example, if X admits a proper biholomorphic embedding into $\mathbb{C}^n$ for some n (that is, X is a *Stein surface*) then for the distance function $f = || \, . \, - p||^2 \colon X \to [0, \infty)$ (for $p \in \mathbb{C}^n$ generic) the submanifolds $f^{-1}(t)$ will be convex, hence contact away from the critical points. In fact, this property characterizes Stein surfaces:

Theorem 1.1 ([19])**.** *The (noncompact) complex surface X is Stein if and only if there is a proper Morse function $f \colon X \to [0, \infty)$ such that away from the critical points the submanifolds $M_t = f^{-1}(t) \subset X$, with the plane fields induced by the complex tangent lines of X in TM_t, are contact 3-manifolds.* □

Suppose that $t \in \mathbb{R}$ is a regular value of the above Morse function $f \colon X \to [0, \infty)$. The manifold (with boundary) $S = f^{-1}[0, t] \subset X$ is called a *Stein domain*, and it can be regarded as the compact version of Stein surfaces. (For more about Stein surfaces and Stein domains see [17, 18].)

Definition 1.2. The contact manifold (M, ξ) is *Stein fillable* if there is a Stein domain S such that (M, ξ) is contactomorphic to ∂S (with the induced contact structure on it). In this case S is called a *Stein filling* of (M, ξ).

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Remark 1.3. *We always assume that M is oriented and ξ respects this orientation through the requirement that $\alpha \wedge d\alpha > 0$ for any 1-form α defining ξ. Since S has a natural orientation (as a complex surface), it induces an orientation on ∂S. We require that the above contactomorphism is orientation preserving.*

It is expected that the knowledge of all contact structures on M will tell us something about its geometry. To achieve this goal it seems reasonable to study all Stein fillings of a given 3-manifold. On the other hand, Stein domains can be regarded as analogues of minimal complex surfaces of general type in the category of manifolds with boundary. Therefore the study of Stein domains is interesting from the 4-dimensional point of view as well. The *geography problem* for surfaces of general type asks the possible values of b_1, c_1^2 and c_2 of such manifolds. Extending this problem we get:

Problem 1.4 (The geography problem for Stein domains)**.** Fix a contact 3-manifold (M, ξ) and describe characteristic numbers of Stein fillings of it.

In this paper we will address the problem of describing Stein domains with the 3-torus T^3, $-\Sigma(2,3,11)$ and $\pm\Sigma(2,3,5)$ as contact boundary. (For a possible definition of these Seifert fibered manifolds see Figure 2.) The problem of Stein fillability (and more generally, symplectic fillability) of contact 3-manifolds has been extensively studied recently, see for example [4, 6, 22, 23, 24, 26]. We only mention a prototype result here:

Theorem 1.5 ([6])**.** *If W is a Stein domain with $\partial W = S^3$ the 3-dimensional sphere then W is diffeomorphic to the 4-dimensional disk D^4.* □

In the following we will prove a similar (but substantially weaker) statement for the 3-torus T^3 and for the Seifert fibered 3-manifolds $-\Sigma(2,3,11)$ and $\pm\Sigma(2,3,5)$. Our main result determines topological properties of Stein fillings of the 3-torus T^3.

Theorem 1.6. *If S is a Stein filling of T^3 then S is homeomorphic to $D^2 \times T^2$.*

The proof of Theorem 1.6 rests on the following result. (The intersection form of a 4-manifold X will be denoted by Q_X.)

Theorem 1.7. *If S is a Stein filling of $-\Sigma(2,3,11)$ for some contact structure on it, then $b_1(S) = 0$ and $Q_S = H$. Moreover, there is a Stein domain S with $b_1(S) = 0$, $Q_S = H$ and $\partial S = -\Sigma(2,3,11)$.*

Here, as usual, H denotes the hyperbolic form $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Below E_8 stands for the symmetric bilinear form defined by the negative definite Cartan matrix of the exceptional Lie algebra E_8. For the fixed orientation of the 3-manifolds $\Sigma(2,3,5)$ and $\Sigma(2,3,11)$ see text following Theorem 2.3. Similar arguments as applied in the above theorem show the following — these statements were already known [3, 22, 31].

Theorem 1.8. *If S is a Stein filling of $\Sigma(2,3,5)$ then S is homeomorphic to the E_8-plumbing E given by Figure 1. The 3-manifold $-\Sigma(2,3,5)$ admits no Stein filling.*

In the proof of Theorem 1.8 we will make use of the knowledge of topological properties of Stein fillings of S^3 and $\mathbb{RP}^3$. For the sake of completeness we include a proof of

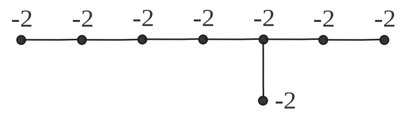

Figure 1. The negative definite E_8-plumbing

Proposition 1.9. *(a) If S is a Stein filling of the 3-sphere S^3 then it is homeomorphic to D^4.*
(b) If S is a Stein filling of $\mathbb{RP}^3$ then S is homeomorphic to the unit disk bundle of the cotangent bundle of the 2-sphere S^2.

In computing the first Betti numbers of various Stein fillings we will verify the following more general statement:

Proposition 1.10. *If S is a Stein filling of a contact 3-manifold (M, ξ) then the homomorphism $i_*\colon \pi_1(M) \to \pi_1(S)$ induced by the inclusion $i\colon M \to S$ is a surjection. Consequently $i_*\colon H_1(M;\mathbb{Z}) \to H_1(S;\mathbb{Z})$ is onto, hence $b_1(S) \leq b_1(M)$.*

In proving Theorem 1.7 we will use recent results in gauge theory. The relevant theorems and constructions will be summarized in Section 2. Section 3 deals with fillings of $-\Sigma(2,3,11)$ while Section 4 contains the proof of our main result Theorem 1.6. In the final section we prove Theorem 1.8.

2. Gauge theoretic background

We will frequently invoke the following celebrated result of Donaldson:

Theorem 2.1 ([5])**.** *If X is a smooth, closed 4-manifold with negative definite intersection form, then Q_X is standard, that is, isomorphic to $\oplus_1^{b_2(X)}\langle -1\rangle$. If X is a smooth, simply connected spin 4-manifold with $b_2^+(X) = 1$ then Q_X is isomorphic to H.* □

Remark 2.2. Using the monopole equations rather the instantons Donaldson originally used in his proof, Furuta [14] extended Theorem 2.1 by showing that if a smooth spin 4-manifold X has $Q_X = 2kE_8 \oplus lH$ then $l \geq 2|k| + 1$.

At one point we will appeal to the following famous result of Rohlin:

Theorem 2.3 ([34])**.** *If X is a smooth spin 4-manifold then the signature $\sigma(X)$ of X is divisible by 16.* □

The 3-manifolds $\Sigma(2,3,5)$ and $\Sigma(2,3,11)$ are defined as oriented boundaries of the complex manifolds $M_c(2,3,5) = \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = \varepsilon,\ |x|^2 + |y|^2 + |z|^2 \leq 1\}$ and $M_c(2,3,11) = \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^{11} = \varepsilon,\ |x|^2 + |y|^2 + |z|^2 \leq 1\}$ (with $|\varepsilon| \ll 1$), i.e., as the boundaries of the corresponding (compactified) Milnor fibers. Alternatively, $-\Sigma(2,3,5)$ and $-\Sigma(2,3,11)$ are the oriented boundaries of the the nuclei $N_1 \subset E(1)$ and $N_2 \subset E(2)$, where N_1 and N_2 are given by the Kirby diagrams of Figure 2.

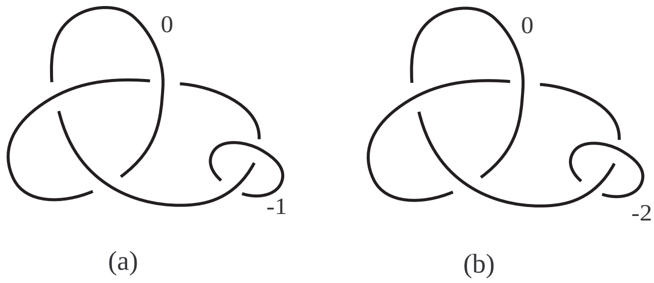

Figure 2. Kirby diagrams for (a) N_1 and (b) N_2

In our subsequent arguments we will apply product formulae for Seiberg-Witten invariants when we cut 4-manifolds along $\Sigma(2,3,5)$ or $\Sigma(2,3,11)$. For the sake of completeness we sketch the definition of Seiberg-Witten invariants and analyze the 3-dimensional equations for $\Sigma(2,3,5)$ and $\Sigma(2,3,11)$ more carefully.

Suppose that X is a smooth, closed, oriented 4-manifold with no 2-torsion in $H_1(X;\mathbb{Z})$. A characteristic element K in $H^2(X;\mathbb{Z})$ uniquely determines a spin^c structure on X, and once a connection A on the line bundle L with $c_1(L)=K$ is fixed, a metric on X gives rise to a twisted Dirac operator $\partial\!\!\!/_A$. The space of connections on the complex line bundle L will be denoted by $\mathcal{A}_L$, the curvature of $A\in\mathcal{A}_L$ is F_A, and F_A^+ stands for its self-dual part. Let W^+ denote the positive spinors corresponding to the spin^c structure defined by K. Then the Seiberg-Witten equations for a pair $(A,\psi)\in\mathcal{A}_L\times\Gamma(W^+)$ read as

$$(*) \qquad \partial\!\!\!/_A\psi = 0 \quad \text{and} \quad F_A^+ = iq(\psi).$$

(Here $q\colon\Gamma(W^+)\to\Gamma(\Lambda^+)$ is a certain quadratic map.) If $K^2-(3\sigma(X)+\chi(X))=0$, the solution space $\mathcal{M}_K$ of the equations (mod symmetries of the equations) is a compact 0-dimensional manifold. An orientation of $H^2_+(X;\mathbb{R})\otimes H^1(X;\mathbb{Z})$ fixes an orientation on this solution space, and provided $b_2^+(X)>1$, the algebraic sum of the points of $\mathcal{M}_K$ turns out to be a smooth invariant of X, denoted by $SW_X(K)\in\mathbb{Z}$. Similar, but somewhat more complicated procedure provides $SW_X(K)\in\mathbb{Z}$ for K with $K^2-(3\sigma(X)+2\chi(X))>0$. (If $K^2-(3\sigma(X)+2\chi(X))<0$ or K is not characteristic, then $SW_X(K)=0$ by definition.) The class K is called a *basic class* of X if $SW_X(K)\neq 0$. It can be shown that $SW_X(K)=(-1)^{\frac{\sigma(X)+\chi(X)}{4}}SW_X(-K)$, therefore K and $-K$ are basic classes at the same time. We say that X is of *simple type* if for a basic class K of X the equation $K^2=3\sigma(X)+2\chi(X)$ holds. Following ideas of Fintushel and Stern [11] we can associate a formal series to any 4-manifold X with $b_2^+(X)>1$: If $\{\pm K_1,...,\pm K_n\}$ are the nonzero basic classes of X, then take

$$\mathcal{SW}(X) = SW_X(0) + \sum_{i=1}^{n} SW_X(K_i)\exp(K_i) + SW_X(-K_i)\exp(-K_i).$$

It has been proved [39] that if X is a minimal surface of general type then it has two basic classes $\pm c_1(X)$, and $SW_X(\pm c_1(X)) = \pm 1$. Therefore in that case $\mathcal{SW}(X) = \pm(\exp(c_1(X)) + (-1)^{\frac{\sigma(X)+\chi(X)}{4}} \exp(-c_1(X))$. (For other complex surfaces we only know that $\pm c_1(X)$ are basic classes.) It is also known that the K3-surface Y has a single basic class which is $c_1(Y) = 0$, hence $\mathcal{SW}(Y) = 1$. For a more thorough study of Seiberg-Witten theory, see [18, 28, 38].

In a similar vein the 3-dimensional analogue of Seiberg-Witten equations can be defined. For a 3-manifold M the spin^c structures are parametrized by $H^2(M;\mathbb{Z})$ and if $W \to M$ denotes the spinor bundle then the Seiberg-Witten equations for $(A,\psi) \in \mathcal{A}_{\det W} \times \Gamma(W)$ are

$$(**) \qquad \partial\!\!\!/_A \psi = 0 \quad \text{and} \quad *F_A = iq(\psi).$$

(As usual, F_A denotes the curvature 2-form of the connection $A \in \mathcal{A}_{\det W}$, and $*$ stands for the Hodge $*$-operator given by a metric on M. Now q maps from $\Gamma(W)$ to $\Gamma(\Lambda^1 M)$.) These equations have been solved for $\Sigma(2,3,11)$ in [30]. Notice that since $\Sigma(2,3,11)$ is an integral homology sphere, it admits a unique spin^c structure. After substituting the Levi-Civita connection with a suitable connection in the definition of $\partial\!\!\!/_A$, in [30] it was shown that $(**)$ admits 3 solutions (up to gauge equivalence): one of them is the trivial solution θ, which is the trivial connection with vanishing spinor field; the other two will be denoted by α and $\overline{\alpha}$.

Remarks 2.4.
- Such a perturbation of the Seiberg-Witten equations over a three-manifold can be naturally extended to give a perturbation of the Seiberg-Witten moduli space over 4-manifolds containing long necks. It is proved [32] that this perturbation over a smooth closed 4-manifold with $b_2^+ > 1$ gives a compact moduli space which is smoothly cobordant to the unperturbed Seiberg-Witten moduli space. This implies that such a perturbation can be used to compute the Seiberg-Witten invariants.
- Because of the presence of a positive scalar curvature metric, one can easily show that the Seiberg-Witten equations on $\Sigma(2,3,5)$ admit a unique solution θ, which is the trivial connection with vanishing spinor field.

By finding relations between the L^2 moduli spaces of Seiberg-Witten solutions over a 4-manifold X with boundary diffeomorphic to $\pm\Sigma(2,3,11)$, in [35] relative invariants, relative basic classes and the (formal) series $\mathcal{SW}(X)$ has been defined for a compact, smooth 4-manifold X with boundary diffeomorphic to $\pm\Sigma(2,3,11)$. The relation between absolute and relative invariants is given by

Theorem 2.5 ([35]). *If the closed 4-manifold Z decomposes as $Z = X \cup_{\Sigma(2,3,11)} Y$ with $b_2^+(X), b_2^+(Y) > 0$ then $\mathcal{SW}(Z) = \mathcal{SW}(X) \cdot \mathcal{SW}(Y)$, that is, the product of the relative invariants equals the absolute invariant of the closed 4-manifold Z.* □

There are three more important ingredients of the proofs we will describe in the following sections. The first theorem (due to Lisca and Matić) provides a Kähler embedding of a Stein domain into a minimal surface of general type, more precisely

Theorem 2.6 ([24]). *For a Stein domain S there exists a minimal surface X of general type and a Kähler embedding $f\colon S \to X$. Moreover, we can assume that $X - f(S)$ is not spin and $b_2^+(X - f(S)) > 1$.* □

The next theorem is a special case of a result of Ozsváth and Szabó which describes restrictions on the embedding of certain circle bundles over surfaces. Suppose that $M_{e,1}$ is a circle bundle over the torus with Euler number e.

Theorem 2.7 ([33]). *If the minimal surface X of general type decomposes as $X = X_1 \cup_{M_{e,1}} X_2$ along the 3-manifold $M_{e,1}$ with $|e| \geq 1$ then either $b_2^+(X_1) = 0$ or $b_2^+(X_2) = 0$.* □

Combining Theorems 2.6 and 2.7 we get (see also [3]):

Corollary 2.8 ([3]). *If S is a Stein domain with $\partial S = M_{e,1}$ and $|e| \geq 1$ then $b_2^+(S) = 0$.* □

Finally we invoke a result of Morgan and Szabó which characterizes homotopy K3-surfaces:

Theorem 2.9 ([29]). *Suppose that X is a simply connected spin 4-manifold of simple type with $Q_X = 2kE_8 \oplus lH$ and $SW_X(0) = \pm 1$. Then $Q_X = 2E_8 \oplus 3H$.* □

(Notice that since X is of simple type and 0 is a basic class, it follows that $l = 4k - 1$. The theorem of Morgan and Szabó proves that $SW_X(0)$ is even once $k > 1$.)

3. Fillings of $-\Sigma(2,3,11)$

We begin our study of Betti numbers of Stein fillings by proving an estimate on b_1.

Proof of Proposition 1.10. It is well-known that a Stein domain S can be built up using 0-, 1- and 2-handles only (cf. [27]); for such manifolds the surjectivity of $\pi_1(\partial S) \to \pi_1(S)$ is obvious. This surjection now trivially implies that $H_1(\partial S; \mathbb{Z}) \to H_1(S; \mathbb{Z})$ is also a surjection, hence $b_1(S) \leq b_1(\partial S)$. □

Remark 3.1. The same surjectivity can be seen from another perspective: As S is a Stein domain, it admits a Lefschetz fibration structure [2, 25]. For such a structure it is well-known that the fiber carries the fundamental group; choosing the fiber in the boundary we get that $\pi_1(\partial S) \to \pi_1(S)$ is onto.

Since $\pm\Sigma(2,3,5)$ and $-\Sigma(2,3,11)$ are integral homology spheres, the above theorem shows that Stein fillings of these Seifert fibered 3-manifolds have trivial first homology, hence vanishing first Betti number.

Next we consider intersection forms of fillings of $-\Sigma(2,3,11)$.

Proposition 3.2. *If X is a smooth 4-manifold with $\partial X = -\Sigma(2,3,11)$ then $b_2^+(X) > 0$.*

Proof. It is a standard fact that the K3-surface Y contains three disjoint copies of the nucleus N_2 (recall that $\partial N_2 = -\Sigma(2,3,11)$); and the intersection form of the manifold $Y - 3 \text{ int } N_2$ is negative definite and nonstandard. (To see this, decompose Y as the fiber sum of two rational elliptic surfaces and analyze the (-2)-spheres and their dual tori created via the gluing process, cf. [18].) So if X is negative definite with $\partial X = -\Sigma(2,3,11)$ then the 4-manifold $(Y - 3N_2) \cup 3X$ we get by replacing the nuclei in Y by X is a smooth manifold with nonstandard negative definite intersection form. The existence of such a manifold, however, contradicts Donaldson's Theorem 2.1, showing that X is not negative definite. □

Proof of Theorem 1.7. Let S be a Stein filling of $-\Sigma(2,3,11)$ and consider the Kähler embedding $S \to X$ where X is a minimal surface of general type — the existence of such an embedding is guaranteed by Theorem 2.6. Recall that we can assume that $X - S$ is nonspin with $b_2^+(X - S) > 1$. The product formula of Theorem 2.5 shows that $\mathcal{SW}(S) = \pm 1$: We use the fact that X, as a minimal surface of general type, has only two basic classes $\pm c_1(X)$; therefore $\mathcal{SW}(X)$ is nondivisible, but since $X - S$ is nonspin, 0 is not characteristic and so $\mathcal{SW}(X - S) \neq 1$. Notice that this computation shows that $c_1(S) = 0$ is the unique basic class for S, in particular, S is spin. (The product formula of Theorem 2.5 applies since $b_2^+(S) > 0$ by Proposition 3.2 and $b_2^+(X - S) > 0$ by Theorem 2.6.) Now consider $Z = (Y - N_2) \cup S$ — as before, Y stands for the K3-surface. The product formula $\mathcal{SW}(Z) = \mathcal{SW}(Y - N_2) \cdot \mathcal{SW}(S)$ shows that $\mathcal{SW}(Z) = \pm 1$. (We used the fact that $\mathcal{SW}(Y) = \pm 1$, hence $\mathcal{SW}(Y) = \mathcal{SW}(Y - N_2) \cdot \mathcal{SW}(N_2) = \pm 1$ implies $\mathcal{SW}(Y - N_2) = \pm 1$ and also $\mathcal{SW}(N_2) = \pm 1$.) Easy handle calculus verifies that Z is simply connected: $Y - N_2$ is simply connected and we can build Z on the top of it by adding only 2-, 3- and 4-handles, since S is Stein. In conclusion, for the simply connected spin manifold Z we have that $SW_Z(0) = \pm 1$; applying Theorem 2.9 of Morgan and Szabó this fact implies that $Q_Z = 2E_8 \oplus 3H$, and since $Q_{Y-N_2} = 2E_8 \oplus 2H$ we get that $Q_S \cong Q_{N_2} = H$. □

Remarks 3.3. 1. Figure 3 demonstrates that, in fact, N_2 carries a Stein structure. (For handle calculus of Stein domains, see [17, 18].) This provides a Stein filling of $-\Sigma(2,3,11)$ as stated. (The framings with which the 2-handles along the curves of Figure 3 are glued, are $tb - 1$, where tb is the Thurston-Bennequin invariant of the Legendrian knot. It can be easily read off from the projection as the difference of the writhe and the number of left cusps, see for example [17, 18]. In Figure 3, the Thurston-Bennequin invariants tb are: 1 for the upper trefoil and -1 for the lower unknot.)

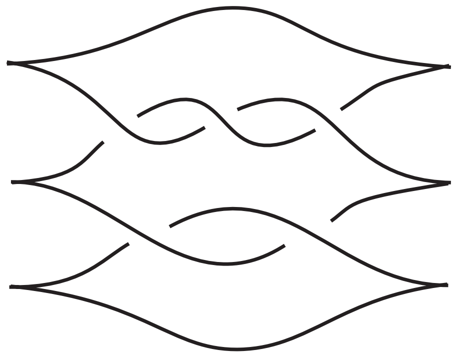

Figure 3. Stein structure on N_2

2. It is known [15, 20] that $-\Sigma(2,3,11)$ carries a unique Stein fillable (in fact, a unique tight) contact structure.
3. It seems natural to conjecture that the Stein filling S of $-\Sigma(2,3,11)$ is diffeomorphic to N_2, although the techniques applied in the above proof seem to be weak to verify such a conjecture.

4. Stein fillings of T^3

Using a famous result of Eliashberg [8] together with the classification result due to Giroux and Kanda (given below), now we can prove our result regarding Stein fillings of the 3-torus T^3. Our proof of Theorem 1.6 will rely on Theorem 1.7.

Theorem 4.1 ([8, 16, 21])**.** *The contact structures* $\xi_n = \ker(\cos(2\pi nt)dx - \sin(2\pi nt)dy)$ *on* T^3 *(in coordinates* (x,y,t) *on* T^3*,* $n = 1,\dots$*) are all noncontactomorphic and comprise a complete list of tight contact structures on the 3-torus* T^3*. If* (T^3, ξ_n) *admits a Stein filling then* $n = 1$*.* □

The contact structure ξ_1 can be given as the boundary of the Stein domain given by Figure 4(i). (For the relation of Kirby diagrams and Stein structures and for the definition of the Thurston-Bennequin invariant tb of a Legendrian knot see [17, 18].) Before turning to the proof of Theorem 1.6 we need a lemma.

Lemma 4.2. *(a) Gluing a 2-handle along the fine curve a (or b) of Figure 4(ii) with framing* $tb(a) - 1 = -1$ *results a handlebody with boundary* $M_{1,1}$ *of Theorem 2.7.*
(b) Gluing 2-handles along the three fine curves a,b,c of Figure 4(ii) (with framings $-1, -1$ *and* -2*, resp.) results a handlebody with boundary* $-\Sigma(2,3,11)$*.*

Proof. The proof of (a) follows from the fact that by gluing the 2-handle along a, topologically we get the same 4-manifold as pictured by Figure 10. in the appendix of [3]. In that same figure the authors show that the boundary of the resulting handlebody is diffeomorphic to $M_{1,1}$.

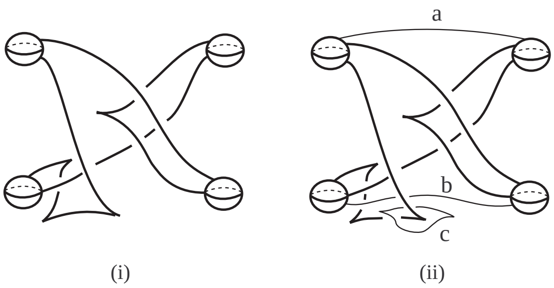

Figure 4. Stein domain with boundary T^3

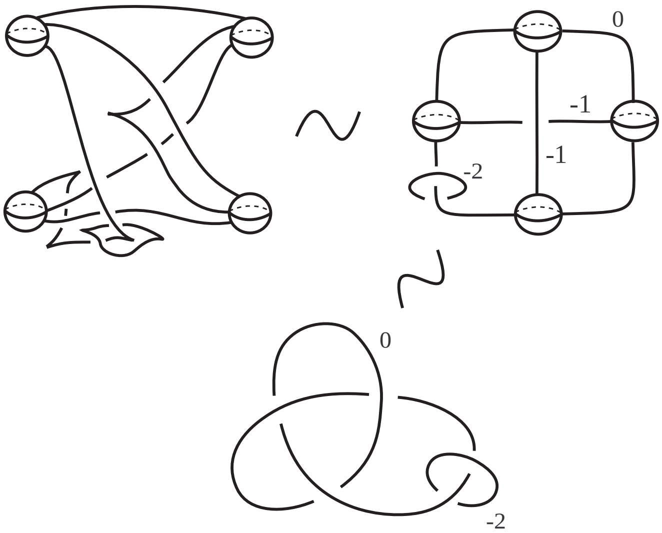

Figure 5. Proof of Lemma 4.2(b) with Kirby diagrams

For the proof of (b), see Figure 5 and recall the Kirby diagram of $-\Sigma(2,3,11)$ as the boundary of N_2 presented by Figure 2(b).

□

The next lemma provides the first step towards proving Theorem 1.6:

Lemma 4.3. *For a Stein filling S of T^3 we have $\chi(S) = \sigma(S) = 0$ and $\pi_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Fix a contactomorphism between ∂S and the boundary of the Stein domain of Figure 4(i). For determining $\chi(S)$ consider $W = S \cup$ three 2-handles attached along

a, b, c with framing $\mathrm{tb} - 1$. The result is a Stein filling of $-\Sigma(2,3,11)$, and according to Theorem 1.7 it has Euler characteristic 3. (Since we glue the 2-handles along Legendrian curves with framing $\mathrm{tb}-1$, the existence of a Stein structure on W follows from by now standard arguments discussed in [17, 18].) Removing the three 2-handles from W we arrive to the conclusion $\chi(S) = 0$. This fact implies that $b_1(S) \geq 1$, since $b_1(S) = 0$ and $\chi(S) = 0$ would imply $b_2(S) = -1$. Therefore S admits unramified covers of any degree. Notice that since $b_1(S) \leq 3$ by Proposition 1.10 (and so $b_2(S) \leq 2$), we get that $|\sigma(S)| \leq 2$. Consider a 3-fold unramified cover of S — the result is a Stein filling $\overline{S}$ of the 3-fold cover of ∂S, which is T^3 again. Therefore all the above said — in particular $|\sigma(\overline{S})| \leq 2$ — holds for $\overline{S}$. Extending the triple cover to $S \cup (D^2 \times T^2)$ we get that $\sigma(\overline{S} \cup D^2 \times T^2) = 3\sigma(S \cup D^2 \times T^2)$. (Notice that this triple cover of $D^2 \times T^2$ is just $D^2 \times T^2$ itself.) By Novikov additivity this implies $\sigma(\overline{S}) = 3\sigma(S)$, showing that $\sigma(S) = 0$.

Finally we show that $\pi_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ for a Stein filling S of T^3. According to Proposition 1.10 the map $H_1(T^3;\mathbb{Z}) \to H_1(S;\mathbb{Z})$ is onto — we claim that the (image of the) circles a and $b \subset T^3$ of Figure 4(ii) remain essential in $H_1(S;\mathbb{Q})$ while c becomes 0. Since ξ_n can be given as a pull-back of ξ_1 under unramified cover along the t coordinate, if $i_*(c) \neq 0$ in $\pi_1(S) = H_1(S;\mathbb{Z})$ then a corresponding n-fold cover of S provides a Stein filling of T^3 equipped with ξ_n. This contradicts Theorem 4.1 once $n > 1$, hence $c = 0$ in $H_1(S;\mathbb{Z})$. If $a = 0$ in $H_1(S;\mathbb{Q})$ then attaching a 2-handle along a with $\mathrm{tb}(a) - 1$ we get a Stein domain $\tilde{S}$ with $\partial\tilde{S} = M_{1,1}$ of Lemma 2.7 and $b_2^+(\tilde{S}) > 0$, since the surface in S with boundary a together with the core of the handle and the dual torus of a in T^3 give a hyperbolic pair in $\tilde{S}$. This fact contradicts Corollary 2.8, therefore $a \neq 0$ in $H_1(S;\mathbb{Q})$. The role of b is analogous, hence $b \neq 0$ in $H_1(S;\mathbb{Q})$. It follows now that S admits a CW decomposition with two 1-cells, and so the number of 2-cells is one in this decomposition (since $\chi(S) = 0$). Since $\pi_1(S)$ is Abelian (being a factor of $\pi_1(T^3) \cong \mathbb{Z}^3$), we get that the attaching circle of this 2-cell is homologically trivial, therefore $H_1(S;\mathbb{Z}) \cong \pi_1(S) \cong \mathbb{Z}\oplus\mathbb{Z}$. In particular, $b_1(S) = 2$, implying $b_2(S) = 1$, and so the intersection form Q_S can be easily identified with $\langle 0 \rangle$. ☐

In order to show that the Stein filling S is homeomorphic to $D^2 \times T^2$ we determine further homotopy groups of it.

Proposition 4.4. *If S is a Stein filling of T^3 then $\pi_2(S) = \pi_3(S) = 0$.*

Proof. Being a Stein domain with $\chi(S) = 0$, S admits a handle decomposition with a single 0-handle, k 1-handles and $(k-1)$ 2-handles. Since $H_2(S;\mathbb{Z}) \cong \mathbb{Z}$, we can slide $(k-2)$ of the 2-handles to algebraically cancel $(k-2)$ 1-handles. Since the attaching circles of these 2-handles are nontrivial in homology, easy induction shows that by attaching them we do not change π_2 (and H_2), which is trivial for the 1-skeleton. The last 2-handle will create the relation implying that the cores of the two remaining 1-handles commute in $\pi_1(S)$ — therefore although its attaching circle is homologically trivial in $S_1 = $ 0-handle$\cup$1-handles, homotopically it is nontrivial. This explains the fact that this last 2-handle creates nontrivial element in H_2 but not in π_2. Finally, in general $\pi_1 \cong H_1$ implies that $\pi_2 \to H_2$ is onto and there is a group $\Gamma(\pi_2(W))$ such that $\Gamma(\pi_2(W)) \to$

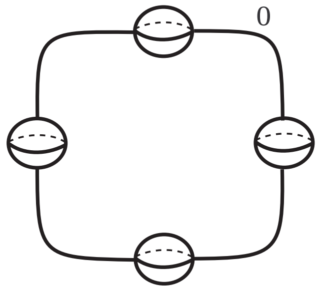

Figure 6. Kirby diagram for $D^2 \times T^2$

$\pi_3(W) \to H_3(W;\mathbb{Z})$ is exact [37]. Since $\pi_2(S) = 0$, it follows that $\Gamma(\pi_2(S)) = 0$, and since S can be decomposed without 3-handles, we get that $H_3(S;\mathbb{Z}) = 0$, implying $\pi_3(S) = 0$. Now the proof is complete. □

The next lemma provides a map between S and $D^2 \times T^2$, which we can modify to a homeomorphism.

Lemma 4.5. *For S as above there exists a strong homotopy equivalence (rel boundary) $f\colon D^2 \times T^2 \to S$.*

Proof. Since the Whitehead group $Wh(\mathbb{Z} \oplus \mathbb{Z})$ is trivial, a homotopy equivalence is automatically a strong homotopy equivalence. Therefore it is enough to find a homotopy equivalence between $D^2 \times T^2$ and S. Let f be chosen to be a contactomorphism $f\colon \partial(D^2 \times T^2) \to \partial S$ mapping the curve $c = \partial D^2 \times \{\text{pt.}\}$ to $f(c)$. Consider the handlebody decomposition of $D^2 \times T^2$ given by Figure 6, and view it upside down, i.e., build $D^2 \times T^2$ on T^3 by adding a 2-handle, two 3-handles and a 4-handle. Since c bounds an immersed disk in W (being trivial in $\pi_1(S)$), the map f can be extended to the 2-handle of $D^2 \times T^2$. According to Proposition 4.4 we have that $\pi_2(S) = \pi_3(S) = 0$, hence this map extends to the 3-handles and the 4-handle, providing a continuous map $f\colon D^2 \times T^2 \to S$. Since the fundamental groups of $D^2 \times T^2$ and S are supported by their boundaries, f induces an isomorphism $f_*\colon \pi_1(D^2 \times T^2) \to \pi_1(S)$. The induced homomorphisms are obviously isomorphisms on π_2 and π_3 (since these groups are trivial), and for the same reason f_* is the isomorphism on $H_i(D^2 \times T^2;\mathbb{Z}) = H_i(S;\mathbb{Z}) = 0$ for $i \geq 3$. Now the relative Hurewitz theorem implies that f induces isomorphisms $f_*\colon \pi_i(D^2 \times T^2) \to \pi_i(S)$ for all $i \geq 1$. This last property of f, however, implies that it is a homotopy equivalence. □

Proof of Theorem 1.6. From the surgery exact sequence [38] now it is easy to see that a strong homotopy equivalence $D^2 \times T^2 \to S$ can be homotoped to a homeomorphism (since, in the terminology of [38] $\mathcal{S}(D^2 \times T^2, \partial(D^2 \times T^2)) = 0$). Therefore the above lemma implies that the Stein filling S is homeomorphic to $D^2 \times T^2$. □

Remark 4.6. Notice that the above proof, in fact, showed that a Stein filling of T^3 embeds into a Stein filling of $-\Sigma(2,3,11)$. Since for this latter 3-manifold all Stein fillings are spin, we conclude that a Stein filling of T^3 is spin. Since $\sigma(S)=0$, it follows that the induced spin structure on ∂S is diffeomorphic to the one $\partial(Y-\nu F)$ inherits from $Y-\nu F$; here Y is the K3-surface and F is a regular fiber in an elliptic fibration on Y. Therefore $Z=S\cup_{T^3}(Y-\nu F)$ is a (simply connected) spin 4-manifold, and using the gauge theoretic results discussed in [36] one can verify that $SW_Z(0)$ is ± 1. Consequently Theorem 2.9 implies that Z is homeomorphic to Y and $\mathcal{SW}_Z=\mathcal{SW}_Y$. Motivated by this identity one can conjecture that the Stein filling S is diffeomorphic to $D^2\times T^2$ — the Stein filling shown by Figure 4(i).

5. Appendix: Stein fillings of $\pm\Sigma(2,3,5)$

Now we turn to the proof of Theorem 1.8. The theorem was already proved by various authors (see Remark 5.6); we include it here because the proof given below is very similar in spirit to the proof of Theorem 1.7. We begin our proof with the following "folk" theorem. (For a complete proof see [22, 31].)

Lemma 5.1. *If S is a Stein filling of a 3-manifold which admits positive scalar curvature then S is negative definite.*

Proof (sketch). Consider the Kähler embedding $S\to X$ where X is a minimal surface of general type with $b_2^+(X-S)>1$. Since a 3-manifold with positive scalar curvature metric cannot divide a 4-manifold with nonzero Seiberg-Witten invariants into two pieces both with $b_2^+>0$, the lemma follows. □

Notice that $\pm\Sigma(2,3,5)$ and $\mathbb{RP}^3$ (as quotients of S^3) admit metrics of positive scalar curvature.

Proposition 5.2. *If S is a Stein filling of S^3, $\mathbb{RP}^3$ or $\Sigma(2,3,5)$ (for some contact structure) then S is negative definite and spin.*

Proof. We give the proof for $\Sigma(2,3,5)$ — the other cases follow the exact same pattern.

The fact that $b_2^+(S)=0$ follows from Lemma 5.1. Recall that the Seiberg-Witten equations admit a unique (up to gauge equivalence) solution on $\Sigma(2,3,5)$. Now consider an embedding $S\to X$ where X is a minimal surface of general type. Grafting solutions for the spin^c structures $\pm c_1(S)$ and $c_1(X-S)$ together we get 4 basic classes unless $c_1(S)=0$ or $c_1(X-S)=0$. Since X has exactly two basic classes and $c_1(X-S)$ is nonspin (therefore $c_1(X-S)\neq 0$) we get that $c_1(S)=0$, consequently S is spin. □

Next we prove Proposition 1.9. The proof uses (by now) standard facts from gauge theory; we will use these facts without reference.

Proof of Proposition 1.9(a). Consider a Stein filling S of S^3. According to the above said, the compact 4-manifold S, and therefore the closed 4-manifold $S\cup D^4$ is negative definite and spin. Therefore Donaldson's Theorem implies that $b_2(S\cup D^4)=b_2(S)=0$. Notice that Proposition 1.10 implies that $\pi_1(S)=1$. Now according to Freedman's Theorem (extended to simply connected 4-manifolds with homology sphere boundary, see [12]), the fact that S is simply connected, $\partial S = S^3$ and $Q_S = \emptyset$ implies that S is homeomorphic to D^4. □

Remark 5.3. It is worth noting that S^3 admits a unique Stein fillable (in fact, unique tight) contact structure, see [9]. Recall that the above Stein filling is proved to be diffeomorphic to D^4, see [6] and Theorem 1.5.

Proof of Proposition 1.9(b). Suppose that S is a Stein domain with $\partial S = \mathbb{RP}^3$. It is easy to see that $\pi_1(S) = 1$, otherwise taking the double cover of S we find a Stein filling of S^3 with even Euler characteristic. (Notice that since $\pi_1(\partial S) \to \pi_1(S)$ is onto, $\pi_1(S)$ is either $\mathbb{Z}_2$ or trivial.) Lemma 5.1 and Proposition 5.2 show that S is negative definite and spin. Now delete the neighborhood νT of a (-2)-sphere T in the K3-surface Y and replace it with S. The gluing map can be chosen so that $(Y - \nu T) \cup S$ is spin. Now a result of Furuta (cf. Remark 2.2) implies that $b_2(S) = 1$, hence $Q_S = \langle -2 \rangle$. Freedman's Theorem then produces the desired homeomorphism between S and νT. □

Remark 5.4. According to a result of McDuff [26] the Stein filling S of $\mathbb{RP}^3$ is actually diffeomorphic to νT.

Proposition 5.5. *If S is a Stein filling of $\Sigma(2,3,5)$ then $\pi_1(S) = 1$.*

Proof. It is well-known that $\pi_1(\Sigma(2,3,5))$ is a perfect group of order 120, and according to Proposition 1.10 the fundamental group $\pi_1(S)$ is the homomorphic image of it. Since all groups of order less than 60 are solvable and there is no onto map from a perfect group to a (nontrivial) solvable group, we get that $\pi_1(S)$ is either isomorphic to $\pi_1(\Sigma(2,3,5))$, it is of order 60 (and then isomorphic to A_5) or $\pi_1(S) = 1$. Next we will exclude the first two possibilities. Consider the universal cover $\tilde{S}$ of S. Since a finite cover of a Stein domain is Stein, we get a Stein filling of the corresponding cover of the boundary ∂S. If $i_* \colon \pi_1(\Sigma(2,3,5)) \cong \pi_1(S)$, then we get a Stein filling of S^3 (the universal cover of $\Sigma(2,3,5)$) with nontrivial second homology, contradicting Proposition 1.9(a). For $\pi_1(S) \cong A_5$ the universal cover $\tilde{S}$ provides a Stein filling of the 60-fold cover of $\Sigma(2,3,5)$, which is $\mathbb{RP}^3$. Since $\tilde{S}$ is a 60-fold (unbranched) cover, we get that $\chi(\tilde{S})$ is divisible by 60. On the other hand, the proof of Proposition 1.9(b) shows that a Stein filling of $\mathbb{RP}^3$ has Euler characteristic 2. This contradiction shows that $\pi_1(S) = 1$, concluding the proof of the proposition. □

Now we are ready to turn to the proof of Theorem 1.8:

Proof of Theorem 1.8. It can be easily verified that the negative definite E_8-plumbing E (see Figure 1) embeds in the K3-surface Y. For a Stein filling S consider $Z = S \cup (Y - E)$. Since S is spin (and $\partial S = \Sigma(2,3,5)$ admits a unique spin structure), Z is spin and $\pi_1(Z) = 1$. Since S is negative definite, we have that $Q_Z = (k+1)E_8 \oplus 3H$. Furuta's Theorem (see Remark 2.2) shows that $k \leq 1$; since Rohlin's famous Theorem 2.3 excludes

$k = 0$, we conclude that $Q_S = E_8$. Freedman's Theorem says that if S is a smooth, spin, simply connected 4-manifold with homology sphere boundary then its intersection form determines it up to homeomorphism. Therefore (using the result of Theorem 5.5) we get that S is homeomorphic to the E_8-plumbing E, which proves the theorem. The part of Theorem 1.8 about $-\Sigma(2,3,5)$ follows from the fact that $-\Sigma(2,3,5)$ does not bound negative definite 4-manifold at all: If $\partial X = -\Sigma(2,3,5)$ and $b_2^+(X) = 0$ then $X \cup_{-\Sigma(2,3,5)} E$ violates Donaldson's Theorem 2.1. (This last argument already appeared in [3].) □

Remarks 5.6.

1. A similar proof of the above theorem was already found by Ohta and Ono [31].
2. In the above proof one can also argue that a negative definite spin 4-manifold with boundary diffeomorphic to $\Sigma(2,3,5)$ has intersection form isomorphic to E_8 along the lines developed by Frøyshov, see [13].
3. According to [20], the 3-manifold $\Sigma(2,3,5)$ admits a unique Stein fillable (in fact, a unique tight) contact structure. Notice that in our proof we made no assumption on the contact structure on $\Sigma(2,3,5)$.
4. The E_8-plumbing E supports a Stein structure providing a Stein filling for $\Sigma(2,3,5)$. It seems natural to expect that any Stein filling of $\Sigma(2,3,5)$ is diffeomorphic to E. In fact, recently Ohta and Ono announced a proof of this statement.
5. For $-\Sigma(2,3,5)$ Lisca proved that it does not admit a symplectic semi-filling, while Etnyre and Honda showed [10] that it supports no tight contact structure at all.

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