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Asymptotic Analysis of Singularly Perturbed Abstract Evolution Equations in Banach and Hilbert Spaces

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Abstract

In the current paper, we are concerned with the study of abstract linear evolution equations in Banach spaces in which the time derivative term is multiplied by a small parameter, say ϵ. Such equations arise in the study of radiative transfer and neutron transport in Nuclear Physics. Following works by Krein (cf [9]) and others, Mika (cf [12,13,14,15]) using either the Hilbert method or the Compressed method has shown that the solution of the given singularly perturbed equation may be approximated upto any prescibed order by a sum of two asymptotic expansions that are the outer expansion that is valid "far away" from the Initial layer and the Inner expansion which vanishes out of a certain neighborhood of the Initial layer. Since the terms of the Inner expansion are usually difficult to calculated, these higher order asymptotic approximations often remain formal. The main objectives of the current paper are:

- to locate precisely the Initial layer (cf [7,8])

- to show that making use of the concept of corrector as set by Lions (cf [11]) the outer expansion alone (at the exclusion of the inner expansion) suffices to achieve an approximation upto any prescribed order of precision.

Moreover, these results are reached under hypotheses that are weaker than those usually considered in the literature. The asymptotic solutions are worked out either in the general situation of Banach spaces or in the case of Hilbert spaces.

Key Words: Singular Perturbation, Asymptotic Expansion, Banach space, semi-group, evolution equation, Initial Layer, Corrector, radiative transfer, neutron transport

Mathematics Subject Classification (MSC 2000): 35Q, 47A55, 47F05

1. Introduction

1.1. The Physical Problem

In the study of particle collision phenomena in Nuclear Physics, one may stress on the particle aspect of the electromagnetic radiation by considering the radiation field to be composed of a "photon gas"(cf.[1], Section 3.1.3 and [5] chapter 13 and [18]). Doing so, one is led to describing by a time dependent and frequency dependent unknown function, say I_ν, called the "radiative specific density". ν stands for the frequency. If one neglects the refraction, the polarization and the dispersion effects then the neutron transport equation (or the radiative transfer equation) in the One-speed Transport Theory (or One-velocity Model Theory) may be written as:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{n}.\vec{\nabla(I_\nu)} \;=\; -R_{\nu,a} + R_{\nu,e}. \qquad (1.0a).$$

In equation (1.0a), the notations have the following meanings.

- c stands for the speed of light;
- $\vec{n}$ represents the unit vector in the direction of the velocity vector;
- $R_{\nu,a}$ is the rate of radiant energy absorption per unit phase space volume;
- $R_{\nu,e}$ stands for the rate of radiant energy emitted per unit phase space volume.

Equation (1.0a) is the core subject matter of our current attention. Setting $1/c = \epsilon$ and $I_\nu = x(t)$, we turn this equation into the following equation:

$$\epsilon\partial_t x(t) - Ax(t) \;=\; f(t) \qquad (1.0b)$$

where A is an operator.

1.2. The Mathematical field works

This physical problem correspond in fact at the mathematical study of a singularly perturbed problem that we introduce as follows. ϵ is a given small parameter such that $\epsilon << 1$. $\mathbf{X}$ being a Banach space, we deal with a family of perturbed non-homogeneous Cauchy problems of evolution equations that presents as follows:

$A(t) = A$ is either a time-dependent or a time-independent linear operator in $\mathbf{X}$ with the domain $\mathcal{D}(A)$. We consider a linear operator L_ϵ defined by:

$$L_\epsilon x(t) \;=\; \epsilon\partial_t x(t) \;-\; Ax(t)\,.$$

We consider a given and fixed element $\mu \in \mathcal{D}(A) \subset \mathbf{X}$. These data result in the following singularly perturbed non-homogeneous Cauchy problem defined for $0 < t < T$:

$$\begin{cases} L_\epsilon x(t) \;=\; f(t) \quad t \in]0, T[\\ \\ x(0) \;=\; \mu. \end{cases} \tag{1.1}$$

This type of singularly perturbed problems are usually treated in the literature by the means of the Hilbert method (Matching principle method) and others including those known to be of the compressed method type. As for the Hilbert method, Krein (cf. [9]) and other authors have established that the solution of the considered singularly perturbed problem may be approached by the sum of three asymptotic expansions. Larsen and Keller also (cf. [10]) established the same kind of result using a sum of four asymptotic expansions about the Boltzmann equation in the case of small mean free paths. Mika has shown (cf. [12,13,14,15]) that such an approximation could be achieved at any prescribed order with the sum of only two asymptotic expansions that are the outer expansion and the Inner expansion. Lions (cf. [11]) introduced a concept of corrector to construct an asymptotic approximation to the solution of a given perturbed problem. Unfortunately, the correctors as proposed in this work is reluctant to perform approximations that are better than order one in ϵ. Even though this basic idea governs the current work for which we drop off the boundary layer based correctors to define a family of some more general correctors.

1.3. The advances of the Current paper

When performing a classical asymptotic analysis of a singularly perturbed problem, one may face some difficulties among which are the following:

- the geometric domain of the Inner layer and that of the Outer expansion intersect;

- the equations defining the terms of the inner expansion may be very complicated so that it could be cumbersome to compute those terms that are beyond the first order term.

With regard to these difficulties, the current paper

- realizes a precise localization of the inner Layer (cf. [4,5]) and

- constructs an approximation solution of any prescribed order by making use of the outer expansion alone at the exclusion of any use of the inner expansion.

The technique we are using starts from the regular outer expansion and develops through an iterated use of the classical Matching Principle or Hilbert method until we reach, at the second step of that iteration, a convenient corrector which is used to construct an approximation of any arbitrary and prescribed order.

2. Functional space setting

Consider the Banach space $\mathbf{X}$ equipped with the norm $\|\cdot\|_X$. $\mathbf{C}^0([0,T[,\mathbf{X})$ denotes the space of (classes of) functions $t \to u(t)$ ($u(t) \in \mathbf{X}$) that are continuous in the variable t; $\mathbf{C}^1(]0,T[,\mathbf{X})$ denote the space of (classes of) functions $t \to u(t)$ that are continuously differentiable in the variable t and $\mathbf{C}^2(]0,T[,\mathbf{X})$ stands for the classe of functions $t \to u(t)$ that are twofold differientiable and that together with their derivatives upto order two are continuous.

$\mathbf{L}^p([0,T],\mathbf{X})$ for $p \in \mathbf{N}$; $1 \leq p < +\infty$ stands for the classe of functions $t \to u(t)$ such that $u(t)$ is mesurable, and

$$\int_0^T \|u(t)\|_X^p dt < +\infty .$$

$\mathcal{L}(\mathbf{X})$ stands for the space of the Linear Contineous operators on $\mathbf{X}$ which is equipped with the norm $\|\cdot\|_{\mathcal{L}(\mathbf{X})}$. We start with recalling some well-known results we need of for the sequel of our study.

For more on the Cauchy Problem of evolution equations and the semigroup theory, the interested reader may consult, for instance, [2-4,16].We will restrict ourself to recall some results that are important for the current study.

Lemma 2.1 *For any $\mathbf{C}_0$-semigroup $(G(t))_{t\geq 0}$ there exist constants M and ω such that:*

$$\|G(t)\|_{\mathcal{L}(\mathbf{X})} \leq Me^{\omega t} \quad t \in [0,+\infty[\tag{2.1}$$

If $\omega = 0$ the semigroup $(G(t))_{t\geq 0}$ is said to be uniformly bounded ;
if $\omega < 0$ the semigroup $(G(t))_{t\geq 0}$ is said to be of negative type.

•

Theorem 2.1 *If $(G(t))_{t\geq 0}$ is a strongly continuous semigroup, then $\mathcal{D}(A)$ is dense in $\mathbf{X}$, A is closed, and for all $x_0 \in \mathcal{D}(A)$ we have $G(t)x_0 \in \mathcal{D}(A)$ for $t \geq 0$.*
Moreover, the function $t \to G(t)x_0$ is differentiable for $t \geq 0$ and it satisfies on the interval $]0,+\infty[$ the equation:

$$\partial_t G(t)x_0 = AG(t)x_0. \tag{2.2}$$

•
We make now the point on the existence and the representation of the solution of Problem (1.1).

Theorem 2.2 *If x is the classical solution to equation* (1.1) *and* $f \in \mathbf{L}^1([0,T],\mathbf{X})$ *then x is given by the representation:*

$$x(t) = G(t)x(0) + \int_0^t G(t-s)f(s)ds \quad 0 \leq t \leq T. \tag{2.3}$$

• In the sequel, especially to localize the Initial layer, we will need to use a concept of threshold of acceptance, we following definition.

Definition 2.1 *Consider a non-zero real number η such that $\eta << \epsilon$. η will be a threshold of acceptance if*
For $x \in \mathbf{R}$; if $|x| \leq \eta$, then we set $x = 0$.
•
After these prelimaries, now, we may consider the asymptotic analysis of our given perturbed problems.

3. The Asymptotic Analysis in Banach Spaces

We set the basic hypothesis $(H.1)$ of this section as follows:

(i) *A generates a semigroup* $(G_A(t))_{t\geq 0} = (G(t))_{t\geq 0}$ $(H.1.a)$

(ii) $(G(t))_{t\geq 0}$ *is strongly continuous of negative type* $(H.1.b)$

• We come back to equation (1.1) from which we derive a family of equations depending on a parameter a ; $a \in \mathbf{R}^-$ that we define as follows:

$$\begin{cases} \epsilon\partial_t x(t) - ax(t) = f(t) \quad t \in]0,T[\\ x(0) = \mu \\ a \in \mathbf{R}^- \quad \mu \in \mathbf{R}. \end{cases} \tag{3.1}$$

We now make the following definition.

Definition 3.1 *If we set $a = -1$, into* (3.1)*, we get an equation we call the non-homogeneous Shadow-equation to* (1.1)*, or simply the Shadow equation, if there is no risk*

of confusion. Obviously, the homogeneous Shadow-equation is that one into which we set $f(t) \equiv 0$.
•

The solution of the homogeneous Shadow equation to (3.1) is $u_{se}(t)$ such that

$$\begin{cases} u_{se}(t) = u_s(t).\mu \\ u_s(t) = \exp(-t/\epsilon) \quad 0 \leq t \leq T. \end{cases} \tag{3.2}$$

3.1. Corrector and Inner Layer

We consider now the q-th outer asymptotic expansion to $x = x(t)$ solution of problem (1.1) to be constructed by the means of the Hilbert method, say $\overline{x}(t)$. It may be written as

$$\overline{x}(t) = \sum_{i=0}^{q} \epsilon^i x_i(t). \tag{3.3}$$

We now state the following theorem.

Theorem 3.1 *Set* $W(t) = \overline{x}(t) - x(t)$.
The following assertions hold true:

$$\begin{cases} (i) \quad L_\epsilon \overline{x}(t) = f(t) + \epsilon^{q+1} \partial_t x_q(t); \\ (ii) \quad L_\epsilon W(t) = \epsilon^{q+1} \partial_t x_q(t); \\ (iii) \quad W(0) = \sum_{i=0}^{q} \epsilon^i x_i(0) - \mu. \end{cases}$$

•

Proof. The coefficient functions $x_i(t) \quad 0 \leq i \leq q$ are determined in replacing the function $x(t)$ with the expression of $\overline{x}$ into equation (1.1). Equating the terms of same power in ϵ, we get:

$$\begin{cases} Ax_0(t) \;=\; -f(t) \\ \\ Ax_1(t) \;=\; (\partial_t x_0(t)) \\ \\ Ax_2(t) \;=\; (\partial_t x_1(t)) \\ \\ \ldots \\ \ldots \\ \ldots \\ \\ Ax_q(t) \;=\; (\partial_t x_{q-1}(t)) \end{cases} \tag{3.4}$$

which may be rewritten as:

$$\begin{cases} x_0(t) \;=\; -A^{-1} f(t) \\ \\ x_1(t) \;=\; A^{-1} \left(\partial_t x_0(t)\right) \\ \\ x_2(t) \;=\; A^{-1} \left(\partial_t x_1(t)\right) \\ \\ \ldots \\ \ldots \\ \ldots \\ \\ x_q(t) \;=\; A^{-1} \left(\partial_t x_{q-1}(t)\right) \end{cases} \tag{3.5}$$

From equation (3.5), we draw straightaway that

$$\begin{cases} L_\epsilon x_0(t) \,=\, \epsilon\partial_t x_0(t) \,+\, f(t) \\ \\ \epsilon L_\epsilon x_1(t) \,=\, \epsilon^2\partial_t x_1(t) \,-\, \epsilon\partial_t x_0(t) \\ \\ \cdots \\ \cdots \\ \cdots \\ \\ \epsilon^{q-1}L_\epsilon x_{q-1}(t) \,=\, \epsilon^{q}\partial_t x_{q-1}(t) \,-\, \epsilon^{q-1}\partial_t x_{q-2}(t) \\ \\ \epsilon^{q}L_\epsilon x_{q}(t) \,=\, \epsilon^{q+1}\partial_t x_{q}(t) \,-\, \epsilon^{q}\partial_t x_{q-1}(t). \end{cases} \tag{3.6}$$

We add up all terms on the same side of (3.6) line equations to get

$$L_\epsilon \overline{x}(t) \,=\, \sum_{i=0}^{q} \epsilon^i L_\epsilon x_i(t) \,=\, f(t) \,+\, \epsilon^{q+1}\partial_t x_q(t),$$

that is (i).
Next consider $L_\epsilon W(t)$. We have:
$L_\epsilon W(t) \,=\, L_\epsilon \overline{x}(t) \,-\, L_\epsilon x(t) \,=\, f(t) \,+\, \epsilon^{q+1}\partial_t x_q(t) \,-\, f(t) \,=\, \epsilon^{q+1}\partial_t x_q(t)$,t
that is (ii).
Statement (iii) of Theorem 3.1 is obvious.
• Now, we introduce the notions of Corrector and Strong Stable Asymptotic solution. □

Definition 3.2 *A regular function $\theta(t)$ is said to be a q-th order Corrector to Problem* (1.1) *if*

$$Y(t) \,=\, x(t) \,-\, (\overline{x}(t) - \theta(t))$$

satisfies $Y(0) \,=\, 0$.
•

Definition 3.3 *We say that a regular function $Y_{as}(t)$ ($Y_{as}(t) \,\in\, \mathbf{X}$) is a q-th Strong Stable asymptotic solution to $x(t)$ solution of* (1.1) *if*
(i) $\quad Y_{as}(t)$ *is a Corrector to* $x(t)$

(ii) $\|x(t) - Y_{as}(t)\|_X \leq C\epsilon^{q+1}$ *where* C *is a constant independent of* ϵ.

•

Lemma 3.1 *We claim that the regular function* θ_s *such that:*

$$\theta_s t = u_s(t) \cdot (\mu - \overline{x}(0)) \tag{3.7}$$

is a Corrector to $x(t)$ *solution of* (1.1).

•

Proof. Consider the function $Y_s(t)$ such that:

$$Y_s(t) = x(t) - (\overline{x}(t) - \theta_s(t))$$

according to Definition 3.2, and since $Y_s(0)$ is obviously equal to zero, $Y_s(t)$ is a Corrector to $x(t)$.

•

$u_s(t)$ is an exponentially decaying function in t so the thin neighborhood of $t = 0$ into which its values are not negligible is set to be the Inner Layer. More precisely, with regard to the threshold of acceptance η (cf. Definition 2.4) we state the following definition. □

Definition 3.4 *The Inner Layer, say* Ω_ϵ, *is defined by*

$$\Omega_\epsilon = \{t \in [0, T]\,;\, u_s(t) > \eta\}$$

that is

$$\Omega_\epsilon = \left\{t \in [0, T]\,;\, t < \epsilon \cdot Log\left(\eta^{-1}\right)\right\}.$$

•

3.2. Construction of the Asymptotic Solution

Our aim now is to construct a q-th Strong Stable asymptotic solution to (1.1). We start from the q-th outer expansion $\overline{x}(t)$. Consider $W(t) = \overline{x}(t) - x(t)$. We consider also an auxiliary asymptotic expansion $Z(t)$ having the following expression:

$$Z(t) = \sum_{i=0}^{q} \epsilon^i \lambda_i z_i,$$

where the coefficient constants λ_i ; $0 \leq i \leq q$ and the coefficient functions z_i ; $0 \leq i \leq q$ are to be determined.
Assume that the functions z_i are constant in the variable t that is: $z_i(t) = z_i$ $0 \leq i \leq q$ and set $U(t) = W(t) - Z(t)$. We have:

$$\begin{cases} L_\epsilon z_i = -Az_i \\ \\ L_\epsilon U(t) = L_\epsilon W(t) - L_\epsilon Z(t) = L_\epsilon W(t) - A\sum_{i=0}^{q} \epsilon^i \lambda_i z_i \\ \\ U(0) = W(0) - Z(0) = x_0(0) - \lambda_0 z_0 - \mu + \sum_{i=1}^{q} \epsilon^i (x_i(0) - \lambda_i z_i), \end{cases}$$

so that the function $U(t)$ is completely determined by

$$\begin{cases} L_\epsilon U(t) = \epsilon^{q+1}\partial_t x_q(t) - A\sum_{i=0}^{q} \epsilon^i \lambda_i z_i \\ \\ U(0) = x_0(0) - \lambda_0 z_0 - \mu + \sum_{i=1}^{q} \epsilon^i (x_i(0) - \lambda_i z_i) \,. \end{cases} \tag{3.8}$$

Consider now the outer expansion to $U(t)$ say $V(t)$ with:

$$V(t) = \sum_{i=0}^{q+1} \epsilon^i v_i(t).$$

We have

$$L_\epsilon V = \sum_{i=0}^{q+1} \epsilon^i L_\epsilon(v_i) \,.$$

Since

$$L_\epsilon(v_i) = \epsilon \partial_t v_i - Av_i,$$

then

$$L_\epsilon V = \sum_{i=0}^{q+1} \epsilon^i \big(\epsilon^{i+1}\partial_t v_i - \epsilon^i A v_i\big).$$

We identify the coefficient functions v_i from the equality

$$L_\epsilon V = L_\epsilon U$$

in equating the terms of like powers in the parameter ϵ. With regard to equation (3.8) giving the expression of $L_\epsilon U$, we get

$$\begin{cases} -Av_0(t) = -A\lambda_0 z_0 \\ -Av_1(t) = -A\lambda_1 z_1 - \partial_t v_0 \\ -Av_2(t) = -A\lambda_2 z_2 - \partial_t v_1 \\ \dots \\ \dots \\ \dots \\ -Av_q(t) = -A\lambda_q z_q - \partial_t v_{q-1} \\ -Av_{q+1}(t) = (\partial_t x_q(t)). \end{cases}$$

So remembering that $\partial_t z_i = 0$, we draw that

$$\begin{cases} v_i(t) = \lambda_i z_i \quad 0 \leq i \leq q \\ v_{q+1}(t) = -A^{-1}\partial_t x_q(t). \end{cases} \tag{3.9}$$

Set $S(t) = U(t) - V(t)$. Since V is a qth order outer expansion to U, from Theorem (3.1), we deduce that:

$$L_\epsilon S(t) = \epsilon^{q+2}\partial_t v_{q+1} = \epsilon^{q+2}\partial_t^2 x_q(t). \tag{3.10}$$

From equation (3.9) and the expression of $Z(t)$, we draw that

$$V(t) = Z(t) - \epsilon^{q+1}A^{-1}\partial_t x_q(t),$$

which implies that

$$\begin{cases} S(t) \;=\; W(t) - 2Z(t) + \epsilon^{q+1}A^{-1}\partial_t x_q(t) \;= \\ \\ \overline{x}(t) - x(t) - 2Z(t) + \epsilon^{q+1}A^{-1}\partial_t x_q(t) \end{cases} \tag{3.11}$$

and

$$S(0) \;=\; x_0(0) - 2\lambda_0 z_0 - \mu \;+\; \sum_{i=1}^{q} \epsilon^i (x_i(0) - 2\lambda_i z_i) \;+\; \epsilon^{q+1}A^{-1}\partial_t x_q(0). \tag{3.12}$$

We set the following assignments:

$$\begin{cases} \lambda_i \;=\; 1 \quad 0 \;\le\; i \;\le\; q \\ \\ z_0 \;=\; (x_0(0) - \mu)\,/2 \\ \\ z_i \;=\; (x_i(0))\,/2 \quad 0 \;\le\; i \;\le\; q, \end{cases} \tag{3.13}$$

which cause that in Equation (3.12) $S(0)$ turns to be

$$S(0) \;=\; \epsilon^{q+1}A^{-1}\partial_t x_q(0). \tag{3.14}$$

Finally, putting together Equations (3.10) and (3.14), we get that S is solution to the following Cauchy Problem

$$\begin{cases} L_\epsilon S(t) \;=\; \epsilon^{q+2}\partial_t^2 x_q(t) \\ \\ S(0) \;=\; \epsilon^{q+1}A^{-1}\partial_t x_q(0). \end{cases} \tag{3.15}$$

It is clear that the function $Z(t) \;=\; \sum_{i=0}^{q} \epsilon^i \lambda_i z_i$ is completely determined by Equation (3.13). We make the following statement that is the core result of the current paper.

Theorem 3.2 *Assume that the coefficient function $x_q(t)$ is such that: $t \to x_q(t) \in \mathbf{C}^2\left(]0,T[,\mathbf{X}\right)$. Assume that hypothesis $(H.1)$ holds true. Set*

$$Y_{as}(t) = \overline{x}(t) - 2Z(t) - \epsilon^{q+1} A^{-1}\partial_t x_q(0)\,.$$

Then the regular function $Y_{as}(t)$ defines a q-th order Strong Stable asymptotic approximation solution to $x(t)$ solution of (1.1). More precisely, we have

$$\begin{cases} x(t) \,=\, \left(\overline{x}(t) \,-\, 2Z(t) \,-\, \epsilon^{q+1}k\right) \,+\, \mathcal{O}\left(\epsilon^{q+1}\right) \\ \overline{x}(t) \,=\, \sum_{i=0}^{q} \epsilon^i x_i(t) \\ Z(t) \,=\, \sum_{i=0}^{q} \epsilon^i \lambda_i z_i \\ k \,=\, A^{-1}\partial_t x_q(0), \end{cases} \tag{3.16}$$

where C is a constant independent of ϵ. In other words, we have:

$$x(t) = \overline{x}(t) - 2Z(t) - \epsilon^{q+1}k + \mathcal{O}\left(\epsilon^{q+1}\right). \tag{3.17}$$

•

Proof. Related to equation (3.15), the representation form given in equation (2.3) of Theorem 2.2-where the second member (the forcing term) is $f_\epsilon(s) = \epsilon^{q+2}\partial_s^2 x_q(s)$-turns to be

$$\begin{cases} G_A^\epsilon(t) \,=\, G(t/\epsilon) \\ S(t) \,=\, G_A^\epsilon(t)S(0) \,+\, \int_0^t G_A^\epsilon(t-s)f_\epsilon(s)ds \quad 0 \,\le\, t \,\le\, T \\ S(0) \,=\, \epsilon^{q+1}A^{-1}\partial_t x_q(0) \\ f_\epsilon(s) \,=\, \epsilon^{q+2}\partial_s^2 x_q(s). \end{cases} \tag{3.18}$$

Since the basic hypothesis $(H.1)$ says that the semigroup $(G(t))_{t\geq 0}$ is strongly continuous and is of negative type, we can write

$$\begin{cases} \|G_A^\epsilon(t)\|_{\mathcal{L}(\mathbf{X})} \leq M.\exp(t\omega/\epsilon) \leq M \\ \|G_A(t)S(0)\|_X \leq \|G_A(t)\|_{\mathcal{L}(\mathbf{X})}\|S(0)\|_X \leq \\ \leq M \cdot \|S(0)\|_X \leq M \cdot \epsilon^{q+1} \cdot \|A^{-1}\partial_t \cdot x_q(0)\|_X \leq \\ \leq C_1 \cdot \epsilon^{q+1}, \end{cases} \tag{3.19}$$

where M and C_1 denote constants that are independent of ϵ. Same way, it is easy to see that

$$\begin{cases} \|f(s)\|_X \leq C_2 \cdot \epsilon^{q+2} \\ \|G_A^\epsilon(t-s)f(s)\|_X \leq \|G_A^\epsilon\|_{\mathcal{L}(\mathbf{X})} \cdot \|f(s)\|_X \leq \\ \leq C_3 \cdot \epsilon^{q+2}, \end{cases} \tag{3.20}$$

where C_2 and C_3 are two constants that are independent of ϵ. Together, inequalities (3.19) and (3.20) show that $S(t)$ as expressed by equation (3.18) is bounded in $\mathbf{X}$, and precisely we have

$$\|S(t)\|_X \leq C.\epsilon^{q+1}. \tag{3.21}$$

We know from equation (3.11) that

$$S(t) = U(t) - V(t) = \overline{x}(t) - x(t) - 2Z(t) + \epsilon^{q+1}A^{-1}\partial_t x_q(t),$$

where $\overline{x}(t)$ is a q-th order outer expansion to $x(t)$. This means that :

$$Y_{as}(t) = \overline{x}(t) - 2Z(t) - \epsilon^{q+1}A^{-1}\partial_t x_q(0)$$

is a q-th order Strong Stable approximation to $x(t)$; that is, the first part of Theorem 3.2. □

Equation (3.16) is an expression that is just equivalent to inequality (3.21). This ends the proof of Theorem 3.2.

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We may note:

Remark 3.1 *In addition to obtaining the approximation of any prescribed order q,* Y_{as} *in using only the Outer expansion, we note that we do not impose the second member* $f(t)$ *to be of class* $\mathbf{C}^{q+1}$ *as it is usually done in the literature.*

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4. The Asymptotic Analysis in Hilbert Spaces

Sometimes, the study of particle flux problems call on naturally the use of Hilbert Spaces. Let us remind the Hilbert spaces we are using in the sequel.

$\mathbf{L}^1(\Omega)$ stands for the Lebesgue space of absolute integrable functions over Ω, and $\mathbf{L}^2(\Omega)$ is the Lebesgue Space of Square Integrable functions defined on Ω. The norms over these classical Lebesgue function spaces are set to be respectively

$$| f |_1 = \int_\Omega |f(x)| dx \text{ and } | f |_2 = \left(\int_\Omega |f(x)|^2 dx\right)^{\frac{1}{2}}.$$

We assume that the norm on $\mathbf{L}^2(\Omega)$ derives from the classical scalar product on $\mathbf{L}^2(\Omega)$, denoted $(\cdot,\cdot)$. Consider $\mathbf{X}$ as an Hilbert Sub-space of $\mathbf{L}^2(\Omega)$ equipped with the derived scalar product and the derived norm $\|\cdot\|_X$. $\mathbf{W}\left([0,T[,\mathbf{X}\right)$ denotes the sub-space of $\mathbf{C}^1\left(]0,T[,\mathbf{X}\right)$ equipped with the norm $\|\cdot\|_{\mathbf{W}}$ defined by

$$\| v \|_{\mathbf{W}} = Max\left[\int_0^T | v |_2 dt, \int_0^T | \frac{d}{dt} v |_2 \, dt\right].$$

We consider the following basic hypothesis:

-A is a positive definite and coercive Hermitian operator. *(H.2)*

In other terms, we have

$$\begin{cases} (-Af, g) \;=\; (f, -Ag) \\ (-Af, f) \;\geq\; 0 \\ c|\, f\,|_2 \leq \; (-Af, f) \end{cases} \qquad (H.2bis).$$

From now on and through out the current section, we assume that this hypothesis $(H.1)$ holds true. We will make use of a result from the Theory of Spectral Analysis which may be put as in the following theorem (cf. [5]).

Theorem 4.1 *If A is a positive definite Hermitian operator then A posseses eigenvalues. These eigenvalues are real positives numbers and eigenfunctions associated to two different eigenvalues are orthogonal.*

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We come back to $\overline{x}(t, u)$ the outer expansion to $x(t, u)$, where u stands for the space variable. For convenience, this time we set

$$W(t, u) \;=\; x(t, u) - \overline{x}(t, u).$$

From Thorem 3.1, we know that $W(t, u)$ satisfies

$$\begin{cases} L_\epsilon W(t, u) \;=\; -\epsilon^{q+1} \partial_t x_q(t, u) \\ W(0, u) \;=\; \mu(u) - \displaystyle\sum_{i=0}^{q} \epsilon^i x_i(0, u). \end{cases} \qquad (4.22)$$

We new state the following lemma.

Lemma 4.1 *There exists a function say $Y(t, u)$ such that :*

$$\begin{cases} L_\epsilon Y(t, u) \;=\; 0 \\ Y(0, u) \;=\; -\mu(u) + \displaystyle\sum_{i=0}^{q} \epsilon^i x_i(0, u) \end{cases} \qquad (4.23)$$

•

Proof of Lemma 4.1
Let us set $Y(t,u)$ in the form of variable separated function such that

$$Y(t,u) = T(t)\cdot Q(u).$$

Then we have

$$L_\epsilon Y(t,u) = \epsilon\cdot Q(u)\cdot \partial_t T(t) - T(t)AQ(u) = 0.$$

Using the classical Method of separated variables, we find that there exists a constant say λ such that:

$$\begin{cases} \epsilon\cdot\partial_t T(t) = \lambda\cdot T(t) \\ \\ AQ(u) = \lambda\cdot Q(u). \end{cases}$$

This means that λ is an eigenvalue to the Positive definite Hermitian operator A. From Theorem 4.1 we know that such a constant λ exists as an eigenvalue of A. Then for any given eigenvalue λ of A, we set

$$T(t) = k(u)\cdot\exp(-\lambda t)$$

where $k(u)$ is constant in the variable t and we set $Q(x)$ to be the eigenfunction associated with the selected eigenvalue λ. Eventually, we get:

$$Y(t,u) = Q(u)\cdot k(u)\cdot\exp(-\lambda t)$$

where, by definition $L_\epsilon Y = 0$ and $Y(0,u) = Q(u)k(u) = \mu(u) - \sum_{i=0}^{q}\epsilon^i x_i(0,u)$. For u such that $Q(u)\neq 0$ we determine the values of $k(u)$.
Q.E.D.
• From this the following results.

Theorem 4.2 *The function $-Y(t,u)$ is a $(q+1)$-th corrector to $x(t,u)$ and $\overline{x}(t,u) - Y(t,u)$ is a $(q+1)$-th order approximation solution to $x(t,u)$. More precisely, we have*

$$\| (W(t,u)+Y(t,u)) \|_{\mathbf{W}} \leq C\cdot\epsilon^q,$$

where C is a constant independent of ϵ.
•

Proof. From equations above, we draw that

$$\begin{cases} L_\epsilon(W(t,u)+Y(t,u)) \;=\; -\epsilon^{q+1}\partial_t x_q(t,u) \\ \\ W(0,u)+Y(t,0) \;=0, \end{cases} \tag{4.24}$$

and

$$\epsilon(\frac{d(W+Y)}{dt}, W+Y) \;+\; (-A(W+Y), W+Y) = \;(f, W+Y),$$

where

$$f(t,u) \;=\; -\epsilon^{q+1}\partial_t x_q(t,u).$$

Noting that

$$(\frac{d(W+Y)}{dt}, W+Y) \;=\; (1/2)\frac{d}{dt}|\,W+Y\,|_2^2,$$

and assuming that the coercivity coefficient of the operator $-A$ is greater than ϵ, we get

$$(1/2)\epsilon\left(\int_0^T \frac{d}{dt}|\,W+Y\,|_2^2 dt \;+\; \int_0^T |\,W+Y\,|_2^2 dt \;\le\; C.\int_0^T \epsilon^{(q+1)}|\,W+Y\,|_2 dt,\right.$$

that is, by definition of the norm on $\mathbf{W}\,([0,T[,\mathbf{X})$,

$$\epsilon\|\,W+Y\,\|_{\mathbf{W}}^2 \;\le\; C.\epsilon^{(q+1)}.\|\,W+Y\,\|_{\mathbf{W}}. \tag{4.25}$$

And equation (4.4) implies that

$$\|\,W+Y\,\|_{\mathbf{W}} \;\le\; C\cdot\epsilon^q,$$

where C stands for various constants that are independent of ϵ. □

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