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Solving Fuzzy Linear Programming Problems with Linear Membership Functions

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Abstract

In this paper, we concentrate on two kinds of fuzzy linear programming problems: linear programming problems with only fuzzy technological coefficients and linear programming problems in which both the right-hand side and the technological coefficients are fuzzy numbers. We consider here only the case of fuzzy numbers with linear membership functions. The symmetric method of Bellman and Zadeh [2] is used for a defuzzification of these problems. The crisp problems obtained after the defuzzification are non-linear and even non-convex in general. We propose here the "modified subgradient method" and use it for solving these problems. We also compare the new proposed method with well known "fuzzy decisive set method". Finally, we give illustrative examples and their numerical solutions.

Key Words: Fuzzy linear programming; fuzzy number; modified subgradient method; fuzzy decisive set method.

1. Introduction

In fuzzy decision making problems, the concept of maximizing decision was proposed by Bellman and Zadeh [2]. This concept was adopted to problems of mathematical programming by Tanaka et al. [13]. Zimmermann [14] presented a fuzzy approach to multiobjective linear programming problems. He also studied the duality relations in fuzzy linear programming. Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita [8] and called robust programming. Dubois and Prade [3] investigated linear fuzzy constraints. Tanaka and Asai [12] also proposed a formulation of fuzzy linear programming with fuzzy constraints and gave a method for its solution which bases on inequality relations between fuzzy numbers. Shaocheng [11] considered

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the fuzzy linear programming problem with fuzzy constraints and defuzzificated it by first determining an upper bound for the objective function. Further he solved the so-obtained crisp problem by the fuzzy decisive set method introduced by Sakawa and Yana [10].

In this paper, we first consider linear programming problems in which only technological coefficients are fuzzy numbers and then linear programming problems in which both technological coefficients and right-hand-side numbers are fuzzy numbers. Each problem is first converted into an equivalent crisp problem. This is a problem of finding a point which satisfies the constraints and the goal with the maximum degree. The idea of this approach is due to Bellman and Zadeh [2]. The crisp problems, obtained by such a manner, can be non-linear (even non-convex), where the non-linearity arises in constraints. For solving these problems we use and compare two methods. One of them called the fuzzy decisive set method, as introduced by Sakawa and Yana [10]. In this method a combination with the bisection method and phase one of the simplex method of linear programming is used to obtain a feasible solution. The second method we use, is the "modified subgradient method" suggested by Gasimov [4]. For both kinds of problems we consider, these methods are applied to solve concrete examples. These applications show that the use of modified subgradient method is more effective from point of view the number of iterations required for obtaining the desired optimal solution.

The paper is outlined as follows. Linear programming problem with fuzzy technological coefficients is considered in Section 2. In section 3, we study the linear programming problem in which both technological coefficients and right-hand-side are fuzzy numbers. The general principles of the modified subgradient method are presented in Section 4. In Section 5, we examine the application of modified subgradient method and fuzzy decisive set method to concrete examples.

2. Linear programming problems with fuzzy technological coefficients

We consider a linear programming problem with fuzzy technological coefficients

$$
\begin{aligned}
&\max \sum_{j=1}^{n} c_j x_j \\
&\text{subject to } \sum_{j=1}^{n} \widetilde{a_{ij}} x_j \leq b_i, \qquad 1 \leq i \leq m \\
&x_j \geq 0, \qquad 1 \leq j \leq n
\end{aligned}
\tag{2.1}
$$

where at least one $x_j > 0$.

We will accept some assumptions.

Assumption 1. $\widetilde{a_{ij}}$ is a fuzzy number with the following linear membership function:

$$\mu_{a_{ij}}(x) = \begin{cases} 1 & \text{if} \quad x < a_{ij}, \\ (a_{ij} + d_{ij} - x)/d_{ij} & \text{if} \quad a_{ij} \le x < a_{ij} + d_{ij}, \\ 0 & \text{if} \quad x \ge a_{ij} + d_{ij}, \end{cases}$$

where $x \in R$ and $d_{ij} > 0$ for all $i = 1, ..., m$, $j = 1, ..., n$. For defuzzification of this problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values first. The bounds of the optimal values, z_l and z_u are obtained by solving the standard linear programming problems

$$\begin{aligned} z_1 = \max \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \; i = 1, ..., m, \\ x_j \ge 0, \; j = 1, ..., n, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} z_2 = \max \sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \le b_i \\ x_j \ge 0. \end{aligned} \tag{2.3}$$

The objective function takes values between z_1 and z_2 while technological coefficients vary between a_{ij} and $a_{ij} + d_{ij}$. Let $z_l = \min(z_1, z_2)$ and $z_u = \max(z_1, z_2)$. Then, z_l and z_u are called the lower and upper bounds of the optimal values, respectively.

Assumption 2. The linear crisp problems (2.2) and (2.3) have finite optimal values.

In this case the fuzzy set of optimal values, G, which is a subset of R^n, is defined as (see Klir and Yuan [6]);

$$\mu_G(x) = \begin{cases} 0 & if \quad \sum_{j=1}^{n} c_j x_j < z_l, \\ (\sum_{j=1}^{n} c_j x_j - z_l)/(z_u - z_l) & if \quad z_l \le \sum_{j=1}^{n} c_j x_j < z_u, \\ 1 & if \quad \sum_{j=1}^{n} c_j x_j \ge z_u. \end{cases} \tag{2.4}$$

The fuzzy set of the i th constraint, C_i, which is a subset of R^m, is defined by

$$\mu_{C_i}(x) = \begin{cases} 0 & , \quad b_i < \sum_{j=1}^{n} a_{ij}x_j \\ (b_i - \sum_{j=1}^{n} a_{ij}x_j) / \sum_{j=1}^{n} d_{ij}x_j & , \quad \sum_{j=1}^{n} a_{ij}x_j \le b_i < \sum_{j=1}^{n} (a_{ij}+d_{ij})x_j \\ 1 & , \quad b_i \ge \sum_{j=1}^{n} (a_{ij}+d_{ij})x_j . \end{cases} \tag{2.5}$$

By using the definition of the fuzzy decision proposed by Bellman and Zadeh [2] (see also Lai and Hwang [7]), we have

$$\mu_D(x) = \min(\mu_G(x), \min_i(\mu_{C_i}(x))). \tag{2.6}$$

In this case the *optimal fuzzy decision* is a solution of the problem

$$\max_{x\ge 0}(\mu_D(x)) = \max_{x\ge 0} \min(\mu_G(x), \min_i(\mu_{C_i}(x))). \tag{2.7}$$

Consequently, the problem (2.1) becomes to the following optimization problem

$$\begin{array}{c} \max \lambda \\ \mu_G(x) \ge \lambda \\ \mu_{C_i}(x) \ge \lambda, \qquad 1 \le i \le m \\ x \ge 0, \qquad 0 \le \lambda \le 1. \end{array} \tag{2.8}$$

By using (2.4) and (2.5), the problem (2.8) can be written as

$$\begin{array}{c} \max \lambda \\ \lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 \le 0, \\ \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j - b_i \le 0, \qquad 1 \le i \le m \\ x_j \ge 0, \ j = 1, ..., n, \qquad 0 \le \lambda \le 1. \end{array} \tag{2.9}$$

Notice that, the constraints in problem (2.9) containing the cross product terms λx_j are not convex. Therefore the solution of this problem requires the special approach adopted for solving general nonconvex optimization problems.

3. Linear programming problems with fuzzy technological coefficients and fuzzy right-hand-side numbers

In this section we consider a linear programming problem with fuzzy technological coefficients and fuzzy right-hand-side numbers

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j \\ &\sum_{j=1}^{n} \widetilde{a_{ij}} x_j \leq \widetilde{b_i}, \qquad 1 \leq i \leq m \\ &\quad x_j \geq 0, \end{aligned} \tag{3.1}$$

where at least one $x_j > 0$.

Assumption 3. $\widetilde{a_{ij}}$and $\widetilde{b_i}$ are fuzzy numbers with the following linear membership functions:

$$\mu_{a_{ij}}(x) = \begin{cases} 1 & if \quad x < a_{ij}, \\ (a_{ij} + d_{ij} - x)/d_{ij} & if \quad a_{ij} \leq x < a_{ij} + d_{ij}, \\ 0 & if \quad x \geq a_{ij} + d_{ij}, \end{cases}$$

and

$$\mu_{b_i}(x) = \begin{cases} 1 & if \quad x < b_i, \\ (b_i + p_i - x)/p_i & if \quad b_i \leq x < b_i + p_i, \\ 0 & if \quad x \geq b_i + p_i, \end{cases}$$

where $x \in R$. For defuzzification of the problem (3.1), we first calculate the lower and upper bounds of the optimal values. The optimal values z_l and z_u can be defined by solving the following standard linear programming problems, for which we assume that all they have the finite optimal values.

$$\begin{aligned} &z_1 = \max \sum_{j=1}^{n} c_j x_j \\ &\sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i, \quad 1 \leq i \leq m \\ &\quad x_j \geq 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &z_2 = \max \sum_{j=1}^{n} c_j x_j \\ &\sum_{j=1}^{n} a_{ij} x_j \leq b_i + p_i, \quad 1 \leq i \leq m \\ &\quad x_j \geq 0, \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
z_3 = \max \sum_{j=1}^{n} c_j x_j \\
\sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \le b_i + p_i, \quad 1 \le i \le m \\
x_j \ge 0
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
z_4 = \max \sum_{j=1}^{n} c_j x_j \\
\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad 1 \le i \le m \\
x_j \ge 0.
\end{aligned}
\tag{3.5}
$$

Let $z_l = \min(z_1, z_2, z_3, z_4)$ and $z_u = \max(z_1, z_2, z_3, z_4)$. The objective function takes values between z_l and z_u while technological coefficients take values between a_{ij} and $a_{ij} + d_{ij}$ and the right-hand side numbers take values between b_i and $b_i + p_i$.

Then, the fuzzy set of optimal values, G, which is a subset of R^n, is defined by

$$
\mu_G(x) = \begin{cases} 0 & if \quad \sum_{j=1}^{n} c_j x_j < z_l, \\ (\sum_{j=1}^{n} c_j x_j - z_l)/(z_u - z_l) & if \quad z_l \le \sum_{j=1}^{n} c_j x_j < z_u, \\ 1 & if \quad \sum_{j=1}^{n} c_j x_j \ge z_u. \end{cases}
\tag{3.6}
$$

The fuzzy set of the ith constraint, C_i, which is a subset of R^n is defined by

$$
\mu_{C_i}(x) = \begin{cases} 0 & if \quad b_i < \sum_{j=1}^{n} a_{ij} x_j, \\ (b_i - \sum_{j=1}^{n} a_{ij} x_j)/(\sum_{j=1}^{n} d_{ij} x_j + p_i) & if \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i, \\ 1 & if \quad b_i \ge \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i. \end{cases}
\tag{3.7}
$$

Then, by using the method of defuzzification as for the problem (2.8), the problem

(3.1) is reduced to the following crisp problem:

$$\begin{gathered}\max \lambda \\ \lambda(z_2 - z_1) - \sum_{j=1}^{n} c_j x_j + z_1 \le 0 \\ \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i - b_i \le 0, \qquad 1 \le i \le m \\ x \ge 0, \qquad 0 \le \lambda \le 1. \end{gathered} \tag{3.8}$$

Notice that, the problem (3.8) is also a nonconvex programming problem, similiar to the problem (2.9).

4. The modified subgradient method

In this section, we briefly present an algorithm of the modified subgradient method suggested by Gasimov [4] which can be applied for solving a large class of nonconvex and nonsmooth constrained optimization problems. This method is based on the construction of dual problems by using sharp Lagrangian functions and has some advantages [1], [4], [9]. Some of them are the following:

- The zero duality gap property is proved for sufficiently large class of problems;
- The value of dual function strongly increases at each iteration;
- The method does not use any penalty parameters;
- The presented method has a natural stopping criterion.

Now, we give the general principles of the modified subgradient method. Let X be any topological linear space, $S \subset X$ be a certain subset of X, Y be a real normed space and Y^* be its dual. Consider the primal mathematical programming problem defined as

$$(P) \qquad \begin{array}{rl} Inf\ P = & \inf_{x \in S} f(x) \\ \text{subject to} & g(x) = 0, \end{array}$$

where f is a real-valued function defined on S and g is a mapping of S into Y.

For every $x \in X$ and $y \in Y$ let

$$\Phi(x,y) = \begin{cases} f(x), & \text{if } x \in S \text{ and } g(x) = y \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.1}$$

We define the augmented Lagrange function associated with problem (P) in the following form: (see Azimov and Gasimov [1] and Rockafellar and Wets [9]),

$$L(x,u,c) = \inf_{y \in Y} \{\Phi(x,y) + c\,\|y\| - \langle y, u \rangle\}$$

for $x \in X$, $u \in Y^*$ and $c \geq 0$. By using (4.1) we concretize the augmented Lagrangian associated with (P):

$$L(x,u,c) = f(x) + c\,\|g(x)\| - \langle g(x), u\rangle\,, \tag{4.2}$$

where $x \in S, u \in Y^*$ and $c \geq 0$.

It is easy to show that,

$$Inf\ P = \inf_{x\in S} \sup_{(u,c)\in Y^*\times\mathbb{R}_+} L(x,u,c).$$

The dual function H is defined as

$$H(u,c) = \inf_{x\in S} L(x,u,c) \tag{4.3}$$

for $u \in Y^*$ and $c \geq 0$. Then, a dual problem of (P) is given by

$$(P^*) \qquad\qquad Sup\ P^* = \sup_{(u,c)\in Y^*\times\mathbb{R}_+} H(u,c).$$

Any element $(u,c) \in Y^* \times \mathbb{R}_+$ with $H(u,c) = Sup\ P^*$ is termed a solution of (P^*).

Proofs of the following three theorems can be found in Gasimov [4].

Theorem 1. Suppose in (P) that f and g are continuous, S is compact and a feasible solution exists. Then $InfP = Sup\ P^*$ and there exists a solution to (P). Furthermore, in this case, the function H in (P^*) is concave and finite everywhere on $Y^* \times \mathbb{R}_+$, so this maximization problem is efficiently unconstrained.

Theorem 2. Let $Inf\ P = Sup\ P^*$ and for some $(\overline{u},\overline{c}) \in Y^* \times \mathbb{R}_+$,

$$\inf_{x\in S} L(x,\overline{u},\overline{c}) = f(\overline{x}) + \overline{c}\,\|g(\overline{x})\| - \langle g(\overline{x}), \overline{u}\rangle\,. \tag{4.4}$$

Then $\overline{x}$ is a solution to (P) and $(\overline{u},\overline{c})$ is a solution to (P^*) if and only if

$$g(\overline{x}) = 0. \tag{4.5}$$

When the assumptions of the theorems, mentioned above, are satisfied, the maximization of the dual function H by using the subgradient method will give us the optimal value of the primal problem.

It will be convenient to introduce the following set :

$$S(u,c) = \{\overline{x} \mid \overline{x} \text{ minimizes } f(x) + c\,\|g(x)\| - u'g(x) \text{ over } x \in S\}.$$

Theorem 3. Let S be a nonempty compact set in $\mathbb{R}^n$ and let f and g be continuous so that for any $(\overline{u}, \overline{c}) \in \mathbb{R}^m \times \mathbb{R}_+$, $S(\overline{u}, \overline{c})$ is not empty. If $\overline{x} \in S(\overline{u}, \overline{c})$, then $(-g(\overline{x}), \|g(\overline{x})\|)$ is a subgradient of H at $(\overline{u}, \overline{c})$.

Now we are able to present the algorithm of the modified subgradient method.

Algorithm

Initialization Step. Choose a vector (u^1, c^1) with $c^1 \geq 0$, let $k = 1$, and go to main step.

Main Step.

Step 1. Given (u^k, c^k). Solve the following subproblem :

$$\begin{array}{rc} & \min(f(x) + c^k \|g(x)\| - \langle g(x), u^k \rangle) \\ \text{subject to} & x \in S. \end{array}$$

Let x_k be any solution. If $g(x_k) = 0$, then stop; (u^k, c^k) is a solution to dual problem (P^*), x_k is a solution to primal problem (P). Otherwise, go to Step 2.

Step 2. Let

$$\begin{aligned} u^{k+1} &= u^k - s^k g(x_k) \\ c^{k+1} &= c^k + (s^k + \varepsilon^k) \|g(x_k)\| \end{aligned} \tag{4.6}$$

where s^k and ε^k are positive scalar stepsizes, replace k by $k+1$, and repeat Step 1.

One of the stepsize formulas which can be used is

$$s^k = \frac{\alpha_k (H_k - H(u^k, c^k))}{5 \|g(x_k)\|^3}$$

where H_k is an approximation to the optimal dual value, $0 < \alpha_k < 2$ and $0 < \varepsilon^k < s^k$.

The following theorem shows that in contrast with the subgradient methods developed for dual problems formulated by using ordinary Lagrangians, the new iterate improves the cost for all values of the stepsizes s^k and ε^k.

Theorem 4. Suppose that the pair $(u^k, c^k) \in \mathbb{R}^m \times \mathbb{R}_+$ is not a solution to the dual problem and $x_k \in S(u^k, c^k)$. Then for a new iterate (u^{k+1}, c^{k+1}) calculated from (4.6) for all positive scalar stepsizes s^k and ε^k we have

$$0 < H(u^{k+1}, c^{k+1}) - H(u^k, c^k) \leq (2s^k + \varepsilon^k) \|g(x_k)\|^2 .$$

5. Solution of defuzzificated problems

In this section, we apply the modified subgradient method and the fuzzy decisive set method explained in the previous section to problems considered in sections 2 and 3 and we will compare these methods from point of view their efficiency.

Notice that, the constraints in problem (2.9) and (3.8) are generally not convex. These problems may be solved either by the fuzzy decisive set method, which is presented by Sakawa and Yana [10], or by the linearization method of Kettani and Oral [5].

There are some disadvantages in using these methods. The fuzzy decisive set method takes a long time for solving the problem. On the other hand, the linearization method increases the number of the constraints.

Here we present the modified subgradient method and use it for solving the defuzzificated problems (2.9) and (3.8). This method is based on the duality theory developed by Azimov and Gasimov [1] for nonconvex constrained problems and can be applied for solving a large class of such problems.

5.1. Application of modified subgradient method to fuzzy linear programming problems

For applying the subgradient method to the problem (2.9), we first formulate it with equality constraints by using slack variables p_0 and p_i. Then, problem (2.9) can be written as

$$\begin{aligned} &\max \lambda \\ &g_0(x,\lambda,p_0) = \lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 + p_0 = 0 \\ &g_i(x,\lambda,p_i) = \sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j - b_i + p_i = 0, \qquad 1 \le i \le m \\ &x \ge 0, \qquad p_0, p_i \ge 0, \qquad 0 \le \lambda \le 1. \end{aligned} \tag{5.1}$$

For this problem the set S can be defined as

$$S = \{(x,p,\lambda) \mid x = (x_1,...,x_n),\ p = (p_0,p_1,...,p_n),\ x_i \ge 0,\ p_0,p_i \ge 0, \quad 0 \le \lambda \le 1\}.$$

Since $\max \lambda = -\min(-\lambda)$ and $g(x,\lambda,p) = (g_0, g_1, ..., g_m)$, the augmented Lagrangian associated with the problem (5.1) can be written in the form:

$$\begin{aligned} L(x,u,c) = \ & -\lambda + c[(\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 + p_0)^2 + \sum_{i=1}^{m} (\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j - b_i + p_i)^2]^{1/2} \\ & - u_0(\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 + p_0) - \sum_{i=1}^{m} u_i (\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j - b_i + p_i). \end{aligned}$$

Now, the modified subgradient method may be applied to the problem (5.1) in the following way:

Initialization Step. Choose a vector (u_0^1, u_i^1, c^1) with $c^1 \geq 0$, let $k = 1$, and go to main step.

Main Step.

Step 1. Given (u_0^k, u_i^k, c^k), solve the following subproblem :

$$\min(-\lambda + c\left[\left(\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 + p_0\right)^2 + \sum_{i=1}^{m}\left(\sum_{j=1}^{n}(a_{ij} + \lambda d_{ij})x_j - b_i + p_i\right)^2\right]^{1/2}$$
$$-u_0\left(\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 + p_0\right) - \sum_{i=1}^{m} u_i\left(\sum_{j=1}^{n}(a_{ij} + \lambda d_{ij})x_j - b_i + p_i\right))$$
$$(x, p, \lambda) \in S.$$

Let (x^k, p^k, λ^k) be any solution. If $g(x^k, p^k, \lambda^k) = 0$, then stop; (u_0^k, u_i^k, c^k) is a solution to dual problem, (x^k, p^k, λ^k) is a solution to problem (3.8) so (x^k, λ^k) is a solution to problem (2.9). Otherwise, go to Step 2.

Step 2. Let

$$u_0^{k+1} = u_0^k - s^k\left(\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j + z_2 + p_0\right)$$
$$u_i^{k+1} = u_i^k - s^k\left(\sum_{j=1}^{n}(a_{ij} + \lambda d_{ij})x_j - b_i + p_i\right), \qquad 1 \leq i \leq m$$
$$c^{k+1} = c^k + (s^k + \varepsilon^k)\,\|g(x_k)\|$$

where s^k and ε^k are positive scalar stepsizes and $s^k > \varepsilon^k > 0$, replace k by $k+1$, and repeat Step 1.

5.2. The algorithm of the fuzzy decisive set method

This method is based on the idea that, for a fixed value of λ, the problems (2.9) and (3.8) are linear programming problems. Obtaining the optimal solution λ^* to the problems (2.9) and (3.8) is equivalent to determining the maximum value of λ so that the feasible set is nonempty. The algorithm of this method for the problem (2.9) is presented below. The algorithm for the problem (3.8) is similiar.

Algorithm

Step 1. Set $\lambda = 1$ and test whether a feasible set satisfying the constraints of the problem (2.9) exists or not using phase one of the simplex method. If a feasible set exists, set $\lambda = 1$. Otherwise, set $\lambda^L = 0$ and $\lambda^R = 1$ and go to the next step.

Step 2. For the value of $\lambda = (\lambda^L + \lambda^R)/2$, update the value of λ^L and λ^R using the bisection method as follows :

$\lambda^L = \lambda$ if feasible set is nonempty for λ

$\lambda^R = \lambda$ if feasible set is empty for λ.

Consequently, for each λ, test whether a feasible set of the problem (2.9) exists or not using phase one of the Simplex method and determine the maximum value λ^* satisfying the constraints of the problem (2.9).

Example 1.

Solve the optimization problem

$$\begin{array}{c} \max 2x_1 + 3x_2 \\ \underset{\sim}{1}x_1 + \underset{\sim}{2}x_2 \leq 4 \\ \underset{\sim}{3}x_1 + \underset{\sim}{1}x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{array} \tag{5.2}$$

which take fuzzy parameters as $\underset{\sim}{1} = L(1,1), \underset{\sim}{2} = L(2,3), \underset{\sim}{3} = L(3,2)$ and $\underset{\sim}{1} = L(1,3)$, as used by Shaocheng [11]. That is,

$$(a_{ij}) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \qquad (d_{ij}) = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \qquad \Rightarrow \qquad (a_{ij} + d_{ij}) = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}.$$

For solving this problem we must solve the folowing two subproblems:

$$\begin{array}{c} z_1 = \max 2x_1 + 3x_2 \\ x_1 + 2x_2 \leq 4 \\ 3x_1 + x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{array}$$

and

$$\begin{array}{c} z_2 = \max 2x_1 + 3x_2 \\ 2x_1 + 5x_2 \leq 4 \\ 5x_1 + 4x_2 \leq 6 \\ x_1, x_2 \geq 0. \end{array}$$

Optimal solutions of these subproblems are

$$\begin{array}{lcl} x_1 = 1.6 & & x_1 = 0.82 \\ x_2 = 1.2 & \text{and} & x_2 = 0.47 \\ z_1 = 6.8 & & z_2 = 3.06, \end{array}$$

respectively. By using these optimal values, problem (5.2) can be reduced to the following equivalent non-linear programming problem:

$$\max \lambda$$

$$\frac{2x_1 + 3x_2 - 3.06}{6.8 - 3.06} \geq \lambda$$

$$\frac{4 - x_1 - 2x_2}{x_1 + 3x_2} \geq \lambda$$

$$\frac{6 - 3x_1 - x_2}{2x_1 + 3x_2} \geq \lambda$$

$$\begin{gathered} 0 \leq \lambda \leq 1 \\ x_1, x_2 \geq 0, \end{gathered}$$

that is

$$\begin{gathered} \max \lambda \\ 2x_1 + 3x_2 \geq 3.06 + 3.74\lambda \\ (1 + \lambda)x_1 + (2 + 3\lambda)x_2 \leq 4 \\ (3 + 2\lambda)x_1 + (1 + 3\lambda)x_2 \leq 6 \\ 0 \leq \lambda \leq 1 \\ x_1, x_2 \geq 0, \end{gathered} \tag{5.3}$$

Let's solve problem (5.3) by using the fuzzy decisive set method.

For $\lambda = 1$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 6.8 \\ 2x_1 + 5x_2 \leq 4 \\ 5x_1 + 4x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1$, the new value of $\lambda = (0 + 1)/2 = 1/2$ is tried.

For $\lambda = 1/2 = 0.5$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 4.9294 \\ (3/2)x_1 + (7/2)x_2 \leq 4 \\ 4x_1 + (5/2)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1/2$, the new value of $\lambda = (0 + 1/2)/2 = 1/4$ is tried.

For $\lambda = 1/4 = 0.25$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 3.9941 \\ (5/4)x_1 + (11/4)x_2 \leq 4 \\ (7/2)x_1 + (7/4)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is nonempty, by taking $\lambda^L = 1/4$ and $\lambda^R = 1/2$, the new value of $\lambda = (1/4 + 1/2)/2 = 3/8$ is tried.

For $\lambda = 3/8 = 0.375$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 4.4618 \\ (11/8)x_1 + (25/8)x_2 \leq 4 \\ (15/4)x_1 + (17/8)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is nonempty, by taking $\lambda^L = 3/8$ and $\lambda^R = 1/2$, the new value of $\lambda = (3/8 + 1/2)/2 = 7/16$ is tried.

For $\lambda = 7/16 = 0.4375$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 4.6956 \\ (23/16)x_1 + (53/16)x_2 \leq 4 \\ (31/8)x_1 + (37/16)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 3/8$ and $\lambda^R = 7/16$, the new value of $\lambda = (3/8 + 7/16)/2 = 13/32$ is tried.

For $\lambda = 13/32 = 0.40625$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 4.5787 \\ (45/32)x_1 + (103/32)x_2 \leq 4 \\ (122/32)x_1 + (71/32)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 3/8$ and $\lambda^R = 13/32$, the new value of $\lambda = (3/8 + 13/32)/2 = 25/64$ is tried.

For $\lambda = 25/64 = 0.390625$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 4.5202 \\ (89/64)x_1 + (203/64)x_2 \leq 4 \\ (242/64)x_1 + (139/64)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is nonempty, by taking $\lambda^L = 25/64$ and $\lambda^R = 13/32$, the new value of $\lambda = (25/64 + 13/32)/2 = 51/128$ is tried.

For $\lambda = 51/128 = 0.3984375$, the problem can be written as

$$\begin{gathered} 2x_1 + 3x_2 \geq 4.5494 \\ (179/128)x_1 + (409/128)x_2 \leq 4 \\ (486/128)x_1 + (281/128)x_2 \leq 6 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 25/64$ and $\lambda^R = 51/128$, the new value of $\lambda = (25/64 + 51/128)/2 = 101/256$ is tried.

The following values of λ are obtained in the next thirteen iterations :

$$\begin{array}{llll} \lambda & = & 101/256 & = 0.39453125 \\ \lambda & = & 203/512 & = 0.396484325 \\ \lambda & = & 407/1024 & = 0.397460937 \\ \lambda & = & 813/2048 & = 0.396972656 \\ \lambda & = & 1627/4096 & = 0.397216796 \\ \lambda & = & 3255/8192 & = 0.397338867 \\ \lambda & = & 6511/16384 & = 0.397399902 \\ \lambda & = & 13021/32768 & = 0.397369384 \\ \lambda & = & 26043/65536 & = 0.397384643 \\ \lambda & = & 52085/131072 & = 0.397377014 \\ \lambda & = & 104169/262144 & = 0.3973731 \\ \lambda & = & 208337/524288 & = 0.3973713 \\ \lambda^* & = & 416675/1048576 & = 0.3973723 \end{array}$$

Consequently, we obtain the optimal value of λ at the twenty first iteration by using the fuzzy decisive set method.

Now, let's solve the same problem by using the modified subgradient method. Before solving the problem, we first formulate it in the form

$$\begin{gathered}\max\lambda = -\min(-\lambda)\\ 3.74\lambda - 2x_1 - 3x_2 + 3.06 + p_0 = 0\\ (1+\lambda)x_1 + (2+3\lambda)x_2 - 4 + p_1 = 0\\ (3+2\lambda)x_1 + (1+3\lambda)x_2 - 6 + p_2 = 0\\ 0 \le \lambda \le 1\\ x_1, x_2 \ge 0,\\ p_0, p_1, p_2 \ge 0,\end{gathered}$$

where p_0, p_1 and p_2 are slack variables. The augmented Lagrangian function for this problem is

$$\begin{aligned}L(x,u,c) = {} & \text{-}\lambda + c[(3.74\lambda - 2x_1 - 3x_2 + 3.06 + p_0)^2 + ((1+\lambda)x_1 + (2+3\lambda)x_2 - 4 + p_1)^2\\ & + ((3+2\lambda)x_1 + (1+3\lambda)x_2 - 6 + p_2)^2]^{1/2}\text{-}u_0\,(3.74\lambda - 2x_1 - 3x_2 + 3.06 + p_0)\\ & -u_1\,((1+\lambda)x_1 + (2+3\lambda)x_2 - 4 + p_1)\,\text{-}u_2\,((3+2\lambda)x_1 + (1+3\lambda)x_2 - 6 + p_2)\,.\end{aligned}$$

Let the initial vector is $(u_0^1, u_1^1, u_2^1, c^1) = (0,0,0,0)$ and let's solve the following subproblem

$$\begin{gathered}\min L(x,0,0)\\ 0 \le \lambda \le 1\\ 0.82 \le x_1 \le 1.6\\ 0.47 \le x_2 \le 1.2.\end{gathered}$$

The optimal solutions of subproblem are obtained as

$$\begin{aligned}x_1 &= 1\\ x_2 &= 0\\ \lambda &= 1\\ g_1(x^1,p^1,\lambda^1) &= 4.8\\ g_2(x^1,p^1,\lambda^1) &= -2\\ g_3(x^1,p^1,\lambda^1) &= -1.\end{aligned}$$

Since $g(x^1,p^1,\lambda^1) \ne 0$, we calculate the new values of Lagrange multipliers $(u_0^2, u_1^2, u_2^2, c^2)$ by using Step 2 of the modified subgradient method. The solutions of the second iteration are obtained as

$$\begin{aligned}x_1 &= 1.1475877\\ x_2 &= 0.75147\\ \lambda^* &= 0.3973723\\ g_1(x^2,p^2,\lambda^2) &= 9 \times 10^{-6}\\ g_2(x^2,p^2,\lambda^2) &= -3.8 \times 10^{-6}\\ g_3(x^2,p^2,\lambda^2) &= 2.31 \times 10^{-6}.\end{aligned}$$

Since $\|g(x)\|$ is quite small, by Theorem 2 $x_1^* = 1.1475877$, $x_2^* = 0.75147$ and $\lambda^* = 0.3973723$ are optimal solutions to the problem (5.3). This means that, the vector (x_1^*, x_2^*) is a solution to the problem (5.2) which has the best membership grade λ^*.

Note that, the optimal value of λ found at the second iteration of the modified subgradient method is approximately equal to the optimal value of λ calculated at the twenty first iteration of the fuzzy decisive set method.

Example 2.

Solve the optimization problem

$$\begin{array}{c} \max x_1 + x_2 \\ \underset{\sim}{1} x_1 + \underset{\sim}{2} x_2 \leq \underset{\sim}{3} \\ \underset{\sim}{2} x_1 + \underset{\sim}{3} x_2 \leq \underset{\sim}{4} \\ x_1, x_2 \geq 0, \end{array} \tag{5.4}$$

which take fuzzy parameters as; $\underset{\sim}{1} = L(1,1), \underset{\sim}{2} = L(2,1), \underset{\sim}{2} = L(2,2), \underset{\sim}{3} = L(3,2), b_1 = \underset{\sim}{3} = L(3,2)$ and $b_2 = \underset{\sim}{4} = L(4,3)$ as used by Shaocheng [11]. That is,

$$(a_{ij}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad (d_{ij}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \Rightarrow \qquad (a_{ij} + d_{ij}) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$(b_i) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad (p_i) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \Rightarrow \qquad (b_i + p_i) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

To solve this problem, first, we must solve the folowing two subproblems

$$\begin{array}{c} z_1 = \max x_1 + x_2 \\ 2x_1 + 3x_2 \leq 3 \\ 4x_1 + 5x_2 \leq 4 \\ x_1, x_2 \geq 0, \end{array}$$

and

$$\begin{array}{c} z_2 = \max x_1 + x_2 \\ x_1 + 2x_2 \leq 5 \\ 2x_1 + 3x_2 \leq 7 \\ x_1, x_2 \geq 0. \end{array}$$

Optimal solutions of these subproblems are

$$\begin{array}{ccc} x_1 = 1 & & x_1 = 3.5 \\ x_2 = 0 & \text{and} & x_2 = 0 \\ z_1 = 1 & & z_2 = 3.5, \end{array}$$

respectively. By using these optimal values, the problem (5.4) can be reduced to the following equivalent non-linear programming problem:

$$\max \lambda$$

$$\frac{x_1 + x_2 - 1}{3.5 - 1} \geq \lambda$$

$$\frac{3 - x_1 - 2x_2}{x_1 + x_2} \geq \lambda$$

$$\frac{4 - 2x_1 - 3x_2}{2x_1 + 2x_2} \geq \lambda$$

$$0 \leq \lambda \leq 1$$
$$x_1, x_2 \geq 0,$$

that is

$$\begin{gathered} \max \lambda \\ x_1 + x_2 \geq 1 + 2.5\lambda \\ (1+\lambda)x_1 + (2+\lambda)x_2 \leq 3 - 2\lambda \\ (2+2\lambda)x_1 + (3+2\lambda)x_2 \leq 4 - 3\lambda \\ 0 \leq \lambda \leq 1 \\ x_1, x_2 \geq 0. \end{gathered} \tag{5.5}$$

Let's solve the problem (5.5) by using the fuzzy decisive set method.

For $\lambda = 1$, the problem can be written as

$$\begin{gathered} x_1 + x_2 \geq 3.5 \\ 2x_1 + 3x_2 \leq 1 \\ 4x_1 + 5x_2 \leq 1 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1$, the new value of $\lambda = (0+1)/2 = 1/2$ is tried.

For $\lambda = 1/2 = 0.5$, the problem can be written as

$$\begin{gathered} x_1 + x_2 \geq 2.25 \\ (3/2)x_1 + (5/2)x_2 \leq 2 \\ 3x_1 + 4x_2 \leq 5/2 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1/2$, the new value of $\lambda = (0 + 1/2)/2 = 1/4$ is tried.

For $\lambda = 1/4 = 0.25$, the problem can be written as

$$\begin{gathered} x_1 + x_2 \geq 1.625 \\ (5/4)x_1 + (9/4)x_2 \leq 5/2 \\ (5/2)x_1 + (7/2)x_2 \leq 13/4 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is empty, by taking $\lambda^L = 0$ and $\lambda^R = 1/4$, the new value of $\lambda = (0 + 1/4)/2 = 1/8$ is tried.

For $\lambda = 1/8 = 0.125$, the problem can be written as

$$\begin{gathered} x_1 + x_2 \geq 1.3125 \\ (9/8)x_1 + (17/8)x_2 \leq 22/8 \\ (9/4)x_1 + (13/4)x_2 \leq 29/8 \\ x_1, x_2 \geq 0, \end{gathered}$$

and since the feasible set is nonempty, by taking $\lambda^L = 1/8$ and $\lambda^R = 1/4$, the new value of $\lambda = (1/8 + 1/4)/2 = 3/16$ is tried.

The following values of λ are obtained in the next twenty one iterations:

$$\begin{aligned} \lambda &= 3/16 &&= 0.1875 \\ \lambda &= 5/32 &&= 0.15625 \\ \lambda &= 11/64 &&= 0.171875 \\ \lambda &= 23/128 &&= 0.1796875 \\ \lambda &= 47/256 &&= 0.18359375 \\ \lambda &= 93/512 &&= 0.181640625 \\ \lambda &= 187/1024 &&= 0.182617187 \\ \lambda &= 375/2048 &&= 0.183105468 \\ \lambda &= 751/4096 &&= 0.183349609 \\ \lambda &= 1501/8192 &&= 0.183227539 \\ \lambda &= 3001/16384 &&= 0.183166503 \\ \lambda &= 6003/32768 &&= 0.183197021 \\ \lambda &= 12007/65536 &&= 0.18321228 \\ \lambda &= 24015/131072 &&= 0.183219909 \end{aligned}$$

$$
\begin{array}{lcl}
\lambda & = & 48029/262144 & = & 0.183216095 \\
\lambda & = & 96057/524288 & = & 0.183214187 \\
\lambda & = & 192115/1048576 & = & 0.183215141 \\
\lambda & = & 384231/2097152 & = & 0.183215618 \\
\lambda & = & 768463/4194304 & = & 0.183215856 \\
\lambda & = & 1536927/8388608 & = & 0.183215975 \\
\lambda^* & = & 3073853/16777216 & = & 0.183215916
\end{array}
$$

Consequently, we obtain the optimal value of λ at the twenty fifth iteration of the fuzzy decisive set method.

Now, let's solve the same problem by using the modified subgradient method. Before solving the problem, we first formulate it in the form

$$
\begin{gathered}
\max \lambda = -\min(-\lambda) \\
2.5\lambda - x_1 - x_2 + 1 + p_0 = 0 \\
(1+\lambda)x_1 + (2+\lambda)x_2 - 3 + p_1 = 0 \\
(2+2\lambda)x_1 + (3+2\lambda)x_2 - 4 + p_2 = 0 \\
0 \leq \lambda \leq 1 \\
x_1, x_2 \geq 0, \\
p_0, p_1, p_2 \geq 0,
\end{gathered}
$$

where p_0, p_1 and p_2 are slack variables. The augmented Lagrangian function for this problem is

$$
\begin{aligned}
L(x,u,c) = \; & -\lambda + c[(2.5\lambda - x_1 - x_2 + 1 + p_0)^2 + ((1+\lambda)x_1 + (2+\lambda)x_2 - 3 + p_1)^2 \\
& + ((2+2\lambda)x_1 + (3+2\lambda)x_2 - 4 + p_2)^2]^{1/2} - u_0\,(2.5\lambda - x_1 - x_2 + 1 + p_0) \\
& - u_1\,((1+\lambda)x_1 + (2+\lambda)x_2 - 3 + p_1) - u_2\,((2+2\lambda)x_1 + (3+2\lambda)x_2 - 4 + p_2)\,.
\end{aligned}
$$

Let the initial vector be $(u_0^1, u_1^1, u_2^1, c^1) = (0,0,0,0)$ and let's solve the following subproblem

$$
\begin{gathered}
\min L(x,0,0) \\
0 \leq \lambda \leq 1 \\
1 \leq x_1 \leq 3.5 \\
0 \leq x_2 \leq 0.
\end{gathered}
$$

The optimal solutions of this problem are obtained as

$$
\begin{array}{lcl}
x_1 & = & 1 \\
x_2 & = & 0 \\
\lambda & = & 1 \\
g_1(x^1, p^1, \lambda^1) & = & 2.5 \\
g_2(x^1, p^1, \lambda^1) & = & 1 \\
g_3(x^1, p^1, \lambda^1) & = & 3.
\end{array}
$$

Since $g(x^1, p^1, \lambda^1) \neq 0$, we calculate the new values of Lagrange multipliers $(u_0^2, u_1^2, u_2^2, c^2)$ by using Step 2 of the modified subgradient method. The solutions of the second iteration are obtained as

$$\begin{aligned}
x_1^* &= 1.45804 \\
x_2^* &= 7.8 \times 10^{-8} \\
\lambda^* &= 0.1832159 \\
g_1(x^2, p^2, \lambda^2) &= 3.28 \times 10^{-7} \\
g_2(x^2, p^2, \lambda^2) &= 8.2 \times 10^{-8} \\
g_3(x^2, p^2, \lambda^2) &= -7.83 \times 10^{-8}.
\end{aligned}$$

Since $\|g(x)\|$ is quite small, by Theorem 2 $x_1^* = 1.45804$, $x_2^* = 7.8 \times 10^{-8} \simeq 0$ and $\lambda^* = 0.1832159$ are optimal solutions to the problem (5.5). This means that, the vector (x_1^*, x_2^*) is a solution to the problem (5.4) which has the best membership grade λ^*.

Note that, the optimal value of λ found at the second iteration of the modified subgradient method is approximately equal to the optimal value of λ calculated at the twenty fifth iteration of the fuzzy decisive set method.

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