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# Left Adjoint of Pullback Cat$^1$- profinite groups

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**Abstract**

In this paper, we present a brief review crossed modules [9], cat$^1$-groups [7], profinite crossed modules [6], cat$^1$-profinite groups [6], pullback profinite crossed modules [6] and also the pullback cat$^1$- profinite groups [2]. We prove that the pulback cat$^1$-profinite group has a left adjoint which is the induced cat$^1$-group.

**Key Words:** Crossed modules, Cat$^1$-groups, pullback, Profinite groups, Left, Right Adjoint, Cocomplete category.

## 1. Introduction

Crossed modules were introduced by J. H. C. Whitehead in [9]. Loday in [7] defined cat$^1$-groups and showed that the XMod category of crossed modules is equivalent to the Cat category of cat$^1$-groups. The manipulation of Crossed modules and cat$^1$-groups structures have been computer implemented using the group theory language GAP [8][1] in [3]. Enumeration of cat$^1$-groups of low order was also presented in [4]. Profinite crossed modules were defined Korkes and Porter in [6]. They also gave some useful completions about profinite crossed modules and cat$^1$-profinite groups in [6]. Pullback cat$^1$-profinite groups were defined in [2].

Our aim is to show that the equivalence between cat$^1$-groups and crossed modules due to Loday [7] takes pullback cat$^1$-profinite groups to the pullback profinite crossed modules.

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[1]Further information can be found at the following internet web site: www-gap.dcs.st-andrews.ac.uk/∼gap/

## 2. Crossed Modules and Cat$^1$-Groups

In this section we recall the descriptions of two equivalent categories: the category of crossed modules and their morphisms; and the category of cat$^1$-groups and their morphisms.

A crossed module $\mathcal{X} = (\partial : S \to R)$ consists of a group homomorphism $\partial$, called the *boundary* of $\mathcal{X}$, together with an action $\alpha : R \to \mathrm{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\begin{array}{lrcl} \textbf{CM1:} & \partial(s^r) & = & r^{-1}(\partial s)r \\ \textbf{CM2:} & s^{\partial s'} & = & s'^{-1}ss'. \end{array}$$

The standard examples of crossed modules are:

1. Any homomorphism $\partial : S \to R$ of abelian groups with $R$ acting trivially on $S$ may be regarded as a crossed module.
2. A *conjugation crossed module* is an inclusion of a normal subgroup $S \trianglelefteq R$, where $R$ acts on $S$ by conjugation.
3. A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on $S$ by conjugation with $\partial^{-1}r$.
4. An *automorphism crossed module* has as range a subgroup $R$ of the automorphism group $\mathrm{Aut}(S)$ of $S$ which contains the inner automorphism group of $S$. The boundary maps $s \in S$ to the inner automorphism of $S$ by $s$.
5. An *R-Module crossed module* has an $R$-module as source and $\partial$ is the zero map.
6. The direct product $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with $R_1,\ R_2$ acting trivially on $S_2,\ S_1$ respectively.
7. An important motivating topological example of crossed module due to Whitehead [9] is the boundary $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$ from the second relative homotopy group of a based pair $(X, A, x)$ of topological spaces, with the usual action of the fundamental group $\pi_1(A, x)$.

A morphism between two crossed modules $\mathcal{X} = (\partial : S \to R)$ and $\mathcal{X}' = (\partial' : S' \to R')$ is a pair $(\sigma, \rho)$, where $\sigma : S \to S'$ and $\rho : R \to R'$ are homomorphisms satisfying

$$\partial' \sigma = \rho\partial, \quad \sigma(s^r) = (\sigma s)^{\rho r}.$$

In [7] Loday reformulated the notion of a crossed module as a cat$^1$-group, namely a group $G$ with a pair of homomorphisms $t, h : G \to G$ having a common image $R$ and satisfying certain axioms. We find it convenient to define a cat$^1$-group $\mathcal{C} = (e; t, h : G \to R)$ as a group $G$, two surjections $t, h : G \to R, t(r,s) = r, h(r,s) = r(\partial s)$ and an embedding $e : R \to G, er = (r, 1)$ satisfying:

$$\begin{array}{ll} \textbf{CAT1:} & te = he = \mathrm{id}_R, \\ \textbf{CAT2:} & [\ker t, \ker h] = \{1_G\}. \end{array}$$

The maps $t, h$ are often called the *source* and *target*, but we choose to call them *tail* and *head* of $\mathcal{C}$, because *source* is the GAP term for the domain of a function. A morphism $\mathcal{C} \to \mathcal{C}'$ of cat$^1$-groups is a pair $(\gamma, \rho)$ where $\gamma : G \to G'$ and $\rho : R \to R'$ are homomorphisms satisfying

$$h'\gamma = \rho h, \;\; t'\gamma = \rho t, \;\; e'\rho = \gamma e.$$

## 3. Profinite crossed modules and cat$^1$-profinite groups

A profinite crossed module [6] $\mathcal{PX} = (\partial : S \to R)$ is a crossed module in which $S$ and $R$ profinite groups, $S$ acts continuously on $R$ and $\partial$ is a continuous group homomorphism.

The following are examples of profinite crossed modules [6]:

1. Let $H$ be a closed normal subgroup of profinite group $R$ with $i : H \to R$ the inclusion. Then we will say $(i : H \to R)$ is a closed normal subgroup pair. In this case, of course, $R$ acts continuously on the left of $H$ by conjugation and the inclusion homomorphism $i$ makes $(i : H \to R)$ into a profinite crossed module.

2. Let $R$ be a finitely generated profinite group, then $\mathrm{Aut}(R)$, the group of continuous automorphisms of $R$, is also profinite in the topology of uniform convergence. Conjugation gives a continuous homomorphism $\partial : R \to \mathrm{Aut(R)}$. Absolutely, $\mathrm{Aut(R)}$ acts continuously on $R$ and $\partial$ is a profinite crossed module.

3. Suppose given a continuous morphism $\theta : M \to N$ of pseudocompact left $R$-modules and form the semidirect product $R \ltimes R$. This is a profinite group which we make act continuously on $M$ via the projection from $R \ltimes N$ to $R$. We define a continuous morphism $\partial : M \to R \ltimes N$ by $\partial(m) = (1, \theta(m))$ where 1 denotes identity in $R$ then $(\partial : M \to R \ltimes N)$ is a profinite crossed module.

If $\mathcal{PX} = (\partial : S \to R)$ and $\mathcal{PX}' = (\partial' : S' \to R')$ are profinite crossed modules and $(\mu, \eta) : (\partial : S \to R) \to (\partial' : S' \to R')$ is a morphism between them in which the pair $(\mu, \eta)$ are both continuous, then the pair $(\mu, \eta)$ is called a morphism of profinite crossed modules.

A cat$^1$-profinite group is a cat$^1$-group $\mathcal{C} = (e; t, h : G \to R)$ in which $G$ is a profinite group and $t$ and $h$ are continuous endomorphisms of $G$.

A morphism of cat$^1$-profinite groups is a morphism $\phi : \mathcal{C} = (e; t, h : G \to R) \to \mathcal{C}' = (e'; t', h' : G' \to R')$ of the underlying cat$^1$-profinite groups such that $\phi$ is a continuous morphism of profinite groups.

To any pre cat$^1$-profinite group there is a canonically associated a cat$^1$-profinite group $\mathcal{C}$, obtained by quotienting the source group by the Peiffer subgroup $[\ker t, \ker h]$. Then the corresponding functor is denoted

$$\text{ass: (pre cat}^1\text{-profinite groups)} \to (\text{cat}^1\text{-profinite groups}),$$

and is clearly the identity when restricted to cat$^1$-profinite groups [5].

## 4. Pullbacks of Profinite Crossed modules and cat$^1$-profinite groups

Let $\mathcal{PX} = (\partial : S \to R)$ be a profinite crossed module and $\iota : Q \to R$ be a continuous homomorphism of profinite groups. Then $\iota^*\mathcal{X} = (\partial^* : \iota^*S \to Q)$ is the pullback of $\mathcal{PX}$ by $\iota$. So that $\iota^*S \subset Q \times S$ is the closed subgroup given by

$$\iota^*S = \{(q, s) \in Q \times S \mid \iota q = \partial s\}$$

and $Q$ acts continuously on the right of $\iota^*S$ by

$$(q_1, s)^q = (q^{-1}q_1q, s^{\iota q})$$

and since $\partial^*(q_1, s) = q_1$.

Let $\mathcal{PC} = (e; t, h : G \to R)$ be a cat$^1$-profinite group and $\iota : Q \to R$ be a continuous homomorphism. Then $e^{**}; t^{**}, h^{**} : \iota^{**}G \to Q$ is a pullback of $G$ where $\iota^{**}G \subset Q \times G \times Q$ and

$$\iota^{**}G = \{(q_1, g, q_2) \in Q \times G \times Q \mid \iota q_1 = tg,\ \iota q_2 = hg\}$$

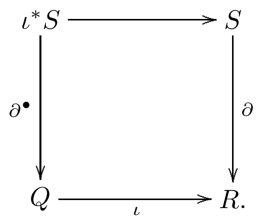
Now we can define tail, head and emmbedding as follows:

$$\begin{aligned} t^{**}(q_1, g, q_2) &= q_1 \\ h^{**}(q_1, g, q_2) &= q_2 \\ e^{**}(q) &= (q, e\iota q, q). \end{aligned}$$

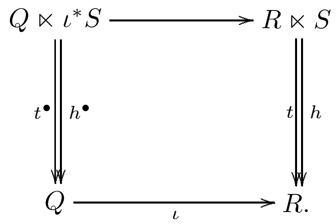
The equivalence between the category of ProfXMod and ProfCat1 was shown in [2].

**Proposition 4.1** If $\iota^*\mathcal{P}\mathcal{X}$ is the pullback of the profinite crossed module $\mathcal{X}$ over $\iota : Q \to R$ and if $\mathcal{C}, \mathcal{D}$ are the cat$^1$-profinite groups obtained from $\mathcal{X}, \iota^*\mathcal{X}$ respectively, then $\mathcal{D} \cong \iota^{**}\mathcal{C}$.

**Proof:**



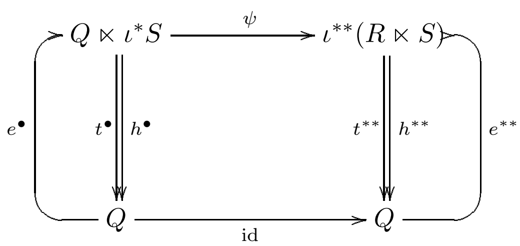

Starting with the pullback profinite crossed module $\iota^*\mathcal{X} = (\partial^\bullet : \iota^*S \to Q)$, the source group of $\mathcal{D}$ is defined as the semi-direct product $Q \ltimes \iota^*S$.




The tail, head and embedding of $\mathcal{D}$ are respectively given by

$$\begin{array}{rcl} t^\bullet(q', (q,s)) & = & q' \\ h^\bullet(q', (q,s)) & = & q'\partial^\bullet(q,s) \\ & = & q'q \\ e^\bullet(q) & = & (q, (1_Q, 1_S)) \end{array}$$

We then define an isomorphism of cat$^1$-profinite groups $(\psi, \mathrm{id}_Q) : \mathcal{D} \to \iota^{**}\mathcal{C}$,



where

$$\psi(q', (q, s)) = (q', (\iota q', s), q'q).$$

First note that $\psi(q', (q, s)) \in \iota^{**}(R \ltimes S)$ because

$$t(\iota q', s) = \iota q'$$

and

$$h(\iota q', s) = (\iota q')(\partial s) = (\iota q')(\iota q) = \iota(q'q).$$

We verify that $\psi$ is a homomorphism as follows:

$$\begin{array}{rcl} \psi((q'_1, (q_1, s_1))(q'_2, (q_2, s_2)) & = & \psi(q'_1 q'_2, (q_1^{q'_2} q_2, s_1^{\iota q'_2} s_2)) \\ & = & (q'_1 q'_2, (\iota(q'_1 q'_2), s_1^{\iota q'_2} s_2), q'_1 q_1 q'_2 q_2) \\ \psi(q'_1, (q_1, s_1))\psi(q'_2, (q_2, s_2)) & = & (q'_1, (\iota q'_1, s_1), q'_1 q_1)(q'_2, (\iota q'_2, s_2), q'_2 q_2) \\ & = & (q'_1 q'_2, (\iota q'_1, s_1)(\iota q'_2, s_2), q'_1 q_1 q'_2 q_2) \\ & = & (q'_1 q'_2, ((\iota q'_1)(\iota q'_2), s_1^{\iota q'_2} s_2), q'_1 q_1 q'_2 q_2). \end{array}$$

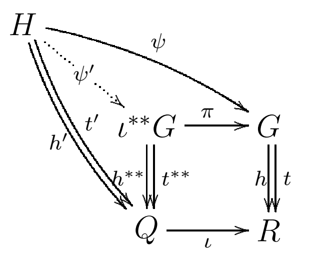
The inverse of $\psi$ is given by $\psi^{-1}(q_1, (r, s), q_2) = (q_1, (q_1^{-1} q_2, s))$.

Then

$$\begin{array}{rcl} t^{**}\psi(q', (q, s)) & = & t^{**}(q', (\iota q', s), q'q) \\ & = & q' \\ & = & t^{\bullet}(q', (q, s)), \\ h^{**}\psi(q', (q, s)) & = & h^{**}(q', (\iota q', s), q'q) \\ & = & q'q \\ & = & h^{\bullet}(q', (q, s)), \\ \psi e^{\bullet}(q) & = & \psi(q, (1_Q, 1_S)) \\ & = & (q, (\iota q, 1_s), q) \\ & = & e^{**}(q), \end{array}$$

so the diagram commutes and the proof is complete. □

The universal property of induced cat$^1$-profinite group is the following. Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat$^1$-group and let $\iota^{**}\mathcal{C} = (e^{**}; t^{**}, h^{**} : \iota^{**}G \to Q)$ be induced by the homomorphism $\iota : Q \to R$, is given by the diagram



where the pair $(\pi, \iota)$ is a morphism of cat$^1$-group such that for any cat$^1$-group $\mathcal{H} = (e'; t', h' : H \to Q)$ and any morphism of cat$^1$-group $(\psi, \iota) : \mathcal{C} \to \mathcal{H}$ there is a unique morphism $((\psi', 1) : \iota^{**}\mathcal{C} \to \mathcal{H}))$ of cat$^1$-profinite groups such that $\pi\psi' = \psi$.

## 5. Construction of the left adjoint

The construction of left adjoint of pullback cat$^1$-groups were given in [1].

**Proposition 5.1** The category of cat$^1$-profinite groups is cocomplete.

**Proof:** Let $F : \mathbf{C} \to (\text{cat}^1 - \text{profinitegroups})$. We wish to construct colim $F$. For each object $c$ of $\mathbf{C}$, we write

$$F(c) = (e_c; t_c, h_c : F_1(c) \to F_0(c), ).$$

Then we form

$$\begin{array}{rcl} F' & = & (e'; t', h' : \text{colim}_c F_1(c) \to \text{colim}_c F_0(c)) \\ & = & (e'; t', h' : F'_1 \to F'_0), \end{array}$$

where $F'_1, F'_0$ are the colimits in the category of profinite groups, so that $t'e' = h'e' = 1$. So $F'$ is a pre-cat$^1$-profinite group. The required colimit is then the associated cat$^1$-profinite group **ass** $F'$,

$$\mathbf{ass}F' = (e''; t'', h'' : F'_1/[\ker t', \ker h'] \to F'_0).$$

□

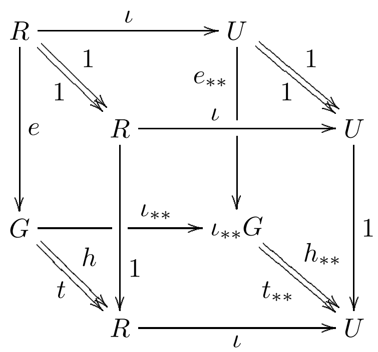
**Proposition 5.2** The functor $\iota^{**} : Cat^1ProGrp/U \to Cat^1ProGrp/R$ has a left adjoint $\iota_{**} : Cat^1ProGrp/R \to Cat^1ProGrp/U$.

**Proof:**

We can give the left adjoint construction as follows. Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat$^1$-profinite group over $R$ and $\iota : R \to U$ is a morphism of profinite groups. Then the induced cat$^1$-profinite group is $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U)$ is given by the pushout

$$\begin{array}{ccc} (1; 1, 1 : R \to R) & \xrightarrow{\iota} & (1; 1, 1 : U \to U) \\ \downarrow & & \downarrow \\ (e; t, h : G \to R) & \longrightarrow & (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U). \end{array}$$

We draw the above diagram as a three dimensional diagram as follows




in the category of cat$^1$-profinite groups. For computational purposes note that, by the previous proposition 5.1, $\iota_{**}G = (G *_R U)/[\ker t_{**}, \ker h_{**}]$, where $*_R$ denotes coproduct of profinite groups, that is, a free product with amalgamation over $R$. □

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