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n–Commutator Groups

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Abstract

A sufficient condition such that any element of G' (the commutator subgroup of G) can be represented as a product of n commutators, was studied in [1]. In this article we study a necessary and sufficient condition such that any element of G' can be represented as a product of n commutators, Let n be the smallest nature number such that any element of finite group G can be represented as a product of n commutators. A group G with this property will be called an n-commutator group, and n will be denoted by $c(G)$. Then $\frac{\ln(|G'|)}{\ln(|G:Z(G)|)} \leq 2c(G)$. In particular, if the all elements of G' can be represented as a commutator, then $|G'| \leq |G : Z(G)|^2$.

Key Words: Commutator subgroup, irreducible characters.

1. Introduction

Let G be a finite group and G' be the commutator subgroup of G. Also, let $Irr(G)$ be the set of all complex irreducible characters of G and $Lin(G) = \{\chi \in Irr(G) | \chi(1) = 1\}$, $Irr_1(G) = Irr(G) - Lin(G)$. We suppose that if $\chi \in Irr(G)$, then $T(\chi) = \{g \in G | \chi(g) = 0\}$.

Definition 1. *Let n be a natural number. Then a finite group G is called an n-commutator group if any element of G' can be represented as a product of n commutators, and no natural number fewer than n have this property. We then denote n by $c(G)$.* ◇

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2. Generalities

Lemma 1. *Let χ be an irreducible character of G. Then, for any g and x in G,*

$$\sum_{t\in G}\chi(g[t,x]) = \frac{|G|}{\chi(1)}\chi(gx)\overline{\chi(x)}. \tag{1}$$

Proof. Let k_s be the class sum for an element $s \in G$. Then

$$k_s = \frac{|G:C_G(s)|}{|G|}\sum_{t\in G} s^t.$$

We extend the character χ by linearity to $\mathbb{C}G$ and define a function

$$\omega_\chi : Z(\mathbb{C}G) \longrightarrow \mathbb{C}$$

by

$$\omega_\chi(z) = \frac{\chi(z)}{\chi(1)}$$

for any $z \in Z(\mathbb{C}G)$. Then it is clear that ω_χ is a homomorphism of $Z(\mathbb{C}G)$. Since

$$\omega_\chi(k_s) = \frac{|G:C_G(s)|}{\chi(1)}\chi(s),$$

it follows that for any $u, x \in G$,

$$\omega_\chi(k_u k_x) = \omega_\chi(k_u)\omega_\chi(k_x) = \frac{|G:C_G(u)||G:C_G(x)|}{\chi(1)^2}\chi(u)\chi(x). \tag{2}$$

Setting $t_2 t_1^{-1} = t$, we get

$$\begin{aligned} \omega_\chi(k_u k_x) &= \tfrac{\chi(k_u k_x)}{\chi(1)} \\ &= \tfrac{|G:C_G(u)||G:C_G(x)|}{\chi(1)|G|^2}\textstyle\sum_{t_1,t_2\in G}\chi(ux^{t_2 t_1^{-1}}) \\ &= \tfrac{|G:C_G(u)||G:C_G(x)|}{\chi(1)|G|}\textstyle\sum_{t\in G}\chi(ux^t), \end{aligned} \tag{3}$$

which, together with (2), yields

$$\sum_{t\in G}\chi(ux^t) = \frac{|G|}{\chi(1)}\chi(u)\chi(x). \tag{4}$$

Replacing x by x^{-1} in (4) and observing that

$$\chi(u(x^{-1})^t) = \chi(u[t,x]x^{-1}) = \chi(x^{-1}u[t,x]),$$

we have

$$\sum_{t\in G}\chi(x^{-1}u[t,x]) = \frac{|G|}{\chi(1)}\chi(u)\overline{\chi(x)}.$$

Taking into account that $\chi(xg) = \chi(gx)$ and replacing $x^{-1}u$ by g in the last equality, we get the required relation. □

Lemma 2. *Let χ be an irreducible character of G. Then any element $g \in G$ and $x_1, x_2, \cdots, x_n \in G$,*

$$\begin{aligned}&\textstyle\sum_{t_1,t_2,\cdots,t_n\in G}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_n,x_n])\\ &= (\tfrac{|G|}{\chi(1)})^n\chi(gx_1x_2\cdots x_n)\overline{\chi(x_1)\chi(x_2)\cdots\chi(x_n)}.\end{aligned} \tag{5}$$

Proof. For $n = 1$, the result is true by Lemma 1. Suppose that the lemma is valid for any $k < n$. Then for any $g, x_1, x_2, \cdots, x_{n-1} \in G$ we have

$$\begin{aligned}&\textstyle\sum_{t_1,t_2,\cdots,t_{n-1}\in G}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_{n-1},x_{n-1}])\\ =\;&(\tfrac{|G|}{\chi(1)})^{n-1}\chi(gx_1x_2\cdots x_{n-1})\overline{\chi(x_1)\chi(x_2)\cdots\chi(x_{n-1})}.\end{aligned}$$

By Lemma 1,

$$\begin{aligned}&\textstyle\sum_{t_1,t_2,\cdots,t_n\in G}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_n,x_n])\\ =\;&\textstyle\sum_{t_1,t_2,\cdots,t_{n-1}\in G}\sum_{t_n}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_{n-1},x_{n-1}][t_n,x_n])\\ =\;&\textstyle\sum_{t_1,t_2,\cdots,t_{n-1}\in G}\frac{|G|}{\chi(1)}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_{n-1},x_{n-1}]x_n)\overline{\chi(x_n)}\\ =\;&\textstyle\frac{|G|}{\chi(1)}\sum_{t_1,t_2,\cdots,t_{n-1}\in G}\chi(x_ng[t_1,x_1][t_2,x_2]\cdots[t_{n-1},x_{n-1}])\overline{\chi(x_n)}.\end{aligned}$$

By induction,

$$\begin{aligned}&\textstyle\sum_{t_1,t_2,\cdots,t_{n-1}\in G}\chi(x_ng[t_1,x_1][t_2,x_2]\cdots[t_{n-1},x_{n-1}])\\ =\;&(\tfrac{|G|}{\chi(1)})^{n-1}\chi(x_ngx_1x_2\cdots x_{n-1})\overline{\chi(x_1)\chi(x_2)\cdots\chi(x_{n-1})},\end{aligned}$$

so that

$$\begin{aligned} & \textstyle\sum_{t_1,t_2,\cdots,t_n\in G}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_n,x_n]) \\ = \; & (\tfrac{|G|}{\chi(1)})^n\chi(gx_1x_2\cdots x_n)\overline{\chi(x_1)\chi(x_2)\cdots\chi(x_n)}. \end{aligned}$$

□

Lemma 3. *Let χ be an irreducible character of G. Then*
a) For any natural number n and $g \in G$

$$\sum_{g_1g_2\cdots g_n=g}\chi(g_1)\chi(g_2)\cdots\chi(g_n) = (\frac{|G|}{\chi(1)})^{n-1}\chi(g).$$

b) For any $g \in G$,

$$\sum_{\substack{t_i,x_i\in G \\ i\in\{1,2,\ldots,n\}}}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_n,x_n]) = (\frac{|G|}{\chi(1)})^{2n}\chi(g). \tag{6}$$

Proof. *a*) The element

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{g\in G}\overline{\chi(g)}g$$

is an idempotent of the algebra $Z(\mathbb{C}G)$. Since $e_\chi^n = e_\chi$, it follows that

$$\begin{aligned} & \tfrac{\chi(1)}{|G|}\textstyle\sum_{g\in G}\overline{\chi(g)}g \\ = \; & \textstyle\prod_{i=1}^n(\tfrac{\chi(1)}{|G|}\sum_{g_i\in G}\overline{\chi(g_i)}g_i) \\ = \; & (\tfrac{\chi(1)}{|G|})^n\textstyle\sum_{g_i\in G}\overline{\chi(g_1)\cdots\chi(g_n)}g_1\cdots g_n \\ = \; & (\tfrac{\chi(1)}{|G|})^n\textstyle\sum_{g\in G}(\sum_{g_1g_2\cdots g_n=g}\overline{\chi(g_1)\chi(g_2)\cdots\chi(g_n)})g. \end{aligned}$$

Comparing the coefficients of g in the first and the last expressions, we get the required result.

b) Summing up equations (5) over $x_1, x_2, \cdots, x_n \in G$ we get

$$\begin{aligned}&\textstyle\sum_{t_i,x_i\in G}\chi(g[t_1,x_1][t_2,x_2]\cdots[t_n,x_n]) =\\ &(\tfrac{|G|}{\chi(1)})^n\textstyle\sum_{x_1,x_2,\cdots,x_n\in G}\chi(gx_1x_2\cdots x_n)\chi(x_1^{-1})\chi(x_2^{-1})\cdots\chi(x_n^{-1}) =\\ &(\textstyle\sum_{x_1,x_2,\cdots,x_n\in G}\chi(gx_1x_2\cdots x_n)\chi(x_n^{-1})\chi(x_{n-1}^{-1})\cdots\chi(x_1^{-1}))(\tfrac{|G|}{\chi(1)})^n.\end{aligned} \tag{7}$$

Put

$$u_1 = gx_1\cdots x_n,\ u_2 = x_n^{-1},\ \cdots,\ u_{n+1} = x_1^{-1}.$$

Then $u_1\cdots u_{n+1} = g$, and the last expression in (7) can be rewritten as

$$(\frac{|G|}{\chi(1)})^n \sum_{u_1\cdots u_{n+1}=g}\chi(u_1)\chi(u_2)\cdots\chi(u_{n+1}),$$

and hence, by part (a), it is equal to

$$(\frac{|G|}{\chi(1)})^{2n}\chi(g),$$

as required. □

Theorem 1. *Let G be a finite group. Then G is an n-commutator group if and only if*

$$|G|^{2n-1}\sum_{\chi\in Irr(G)}\frac{\chi(g)}{\chi(1)^{2n-1}} \tag{8}$$

is a natural number for all $g \in G'$, where n is the smallest natural number with this property.

Proof. Let $\rho = \sum_\chi \chi(1)\chi$ be the regular character of G. Multiplying both sides of (6) by $\chi(1)$ and summing over all $\chi \in Irr(G)$, we get

$$\sum_{\substack{t_i,x_i\in G,\\ i\in\{1,2,\cdots,n\}}}\rho(g[t_1,x_1][t_2,x_2]\cdots[t_n,x_n]) = |G|^{2n}\sum_{\chi\in Irr(G)}\frac{\chi(g)}{\chi(1)^{2n-1}}. \tag{9}$$

Suppose that G is an n-commutator group. We now deduce from the first equality in (9) that if $g \in G'$, then since g can be represented as a product of n commutators, we have

$$|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}} = f_n(g),$$

where $f_n(g)$ is the number of representations of g as a product of n commutators. Since, $f_n(g) \geq 1$ for any $g \in G'$,

$$|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}}$$

is a natural number for any $g \in G'$. If $|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}}$ is a natural number for all $g \in G'$, where n is the smallest natural number with this property, then we deduce from the equality (9) that if $g \in G'$, then g can be represented as a product of n commutators, and n is the smallest natural number with this property. Thus, G is an n-commutator group. □

Remark 1: Let G be a finite group. Then by using the character table of G one can say that whether G is an n-commutator group or not, an observation that follows immediately from Theorem 1.

Gallagher proved in [1] that:

Theorem 2. *(Gallagher) Let $\{x_1, \cdots, x_n\}$ be a complete system of representatives of the sets $T(\chi)(\chi \in Irr_1(G))$. Then any element of G' can be represented as*

$$[g_1, x_1][g_2, x_2] \cdots [g_n, x_n],$$

where $g_i \in G$,$i \in \{1, 2, \cdots, n\}$.

Corollary. *For any finite group G, $c(G) \leq |Irr_1(G)|$.*

Proof. Proof is obvious by Theorem 2. □

Proposition. *Let G be a finite group. Then $\frac{\ln(|G'|)}{\ln(|G:Z(G)|)} \leq 2c(G)$. In particular, if $c(G) = 1$, then $|G'| \leq |G : Z(G)|^2$.*

Proof. If T is a transversal for $Z(G)$ in G, an easy calculation shows that every commutator in G actually has the form $[s, t]$ for elements $s, t \in T$. Thus, by definition of $c(G)$ we have $|G'| \leq (|T|)^{2c(G)} = (|G : Z(G)|)^{2c(G)}$. Thus, $\frac{\ln(|G'|)}{\ln(|G:Z(G)|)} \leq 2c(G)$. □

Question: Let n be a natural number. Does there exist a finite group G such that $c(G) = n$?

Remark 2: Generalize this for some class of simple groups see attached paper. From Theorem 1 we have $c(A_5) = 1$. But A_5 is not solvable. Thus, it is not true that if $c(G) = 1$, then G is solvable.

Conjecture: Let G be a finite solvable group. Then $c(G) \leq$ derived length of G.

References

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