Abelian fibred holomorphic symplectic manifolds

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Abstract

We study holomorphic symplectic manifolds which are fibred by abelian vari-
eties. This structure is a higher dimensional analogue of an elliptic fibration
on a K3 surface. We investigate when a holomorphic symplectic manifold is
fibred in this way, and are led to several natural conjectures. We then study
the geometry of these fibrations. The expectation is that this point of view will
prove useful in understanding holomorphic symplectic manifolds, and possibly
lead to a classification.

1. Introduction

Irreducible holomorphic symplectic manifolds are higher dimensional generalizations
of K3 surfaces. It has been roughly twenty years since Fujiki [12] found the first example,
the Hilbert scheme $S^{[2]}$ of two points on a K3 surface S, and Beauville [1] generalized this
to construct two families, $S^{[n]}$ and the generalized Kummer varieties K_n. Since then there
have been other constructions, but only the two examples of O'Grady [38, 39] have given
us manifolds which are not deformations of Beauville's examples. The purpose of this
article is to describe a framework for understanding irreducible holomorphic symplectic
manifolds, which hopefully will lead towards some kind of classification. The main results
are only conjectural, but we will describe the evidence and motivation behind them, while
surveying special cases which have already been proved.

In studying the moduli space of K3 surfaces, one typically looks at Kummer surfaces
or quartics in $\mathbb{P}^3$, as these are dense but also relatively easy to understand. However,
the structure that will generalize to higher dimensions is a fibration by abelian varieties.
This suggests that we first review elliptic K3s, which also happen to be dense in the
moduli space. We divide this into three main steps. Firstly, we need to know which
K3 surfaces are elliptic. Secondly, we describe the family of elliptic K3s which admit a
section. Thirdly, we describe the relation between elliptic K3s which don't admit sections
and their *relative Jacobians*, which do. There is nothing new here: for the first step we
recall a theorem of Pjateckiĭ-Šapiro and Šafarevič [41] from the 70s (c.f. also Section 5 of
Kodaira [27]), while the second and third steps have been well understood for arbitrary
elliptic surfaces for a long time.

Elliptic K3s have base $\mathbb{P}^1$. There is evidence to suggest that if the $2n$-dimensional
irreducible holomorphic symplectic manifold X admits a non-trivial fibration, then the
fibres must be abelian varieties and the base must be $\mathbb{P}^n$ (a large part of this has been
proved by Matsushita [32]). So the 'right' generalization of an elliptic fibration on a K3

surface appears to be a fibration by n-dimensional abelian varieties over $\mathbb{P}^n$, which we shall call an *abelian fibration*. The aim of this paper is to formulate analogues for abelian fibrations of the three main steps mentioned above for elliptic K3s.

Firstly, when does an irreducible holomorphic symplectic manifold X admit an abelian fibration? Since we are largely interested in classification up to deformation equivalence, we'd like to know whether X can always be deformed to have such a structure. Secondly, can we describe the family of all abelian fibred X which admit sections? By trying to associate a holomorphic symplectic surface (ie. a K3 surface or complex tori) to such a fibration, we can relate or possibly even deform it to Beauville's examples. Thirdly, can we relate abelian fibrations X which don't admit sections to ones that do? Once again, the goal is to deform X to a more familiar manifold, which is more amenable to classification.

The programme we will describe tends to suggest that all irreducible holomorphic symplectic manifolds can be related to Beauville's examples. By "related" we don't simply mean deformed, as O'Grady's examples would contradict that. Indeed it was through trying to understand O'Grady's ten-dimensional example and its relation to the Hilbert scheme $S^{[5]}$ that the author was lead to the ideas in this paper. The author still feels that a proper understanding of this relationship will reveal many of the mysteries of holomorphic symplectic manifolds.

2. Elliptic K3 surfaces

Let S be a K3 surface. The group $\mathrm{H}^2(S, \mathbb{Z})$ is an even unimodular lattice, with quadratic form q given by intersection pairing. It has signature $(3, 19)$, so from the classification of quadratic forms it is isomorphic to $L := 3H \oplus 2(-E_8)$.

Definition 2.1. The *period* of S is the lattice $(\mathrm{H}^2(S, \mathbb{Z}), q)$ together with its weight-two Hodge structure

$$\mathrm{H}^2(S, \mathbb{Z}) \otimes \mathbb{C} = \mathrm{H}^2(S, \mathbb{C}) = \mathrm{H}^{2,0}(S, \mathbb{C}) \oplus \mathrm{H}^{1,1}(S, \mathbb{C}) \oplus \mathrm{H}^{0,2}(S, \mathbb{C}).$$

The holomorphic symplectic form σ on S spans $\mathrm{H}^{2,0}(S, \mathbb{C})$, and its complex conjugate $\bar{\sigma}$ spans $\mathrm{H}^{0,2}(S, \mathbb{C})$. Then $\mathrm{H}^{1,1}(S, \mathbb{C})$ is given by the orthogonal complement to $\mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma}$ with respect to q. Thus after choosing an identification of $(\mathrm{H}^2(S, \mathbb{Z}), q)$ with L, the period is completely determined by the complex line $[\sigma] \in \mathbb{P}(L \otimes \mathbb{C})$. This satisfies $q([\sigma]) = 0$ and $q([\sigma] + [\bar{\sigma}]) > 0$, and hence sits inside a quadric Q in $\mathbb{P}(L \otimes \mathbb{C})$. The following result, known as the Global Torelli Theorem, is originally due to Pjateckiĭ-Šapiro and Šafarevič [41] in the algebraic case; the general case is due to many authors, including Burns, Rapoport, Looijenga, Peters, and Friedman (for example, see [28] or [4]).

Theorem 2.1. *If two K3 surfaces S and S' have isomorphic periods*

$$\psi : \mathrm{H}^2(S, \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}^2(S', \mathbb{Z})$$

and moreover the image of the Kähler cone of S intersects non-trivially the Kähler cone of S', $\psi(\mathcal{K}_S) \cap \mathcal{K}_{S'} \neq \{0\}$, then there exists a unique isomorphism

$$f : S \xrightarrow{\cong} S'$$

which induces ψ.

Recall that the Kähler cone is the open cone in $\mathrm{H}^{1,1}(S, \mathbb{R})$ of all classes that can be represented by Kähler forms. If $\psi(\mathcal{K}_S) \cap \mathcal{K}_{S'} = \{0\}$ then S and S' will still be isomorphic, but the isomorphism f won't induce ψ. In this case, we first have to compose ψ with a period-preserving automorphism of the lattice, so that the assumption on the Kähler cones is satisfied.

We can define a *marked K3 surface* to be S together with an identification of the lattice $(\mathrm{H}^2(S, \mathbb{Z}), q)$ with L (called a *marking*). Two marked K3 surfaces are equivalent if there is an isomorphism between them which commutes with their markings, and we denote by $\mathcal{T}$ the moduli space of marked K3 surfaces up to equivalence. One can show that the period map

$$\mathcal{P} : \mathcal{T} \to Q \subset \mathbb{P}(L \otimes \mathbb{C}),$$

which takes a K3 surface to the complex line $[\sigma]$, is a local isomorphism, and hence $\mathcal{T}$ is locally isomorphic to a 20-dimensional complex manifold. However, it is not Hausdorff. Moreover, the actual moduli space $\mathcal{M}$ of K3 surfaces, which is obtained by quotienting $\mathcal{T}$ by the group of automorphisms of L, will be topologically even less well-behaved.

2.1. A categorization of elliptic K3 surfaces

An elliptic K3 surface S will have base $\mathbb{P}^1$. Many kinds of singular fibres are possible, but generically it will have 24 nodal $\mathbb{P}^1$s. A fibre F will be a nef divisor with $F^2 = 0$. If the fibration has a section E, which is necessarily a (-2)-curve, then $(E+F)^2$ also equals zero; however, $(E+F).E = -1$ so $E+F$ is not nef. Observe that reflection in E recovers the fibre F

$$E + F \mapsto (E + F) + ((E + F).E)E = F.$$

We recall a result of Pjateckiĭ-Šapiro and Šafarevič [41]:

Theorem 2.2. *A projective K3 surface S is elliptic if and only if there exists a non-trivial divisor D with $D^2 = 0$.*

Proof. Here (and throughout) by "non-trivial" we mean not linearly equivalent to the trivial divisor; we write this as $D \neq 0$. Riemann-Roch shows that either D or $-D$ is effective, so assume that D is effective. Fix an identification of $(\mathrm{H}^2(S, \mathbb{Z}), q)$ with L. The group of automorphisms of L which fix the period is generated by

1. the subgroup which preserves the set of nef divisors,
2. the subgroup generated by reflections in (-2)-curves,
3. the subgroup generated by $-\mathrm{Id}$.

We will show there is an automorphism belonging to the second of these subgroups which takes D to a nef divisor. Since S is projective, let H be an ample divisor and consider $D.H \in \mathbb{Z}$; it is positive as D is effective. If D is not nef, there exists an irreducible (effective) divisor A such that $D.A < 0$. This can only happen if A is a component of D and $A^2 = -2$. Thus A is a (-2)-curve and we can define

$$D_1 := D + (D.A)A$$

to be the reflection in A. This reflection preserves the positive half-cone containing H, and therefore D_1 is effective. If D_1 is not nef, we repeat the procedure. Since

$$D_1.H = D.H + (D.A)(A.H) < D.H$$

we get a decreasing sequence of positive integers $D.H$, $D_1.H$, $D_2.H$, ... and hence the procedure must eventually terminate. So we can assume that D is nef.

The next step is to show that the linear system $|D|$ contains a divisor with only one component (possibly non-reduced). This is achieved by showing that there is at least one divisor in the linear system which contains an irreducible component C with $C^2 = 0$; the ample divisor H is used again at this stage. Then one shows that $mC \in |D|$ for some $m > 0$ (see [41] for details).

Since $C^2 = 0$, Riemann-Roch shows that C moves in a pencil and we obtain a rational map $f : S \to \mathbb{P}^1$; there are no base-points as $C^2 = 0$, and hence f is a fibration. Moreover by Bertini's theorem the generic element of $|C|$ is a smooth irreducible curve, which must therefore be elliptic. $\qquad\square$

Remark 2.1. Earlier Kodaira proved that a K3 surface with Picard group generated by a non-trivial divisor D with $D^2 = 0$ must be elliptic (this is Theorem 15 in [27]). Such a surface is necessarily non-projective, as a projective elliptic K3 surface will contain a fibre and an ample divisor, and hence will have Picard number at least two.

Conversely, suppose a K3 surface contains a non-trivial divisor D with $D^2 = 0$, and has Picard number greater than one. Then the Picard group, which is a sublattice of L, must be indefinite and hence contains a divisor E with $E^2 > 0$. The surface is therefore projective.

It follows that Kodaira's result is a proof of Theorem 2.2 in the non-projective case. Another approach is to use the argument suggested in Exercise 6 on page 111 of Beauville's book [2]. Although this argument is for projective surfaces, it applies more generally. For instance, we still have Zariski decomposition for non-projective K3 surfaces by a recent result of Boucksom [6]. The author is grateful to Daniel Huybrechts for explaining this second proof of Theorem 2.2 in the non-projective setting, and to the referee for directing him to the results in Kodaira's paper [27].

Remark 2.2. Suppose we are given $x \in L \setminus \{0\}$ with $x^2 = 0$, and let S be a K3 surface with a marking

$$\phi : \mathrm{H}^2(S, \mathbb{Z}) \xrightarrow{\cong} L.$$

Recall the period of S is determined by $[\phi_{\mathbb{C}}(\sigma)] \in Q \subset \mathbb{P}(L \otimes \mathbb{C})$; then $\phi^{-1}(x)$ represents a divisor in S if and only if it has type $(1,1)$, if and only if $x.\phi_{\mathbb{C}}(\sigma) = 0$. For fixed x, this gives a hypersurface in the moduli space $\mathcal{M}$ of all K3 surfaces, so elliptic K3s occur in 19-dimensional families. Moreover, considering all such $x \in L$, one can show that the collection of all these families is dense in $\mathcal{M}$ (the argument is similar to those appearing in Section 4.7 of [24]). Indeed the denseness of elliptic K3s in the moduli space of all K3 surfaces was first proved by Kodaira [27], by deforming to non-projective elliptic K3 surfaces. We use essentially his argument in our proof of Proposition 2.3 below.

In the proof of Theorem 2.2, the key point in the first paragraph is that if there is a curve A on which D is negative, then A induces an automorphism of the lattice L (preserving the period) which gives a new divisor D_1 with $D_1.A = -D.A$. Moreover, the process terminates after a finite number of steps. This argument will fail in higher dimensions. For this reason, let us state and prove a weaker result that can be more easily generalized (this is Theorem 18 in [27]).

Proposition 2.3. *Any K3 surface can be deformed to an elliptic K3.*

Proof. Deform to a non-projective K3 surface S whose Picard group is generated by a non-trivial primitive divisor D with $D^2 = 0$ (thus D must be irreducible). That such a deformation exists follows from arguments similar to those in Section 4.7 of [24], and only uses the Local Torelli Theorem identifying $\mathcal{T}$ with Q locally. Either D or $-D$ lies on the boundary of the closure of the positive cone. For a non-projective K3 surface, the positive cone coincides with the Kähler cone (this fact has been generalized in higher dimensions by Huybrechts [23]). Thus either D or $-D$ is nef, and S contains an irreducible nef divisor with square zero.

Assuming it is D which is nef, we find

$$\mathrm{h}^2(X, \mathcal{O}(D)) = \mathrm{h}^0(X, \mathcal{O}(-D)) = 0$$

and hence Riemann-Roch shows that $\mathrm{h}^0(X, \mathcal{O}(D)) \geq 2$ and D moves in a pencil. Since $D^2 = 0$ and D is irreducible, the pencil is base-point free. It is now fairly easy to show that it is actually an elliptic fibration over $\mathbb{P}^1$ (this fact has also been partially generalized in higher dimensions by Matsushita [32, 33]). □

Remark 2.3. In higher dimensions the challenge will be to show that a particular divisor moves.

2.2. Elliptic K3 surfaces which admit sections

Every elliptic curve over $\mathbb{C}$ can be written as a plane cubic, but we'd like this description to apply to a $\mathbb{P}^1$ family of elliptic curves. This is possible provided the family has a section. One way to understand this is to regard the family as a single elliptic curve E over the function field $\mathbb{C}(\mathbb{P}^1)$ of the base; if there exists a $\mathbb{C}(\mathbb{P}^1)$-valued point then E can be written as a cubic in the projective plane over $\mathbb{C}(\mathbb{P}^1)$. A $\mathbb{C}(\mathbb{P}^1)$-valued point is precisely a section of the family.

Suppose that S is an elliptic K3 surface with a section. We assume that S has generic singular fibres, meaning that the only singular fibres are nodal or cuspidal $\mathbb{P}^1$s (type I_1 or type II, respectively, according to Kodaira's notation [26]). If this were not the case, we should first blow down all the irreducible components of singular fibres which the section does not pass through. This will leave us with a singular surface, but at least all fibres will be irreducible (note that an elliptic K3 surface cannot have multiple fibres). We can describe S as a family of plane cubics, where the planes will come from the projectivization of some rank-three vector bundle V over $\mathbb{P}^1$. By Birkhoff and Grothendieck's Theorem such a bundle necessarily splits into a direct sum of line bundles; in fact we find

$$S \subset \mathbb{P}(\mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O})$$

is given by a cubic equation

$$y^2 z = 4x^3 - axz^2 - bz^3$$

where a and b are sections of $\mathcal{O}(8)$ and $\mathcal{O}(12)$ respectively. In this equation x, y, and z are projections from V to $\mathcal{O}(4)$, $\mathcal{O}(6)$, and $\mathcal{O}$, so that

$$s := y^2 z - 4x^3 + axz^2 + bz^3$$

is a section of $\mathrm{Sym}^3 V^* \otimes \mathcal{O}(12)$ whose vanishing cuts out a cubic curve in each fibre of $\mathbb{P}(V)$. The section of S is given by $(x, y, z) = (0, 1, 0)$. Furthermore, there is a $\mathbb{C}^*$-action

$$a \mapsto \lambda^4 a \qquad \text{and} \qquad b \mapsto \lambda^6 b$$

such that all pairs (a, b) in the same orbit give isomorphic surfaces S.

The sections a and b belong to 9 and 13-dimensional vector spaces, respectively. Taking into account the $\mathbb{C}^*$-action and the 3-dimensional family $\mathrm{PGL}(2, \mathbb{C})$ of automorphisms of the base $\mathbb{P}^1$, we see that the moduli space of elliptic K3 surfaces which admit a section is 18-dimensional. Moreover, this moduli space is irreducible by a theorem of Seiler (proved in Friedman and Morgan [13], for example).

The description of an elliptic surface as a family of cubics is known as a *Weierstraß model*, and exists for any elliptic surface which admits a section (see [4], [13], [25], or almost any book on complex surfaces).

2.3. Elliptic K3 surfaces which don't admit sections

Next we wish to consider elliptic K3 surfaces which don't necessarily admit sections. Once again, we will assume that S has generic singular fibres (nodal or cuspidal $\mathbb{P}^1$s). Let $i : S_t \hookrightarrow S$ be the inclusion of a smooth fibre, and let $i_* L$ be the push-forward of a degree-zero line bundle L on S_t. Associated to S is its *relative Jacobian J*, defined as the moduli space of pure semi-stable sheaves on S with the same Hilbert polynomial as $i_* L$. Since S has generic singular fibres (in particular, all fibres are irreducible), every element of J is the push-forward of a degree-zero torsion-free sheaf on some fibre. Thus there is

a projection from J to $\mathbb{P}^1$:

$$S \qquad\qquad\qquad\qquad J$$
$$p_S \searrow \qquad \swarrow p_J$$
$$\mathbb{P}^1$$

Each fibre $J_t = \mathrm{Jac}(S_t)$ is isomorphic to the corresponding S_t, and in fact S and J are locally isomorphic as fibrations. However, the relative Jacobian J has a natural section s_0 given by the flat family $\mathcal{O}_S$ (thus $s_0(t) = \mathcal{O}_S|_{S_t} = \mathcal{O}_{S_t} \in J_t$) whereas S itself may not have a section. Let $\{U_i\}$ be an open cover of $\mathbb{P}^1$ with $S_i \cong J_i$ over U_i, where we denote $p_S^{-1}(U_i)$ and $p_J^{-1}(U_i)$ by S_i and J_i respectively. The isomorphism is not unique, but is given by the choice of a local section of S: if $s_i : U_i \to S_i$ is a local section, there exists a local isomorphism $\phi_i : S_i \to J_i$ taking s_i to s_0. If the s_i patch together to give a global section, then the ϕ_i patch together to give a global isomorphism $S \cong J$. Hence we have

Proposition 2.4. *The elliptic fibration S is isomorphic to its relative Jacobian J if and only if it admits a section.*

In general, we can regard S as being a fibration which is locally the same as J, but stuck together in a different way. In other words S is a *torsor* over J, and hence is classified by an element of $\mathrm{H}^1(\mathbb{P}^1, \mathcal{B})$ where $\mathcal{B}$ is the sheaf of local holomorphic sections of $p_J : J \to \mathbb{P}^1$ (or equivalently, the sheaf of translations in fibres, as a local section s gives a family of translations by $s - s_0$). To see this explicitly, observe that above an overlap $U_i \cap U_j$ the local isomorphisms $\phi_i : S_i \to J_i$ and $\phi_j : S_j \to J_j$ differ by a translation, and hence $\phi_i \circ \phi_j^{-1}$ is a local section of $\mathcal{B}$. Together these give a Čech 1-cocycle.

Definition 2.2. The *analytic Tate-Shafarevich group* $\mathrm{III}^{\mathrm{an}}(J)$ is the group which classifies, up to isomorphism, elliptic fibrations S with relative Jacobian J.

The group $\mathrm{III}^{\mathrm{an}}(J)$ classifies *all* such S, including those with multiple fibres. Since these additional surfaces will not be K3 surfaces, we ignore them, and we are left with $\mathrm{H}^1(\mathbb{P}^1, \mathcal{B})$. We can use the Leray spectral sequence to show that

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{B}) \cong \mathrm{H}^2(J, \mathcal{O}^*).$$

The latter is the *analytic Brauer group of J*; it is one-dimensional, but with a strange topology. More precisely, the exponential exact sequence on J gives

$$0 \to \mathrm{H}^1(J, \mathcal{O}^*) \to \mathrm{H}^2(J, \mathbb{Z}) \to \mathrm{H}^2(J, \mathcal{O}) \to \mathrm{H}^2(J, \mathcal{O}^*) \to 0.$$

The first group is the Picard group $\mathrm{Pic}(J)$, the second is the rank 22 lattice L, and the third is isomorphic to $\mathbb{C}$. Thus $\mathrm{H}^2(J, \mathcal{O}^*)$ is the quotient of $\mathbb{C}$ by a lattice which can vary from rank 20, when J has Picard number $\rho = 2$, to rank 2, when $\rho = 20$ (note that $\rho \geq 2$ since J is elliptic with a section, and so always has at least two independent divisors, namely a fibre and a section).

The extra dimension of the Brauer group, added to the 18 of the moduli space of elliptic K3 surfaces which admit sections, gives a 19-dimensional moduli space of elliptic K3 surfaces, as expected.

Remark 2.4. The Brauer group $H^2(J, \mathcal{O}^*)$ is the space of *gerbes* on J. In Section 6 we shall give a description, due to Căldăraru, of how gerbes arise in the classification of elliptic fibrations S with given relative Jacobian J.

Remark 2.5. If S is algebraic then an ample divisor induces a multi-valued section of the fibration $S \to \mathbb{P}^1$. Suppose S corresponds to an element α of the Brauer group $H^2(J, \mathcal{O}^*)$ of its relative Jacobian. If S admits a k-valued section then one can show that α must be k-torsion, and conversely. The *algebraic Brauer group* consists of the torsion elements of $H^2(J, \mathcal{O}^*)$ (equivalently, we can define it as $H^2_{\text{ét}}(J, \mathcal{O}^*)$, where we calculate cohomology in the étale topology). Thus algebraic elliptic fibrations correspond to elements of the algebraic Brauer group, which is dense in the analytic Brauer group.

Once again, this theory is not specific to K3 surfaces, but applies to general elliptic surfaces, and was first developed in this form by Kodaira [26] (alternatively see [4], [13], or [25]).

This concludes what we want to say about elliptic K3 surfaces. In summary, the three main points are

1. any K3 surface can be deformed to an elliptic K3 surface,
2. elliptic K3 surfaces which admit sections are described by an 18-dimensional *connected* moduli space,
3. an elliptic K3 surface which does not admit a section can be connected to its relative Jacobian by a one-dimensional family.

In particular, all K3 surfaces are deformation equivalent. In the rest of this article we will formulate generalizations of these statements for higher dimensional irreducible holomorphic symplectic manifolds.

3. Irreducible holomorphic symplectic manifolds

K3 surfaces are compact, Kähler, simply-connected, and have non-degenerate holomorphic two-forms. All of these notions generalize to higher dimensions.

Definition 3.1. A *holomorphic symplectic manifold* X is a compact Kähler manifold which admits a non-degenerate holomorphic two-form $\sigma \in H^0(X, \Lambda^2 T^*)$, known as the *holomorphic symplectic form*. If X is simply-connected and cannot be written as a product $Y \times Z$ of two manifolds, then we shall say X is *irreducible*.

Suppose X has dimension $2n$. By non-degeneracy, σ^n is a trivialization of the canonical bundle $\mathcal{K} = \Lambda^{2n} T^*$. Therefore $c_1(T) = 0$ and by Yau's theorem X admits a Ricci-flat metric. In fact, this metric is *hyperkähler*, and thus compact hyperkähler manifolds are really the differential geometric equivalents of holomorphic symplectic manifolds.

3.1. Examples

The following two families of examples are due to Beauville [1].

Example 3.1. Obviously a K3 surface S is an irreducible holomorphic symplectic manifold. The *Hilbert scheme of n points on S*, denoted $S^{[n]}$ is the moduli space parameterizing length n zero-dimensional subschemes of S. A typical element consists of n distinct unordered points of S, though the behaviour when some of the points collide is more complicated. For example, $S^{[2]}$ is given by blowing up the diagonal in $S \times S$ and then quotienting by the involution which exchanges the two factors (this was the first higher dimensional holomorphic symplectic manifold, discovered by Fujiki [12]). The Hilbert scheme $S^{[n]}$ is a smooth resolution of $\mathrm{Sym}^n S$. It is an irreducible holomorphic symplectic manifold of dimension $2n$.

Example 3.2. Let T be an abelian surface, or more generally a complex torus of dimension two. The Hilbert scheme $T^{[n+1]}$ is a holomorphic symplectic manifold, but it is not irreducible. There is a map π to T given by composing the map $T^{[n+1]} \to \mathrm{Sym}^{n+1} T$ with the group structure on T. The fibres are all isomorphic. Define the *generalized Kummer variety K_n* to be $\pi^{-1}(0)$. It is an irreducible holomorphic symplectic manifold of dimension $2n$. When $n = 1$, we find K_1 is a Kummer K3 surface, whence the name.

The following example is due to Mukai [34, 35].

Example 3.3. Let S be a K3 surface. We saw that there is a quadratic form q on $\mathrm{H}^2(S, \mathbb{Z})$ given by intersection pairing. We extend this to a quadratic form on $\mathrm{H}^\bullet(S, \mathbb{Z})$ by defining

$$q(v, w) := \int_S (-v_0 w_4 + v_2 w_2 - v_4 w_0)$$

where v_i and $w_i \in \mathrm{H}^i(S, \mathbb{Z})$. For a sheaf $\mathcal{E}$ on S, define the *Mukai vector $v(\mathcal{E}) \in \mathrm{H}^\bullet(S, \mathbb{Z})$* to be the product $\mathrm{ch}(\mathcal{E})\mathrm{Td}^{1/2}$ of the Chern character of $\mathcal{E}$ with the square root of the Todd class of S. For example, if $\mathcal{E}$ is locally free of rank r, with Chern classes c_1 and c_2, then

$$v(\mathcal{E}) = (r, c_1, r + c_1^2/2 - c_2).$$

For $v \in \mathrm{H}^\bullet(S, \mathbb{Z})$, define $\mathcal{M}^s(v)$ to be the moduli space of stable sheaves $\mathcal{E}$ on S with Mukai vector $v(\mathcal{E}) = v$. Then $\mathcal{M}^s(v)$ is smooth of dimension $2n = q(v, v) + 2$ and has a holomorphic symplectic form. It is known that for various choices of v, $\mathcal{M}^s(v)$ is birational to the Hilbert scheme $S^{[n]}$: this is immediate for rank one, was proved for various rank two moduli spaces in [14], and for arbitrary rank and elliptic K3s it was proved in [37] (see also Section 11.3 in [20]). However, in general the moduli space of stable sheaves need not be compact. To compactify we have to add semi-stable sheaves, and this may result in singularities.

However, if v is primitive then every semi-stable sheaf $\mathcal{E}$ with Mukai vector $v(\mathcal{E}) = v$ will automatically be stable. So in this case $\mathcal{M}^s(v)$ is an irreducible holomorphic symplectic manifold. On the other hand, under these conditions it is also known that $\mathcal{M}^s(v)$ is a deformation of the Hilbert scheme $S^{[n]}$, where $2n = q(v, v) + 2$. This follows from the work of Göttsche, Huybrechts, and O'Grady [14, 22, 37] (the most general statement is due to Yoshioka [47]).

Mukai's results also apply to the moduli space of sheaves on an abelian surface. Unfortunately, when we can show that the moduli space is smooth and compact, it also happens to be a deformation of the generalized Kummer variety (see Yoshioka [48], for example).

The first known example not deformation equivalent to a Hilbert scheme $S^{[n]}$ or a generalized Kummer variety K_n is due to O'Grady [38].

Example 3.4. Consider the previous example with Mukai vector $v = (2, 0, -2)$. This vector is not primitive, and hence there exist strictly semi-stable sheaves which we add to $\mathcal{M}^s(v)$ to get the *singular* moduli space of semi-stable sheaves $\mathcal{M}^{ss}(v)$. O'Grady proved that $\mathcal{M}^{ss}(v)$ admits a symplectic desingularization $\widetilde{\mathcal{M}}$, which is a ten-dimensional irreducible symplectic manifold. He also showed that $b_2(\widetilde{\mathcal{M}}) \geq 24$, and hence $\widetilde{\mathcal{M}}$ is not a deformation of the Hilbert scheme $S^{[5]}$ or the generalized Kummer variety K_5, which have second Betti numbers equal to 23 and 7, respectively.

The only other known example comes from a similar construction starting with an abelian surface (see O'Grady [39] for details). It is six-dimensional and has $b_2 = 8$.

3.2. Moduli spaces

Moduli spaces of irreducible holomorphic symplectic manifolds have been extensively studied [22, 24]. Although the Global Torelli Theorem is false in general, the theory is nonetheless very similar to the K3 case. For an arbitrary irreducible holomorphic symplectic manifold X, there exists a quadratic form q_X on $\mathrm{H}^2(X, \mathbb{Z})$ known as the Beauville-Bogomolov form [1]; this generalizes the intersection pairing on $\mathrm{H}^2(S, \mathbb{Z})$ when S is a K3 surface. So $(\mathrm{H}^2(X, \mathbb{Z}), q_X)$ is once again a lattice that we write abstractly as (Γ, q_Γ) (or simply as Γ). As in Definition 2.1, we can define the period of X as $(\mathrm{H}^2(X, \mathbb{Z}), q_X)$ together with its weight-two Hodge structure, and after identifying $\mathrm{H}^2(X, \mathbb{Z})$ with Γ, this is once again determined by the complex span of the holomorphic symplectic form $[\sigma] \in \mathbb{P}(\Gamma \otimes \mathbb{C})$.

As with K3 surfaces, there is a period map

$$\mathcal{P}_\Gamma : \mathcal{T}_\Gamma \to Q_\Gamma := \{x \in \mathbb{P}(\Gamma \otimes \mathbb{C}) | q_\Gamma(x) = 0, \ q_\Gamma(x + \bar{x}) > 0\}$$

from the moduli space $\mathcal{T}_\Gamma$ of *marked* irreducible holomorphic symplectic manifolds to a quadric in $\mathbb{P}(\Gamma \otimes \mathbb{C})$. Beauville showed the period map is a local isomorphism, and hence $\mathcal{T}_\Gamma$ is locally isomorphic to a complex manifold of dimension $b_2 - 2$ (where b_2 is the second Betti number of X), though as with K3 surfaces $\mathcal{T}_\Gamma$ is not Hausdorff.

Huybrechts [22] proved surjectivity of the period map, and also showed that nonseparated points in $\mathcal{T}_\Gamma$ correspond to birational (marked) manifolds X and X'. After quotienting $\mathcal{T}_\Gamma$ by automorphisms of Γ to obtain $\mathcal{M}_\Gamma$ the converse is also true: (unmarked) X and X' are birational if and only if they correspond to non-separated points in $\mathcal{M}_\Gamma$. This also implies the important result that two birational holomorphic symplectic manifolds X and X' are deformation equivalent.

Finally, let us also note that the Global Torelli Theorem as formulated for K3 surfaces (Theorem 2.1) is false in higher dimensions: Namikawa [36] has constructed a counterexample consisting of two non-birational holomorphic symplectic manifolds with isomorphic periods.

3.3. Abelian fibrations

Definition 3.2. By *abelian fibration* on a $2n$-dimensional irreducible holomorphic symplectic manifold X we shall mean the structure of a fibration over $\mathbb{P}^n$ whose generic fibre is a smooth abelian variety of dimension n.

This shall be our higher dimensional analogue of elliptic fibrations on K3 surfaces. At first sight, the definition may appear to be unnecessarily restrictive. For example, maybe we should allow the base to be a more general n-fold than $\mathbb{P}^n$, or even to have dimension different to n. However, this is more-or-less the only fibration structure that can exist on an irreducible holomorphic symplectic manifold, by the following result of Matsushita [32, 33].

Theorem 3.1. *For projective X, let $f : X \to B$ be a proper surjective morphism such that the generic fibre F is connected. Assume that B is smooth and $0 < \dim B < \dim X$. Then*

1. *F is an abelian variety up to a finite unramified cover,*
2. *B is n-dimensional and has the same Hodge numbers as $\mathbb{P}^n$,*
3. *the fibration is Lagrangian with respect to the holomorphic symplectic form.*

In particular, if X is 4-dimensional, we can use the Castelnuovo-Enriques classification of surfaces to deduce that the generic fibre is an abelian surface and the base is $\mathbb{P}^2$ (this was also proved by Markushevich in [30], but under the assumption that the fibration is Lagrangian). Assuming only that B is normal, Matsushita also deduced slightly weaker results.

Matsushita's theorem provides some justification for assuming the base must be $\mathbb{P}^n$. Moreover, most of (perhaps all of) the examples of irreducible holomorphic symplectic manifolds described in Section 3.1 can be deformed to abelian fibrations.

Example 3.5. Both Examples 3.1 and 3.2, the Hilbert scheme of points on a K3 surface and the generalized Kummer variety, are abelian fibrations when the underlying K3 surface S or complex tori T, respectively, is an elliptic surface. For example, if $f : S \to \mathbb{P}^1$ is the fibration on S, we get an induced fibration

$$f^{[n]} : S^{[n]} \to \mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$$

on $S^{[n]}$. However, the fibres in this case are products of n elliptic curves: still n-dimensional abelian varieties, but quite a special case. A similar thing happens for the generalized Kummer variety. We regard these abelian fibrations as rather degenerate; there are 'better' abelian fibrations on these manifolds, as the next two examples show.

Example 3.6. Let S be a K3 surface which contains a smooth genus g curve C. Let L be a degree g line bundle on a smooth curve D in the linear system $|C|$, and consider i_*L where i is the inclusion of D in S. Let Z be the moduli space of rank-one torsion sheaves on S which have the same type as i_*L. In other words, Z is the Mukai moduli space $\mathcal{M}^s(0, [C], 1)$. Since $v = (0, [C], 1)$ is primitive, the moduli space is smooth and compact. Clearly it has the structure of an abelian fibration, with base given by the linear system $|C| \cong \mathbb{P}^g$ and fibre over $D \in |C|$ given by $\mathrm{Pic}^g D$:

$$\begin{array}{ccc} \mathrm{Pic}^g & \hookrightarrow & Z \\ & & \downarrow \\ & & |C| \cong \mathbb{P}^g \end{array}$$

From the comments in Example 3.3, we know that Z is deformation equivalent to the Hilbert scheme $S^{[g]}$. It is also birational to the Hilbert scheme. To see this, note that by Riemann-Roch a generic degree g line bundle on a genus g curve has a unique section up to scale, which vanishes at precisely g points. This gives a map from a (Zariski) open subset U of Z to $S^{[g]}$. To get a map in the opposite direction, recall that for $g \geq 3$, S is embedded in the dual $(\mathbb{P}^g)^\vee$ of the linear system $|C|$ (for $g = 2$, S is a double cover of $(\mathbb{P}^2)^\vee$ branched over a sextic). A generic collection of g points on S defines a hyperplane, which intersects S in a curve $D \in |C|$. The g points lie on this curve and define a degree g line bundle on D. Hence we have a map from an open subset of $S^{[g]}$ to Z.

Remark 3.7. The birational map in Example 3.6 is a *generalized Mukai flop*, and was studied by Markman [29]. Also, recall that two birational holomorphic symplectic manifolds are automatically deformation equivalent, by the results of Huybrechts [22].

Example 3.8. For generalized Kummer varieties we can do something similar. We begin with an abelian surface T which contains a genus $g + 2$ curve C. Let Y' be the moduli space of rank-one torsion sheaves on T, which have the same type as push-forwards of degree $g + 2$ line bundles on smooth curves in the linear system $|C|$. This is the Mukai moduli space $\mathcal{M}^s(0, [C], 1)$ of sheaves on T, and it is smooth and compact. However, Y' is not irreducible. Since a degree $g + 2$ line bundle on a genus $g + 2$ curve C determines, generically, $g + 2$ points on C, we obtain a map π

$$Y' \to \mathrm{Sym}^{g+2}T \to T$$

where the second map is given by the group structure on T. This is just the Albanese map of Y', and the fibres are all isomorphic. Then $Y := \pi^{-1}(0)$ is an irreducible holomorphic symplectic manifold of dimension $2g$. Moreover, it is a deformation of the generalized Kummer variety K_g, by results of Yoshioka [48].

We will show that Y is an abelian fibration. Since the linear system of curves $|C|$ on T is g-dimensional, Y' has a fibration structure

$$\begin{array}{ccc} \mathrm{Pic}^{g+2} & \hookrightarrow & Y' \\ & & \downarrow \\ & & |C| \cong \mathbb{P}^g \end{array}$$

which induces a fibration of Y

$$
\begin{array}{ccc}
Y \cap \mathrm{Pic}^{g+2} & \hookrightarrow & Y \\
& & \downarrow \\
& & |C| \cong \mathbb{P}^g
\end{array}
$$

Moreover, when we restrict π to the fibre $\mathrm{Pic}^{g+2}D$ of $Y' \to \mathbb{P}^g$, $D \in |C|$, it still surjects onto T. Therefore the induced fibre of Y

$$
Y \cap \mathrm{Pic}^{g+2}D = \ker(\pi|_{\mathrm{Pic}^{g+2}D} : \mathrm{Pic}^{g+2}D \to T)
$$

will be a g-dimensional abelian variety. Therefore the generalized Kummer variety K_g can be deformed to an abelian fibration Y.

Remark 3.9. In Example 3.6 the fibres of Z (which is a deformation of $S^{[g]}$) are principally polarized abelian varieties, as they are Jacobians of curves. In Example 3.8 the fibres of Y (which is a deformation of K_g) are not Jacobians; they are given by

$$
Y_D := \ker(\pi|_{\mathrm{Pic}^{g+2}D} : \mathrm{Pic}^{g+2}D \to T)
$$

where $D \in |C|$. Assume that T is polarized by the curve C, and let the type of this polarization be (d_1, d_2) where $d_1|d_2$ (see Birkenhake and Lange [5]). If we assume that C is reduced, then from

$$
d_1 d_2 = \frac{C.C}{2} = g + 1.
$$

we conclude that $(d_1, d_2) = (1, g + 1)$. Therefore the (complementary) polarization of Y_D has type $(1, \ldots, 1, g + 1)$. In particular, the fibres of Y are never principally polarized. We shall return to this point later.

Example 3.10. Let S be a K3 surface which is the double cover of $\mathbb{P}^2$ branched over a sextic. The pull-back of a conic from $\mathbb{P}^2$ is a genus five curve C on S. In [38], O'Grady showed that an open subset of his moduli space $\widetilde{\mathcal{M}}$ (from Example 3.4) is birational to an open subset of the Mukai moduli space $\mathcal{M}^s(0, [C], 2)$ of sheaves on S. The latter is the moduli space of rank-one torsion sheaves with the same type as push-forwards of degree 6 line bundles on smooth curves in $|C|$. In this case $v = (0, [C], 2)$ is not primitive, as curves in $|C|$ can be non-reduced (for example, the pull-back of a double line). So a priori, it is not clear that the moduli space $\mathcal{M}^s(0, [C], 2)$ can be completed to a smooth irreducible holomorphic symplectic manifold. More precisely, adding strictly semi-stable sheaves will give a singular space $\mathcal{M}^{ss}(0, [C], 2)$, and it is not known whether a symplectic desingularization of this space exists (though presumably one can use the same kind of desingularization as for $\widetilde{\mathcal{M}}$).

In any case, the incomplete space $\mathcal{M}^s(0, [C], 2)$ obviously has the structure of an abelian fibration

$$
\begin{array}{ccc}
\mathrm{Pic}^6 & \hookrightarrow & \mathcal{M}^s(0, [C], 2) \\
& & \downarrow \\
& & |C| \cong \mathbb{P}^5
\end{array}
$$

Since birational varieties are deformation equivalent, it appears probable that O'Grady's space $\widetilde{\mathcal{M}}$ can be deformed to an abelian fibration.

There is likely to be a similar description for O'Grady's other space [39]. If we could make these arguments rigorous, it would imply that all the *currently known* examples of irreducible holomorphic symplectic manifolds can be deformed to abelian fibrations. This is the first step in a (conjectural) three-part programme to understand irreducible holomorphic symplectic manifolds, which we now summarize

1. given X, deform it to an abelian fibration,
2. classify those abelian fibrations which admit sections,
3. connect abelian fibrations which don't admit sections to corresponding fibrations which do by one-dimensional families.

We shall address each of these problems in the next three sections.

4. Deforming to abelian fibrations

In this section, we address the question of whether an arbitrary irreducible holomorphic symplectic manifold can be deformed to an abelian fibration.

4.1. Divisors on abelian fibrations

By Theorem 2.2 a (projective) K3 surface S is elliptic if and only if it admits a non-trivial divisor whose square is zero. A fibre F of the elliptic fibration is an example of such a divisor, though as we saw at the beginning of Section 2.1, not the only example. What special divisors exist on an abelian fibred irreducible holomorphic symplectic manifold X^{2n}? The most obvious choice is the pull-back of a hyperplane from the base $\mathbb{P}^n$; call this D. In fact, in the most generic situation the Picard group of X will be generated by D.

Observe that D^n is non-trivial, as it is rationally equivalent to a fibre of X, but $D^{n+1} = 0$. For a divisor E, the largest integer k such that E^k is non-trivial is known as the *numerical Iitaka dimension* of E (this is sometimes referred to as the *numerical Kodaira dimension*, but following Esnault and Viehweg [11], we reserve such terminology for the canonical divisor). Thus D has numerical Iitaka dimension n.

For an arbitrary irreducible holomorphic symplectic manifold X^{2n}, Verbitsky [46] proved that the symmetric product $\mathrm{Sym}^\bullet \mathrm{H}^2(X, \mathbb{C})$ of the second cohomology group injects into the cohomology ring $\mathrm{H}^\bullet(X, \mathbb{C})$ up to the middle dimension $\mathrm{H}^{2n}(X, \mathbb{C})$. It follows that given an arbitrary non-trivial divisor D on X, D^n must be non-trivial. On the other hand, Verbitsky also showed that $D^{n+1} = 0$ if and only if $q_X(D) = 0$, where q_X is the Beauville-Bogomolov quadratic form. Fujiki's formula (see [22], for example) says that

$$\int_X D^{2n} \;\; = \;\; c_X q_X(D)^n \tag{1}$$

for all divisors, where c_X is a positive real scalar (independent of D) known as the *Fujiki constant*. So if $D^k = 0$ for $k \geq n+1$, then $q_X(D) = 0$ and hence $D^{n+1} = 0$. In summary, a divisor D on X^{2n} will have numerical Iitaka dimension equal to

1. 0 if D is trivial,
2. n if $q_X(D) = 0$,
3. $2n$ otherwise.

We saw above that an abelian fibration X has a non-trivial divisor whose square is zero with respect to q_X, just as an elliptic K3 surface S has a divisor whose square is zero with respect to the intersection pairing. Could the converse be true?

Example 4.1. Let $g : S \to \mathbb{P}^2$ be a K3 surface which is a double cover of the plane branched over a sextic. The pull-back of a line from $\mathbb{P}^2$ is a genus two curve C, and we saw in Example 3.6 that the moduli space $\mathcal{M}^s(0, [C], 1)$ is birational to the Hilbert scheme $S^{[2]}$. The birational transformation is an example of a Mukai flop [34]. It is given by first blowing-up the locus

$$\{\mathcal{K}_D | D \in |C|\} \cong (\mathbb{P}^2)^\vee \subset \mathcal{M}^s(0, [C], 1)$$

consisting of canonical bundles (note that $|C|$ is the set of lines in the original $\mathbb{P}^2$, and hence this locus is isomorphic to the dual plane $(\mathbb{P}^2)^\vee$). The exceptional locus of the blow-up is isomorphic to

$$\{(x, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee | x \in l\}$$

and can be blown-down in a different direction to produce the locus

$$G := \{g^{-1}(x) | x \in \mathbb{P}^2\} \cong \mathbb{P}^2 \subset S^{[2]}$$

(more precisely, G is the closure of the set of $g^{-1}(x)$ for x in the complement of the sextic branching curve).

Now $\mathcal{M}^s(0, [C], 1)$ is an abelian fibration over $(\mathbb{P}^2)^\vee$. Choose a point $w \in \mathbb{P}^2$; it will corresponds to a hyperplane in $(\mathbb{P}^2)^\vee$ and thus we obtain a divisor D in $\mathcal{M}^s(0, [C], 1)$ whose square is zero. Since birational maps between holomorphic symplectic manifolds are isomorphisms in codimension two (see for instance [22] or Huybrechts' notes in [15]), there must be a corresponding divisor D' in $S^{[2]}$ and it is easy to see that

$$D' = \{\xi \in S^{[2]} | w, y, \text{ and } z \text{ are collinear, where } \{y, z\} = g(\mathrm{supp}(\xi))\}$$

(once again, D' should really be defined as the closure of the set of such ξ with y and z distinct). From this description, one can see that the rational equivalence class of $(D')^2$ is not base-point free; indeed D' always contains the locus

$$G = \{g^{-1}(x) | x \in \mathbb{P}^2\} \cong \mathbb{P}^2$$

regardless of the choice of w. So $(D')^2$ does not represent a fibre and the structure of an abelian fibration is lost under the birational transform (of course, we have only shown that D' is not the pull-back of a hyperplane from the base of an abelian fibration; $S^{[2]}$ might still have a fibred structure unrelated to D').

Why does D' not induce an abelian fibration on $S^{[2]}$? This appears to be because D' is not nef: any curve in the base-point locus G will intersect D' negatively. On the other hand, D is nef: a curve in $\mathcal{M}^s(0, [C], 1) \to (\mathbb{P}^2)^\vee$ is either contained in a fibre and has zero intersection with D, or is a finite cover of a curve in $(\mathbb{P}^2)^\vee$ and intersects D positively.

This example highlights two important points. Firstly, if X contains a divisor D with $q_X(D) = 0$, and we want to show that X is an abelian fibration, then it will probably help to assume that D is nef. Secondly, if $q_X(D) = 0$ but D is not nef, then it may still be possible to find a birational model X' of X which contains a nef divisor whose square is zero, and which is an abelian fibration. We state this in the following conjectures.

Conjecture 4.1. *An irreducible holomorphic symplectic manifold X is an abelian fibration if and only if it contains a non-trivial nef divisor D whose square with respect to the Beauville-Bogomolov quadratic form is zero, $q_X(D) = 0$.*

Remark 4.2. Clearly an abelian fibration contains such a divisor, namely the pull-back of a hyperplane from the base. We will try to argue (in this subsection and the next) that the right kind of vanishing theorems will imply the converse.

Conjecture 4.2. *Any irreducible holomorphic symplectic manifold X' which contains a non-trivial divisor D' with $q_X(D') = 0$ is birational to an abelian fibration X. Moreover, if D is the divisor corresponding to D', then there exists a period-preserving automorphism τ of the lattice $\mathrm{H}^2(X, \mathbb{Z})$ such that $\tau(D)$ is nef.*

Remark 4.3. In Example 4.1 the divisor D corresponding to D' is already nef, but as we saw in the proof of Theorem 2.2 for elliptic K3s, the automorphism τ is necessary in general.

Conjecture 4.2 is based on observations like in Example 4.1 and on our insight from the K3 case. Let us look instead at Conjecture 4.1. We saw already that a non-trivial divisor D with $q_X(D) = 0$ has numerical Iitaka dimension n. Consider the map to projective space

$$\phi_{mD} : X \to \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}(mD)))$$

given by the sections of $\mathcal{O}(mD)$, for $m > 0$. The dimension of the image of ϕ_{mD}, in the limit as $m \to \infty$, is known as the *Iitaka dimension* of D. We want to show that the Iitaka dimension of D is also n, as it is the map ϕ_{mD} which ought to give the abelian fibration with base space $\mathbb{P}^n$ (of course we also have to prove that ϕ_{mD}, which is a priori only a rational map, is a morphism). The Iitaka dimension of a divisor is never greater than the numerical Iitaka dimension [11], so our aim is to prove the reverse inequality in the case that D is nef.

Because of Matsushita's Theorem 3.1 cited above, a non-trivial fibration of X has to be over an n-dimensional base. So as soon as some $\mathrm{H}^0(X, \mathcal{O}(mD))$ has dimension greater than one, and we show ϕ_{mD} is a genuine morphism, we can conclude that the Iitaka dimension is indeed n. One way to show a line bundle has sections is to show that its higher cohomology vanishes, and we will look at vanishing theorems in the next subsection. First let us indicate why Conjecture 4.1 is important for the study of deformation classes of holomorphic symplectic manifolds.

Proposition 4.3. *Every irreducible holomorphic symplectic manifold X' with second Betti number $b_2(X') \geq 5$ can be deformed to another, X, such that X contains a non-trivial nef divisor D with $q_X(D) = 0$.*

Proof. The idea is essentially the same as the one used to prove Proposition 2.3. The lattice $(\mathrm{H}^2(X', \mathbb{Z}), q_{X'}) \cong (\Gamma, q_\Gamma)$ has rank at least five and is indefinite (more precisely, it has signature $(3, b_2(X') - 3)$). By Meyer's Theorem (see [8], page 75) it is isotropic, ie. it contains a non-trivial element whose square is zero. Using an argument similar to those in Section 4.7 of [24], we deform to a non-projective X whose Picard group is generated by a primitive divisor D with $q_X(D) = 0$. Either D or $-D$ lies on the boundary of the closure of the positive cone, which coincides with the closure of the Kähler cone by Huybrechts' result [23]. Thus either D or $-D$ is nef, and we are done. $\square$

Remark 4.4. The author is grateful to the referee for pointing out Meyer's Theorem. There do exist quadratic forms of signature $(3, 0)$ and $(3, 1)$ which are not isotropic, and therefore the proof fails when $b_2(X')$ is three or four. We don't currently know whether there exist irreducible holomorphic symplectic manifolds with those Betti numbers.

4.2. Vanishing theorems

Huybrechts [22] proved a Riemann-Roch formula for holomorphic symplectic manifolds which says

$$\chi(\mathcal{O}(D)) = \sum_{i=0}^{n} a_{2i}(q_X(D))^i$$

for constants a_{2i}. Thus given D with $q_X(D) = 0$ we have

$$\chi(\mathcal{O}(mD)) = a_0 = \chi(\mathcal{O}_X) = n + 1.$$

Let $\mathrm{h}^i(mD) := \dim \mathrm{H}^i(X, \mathcal{O}(mD))$; then

$$\sum_{i=0}^{2n} (-1)^i \mathrm{h}^i(mD) = n + 1.$$

In the case that $m = 1$ and D is non-trivial and primitive, we want to show that $\mathrm{h}^0(D) = n + 1$, and obviously it suffices to show that $\mathrm{h}^i(D) = 0$ for $i > 0$ (the calculations in the next subsection indicate that this is the expected behaviour). In fact, it is enough to show this for even i, as this would imply $\mathrm{h}^0(D) \geq n + 1$. So we have to prove a vanishing theorem for a line bundle with some special properties (nef and $q_X(D) = 0$) on a holomorphic symplectic manifold.

If D were ample the vanishing result would be immediate. Instead we are in the 'semi-positive' situation of having a nef line bundle. For projective X, the Kawamata-Viehweg vanishing theorem would imply that $\mathrm{h}^i(D) = 0$ for i greater than the numerical Iitaka dimension of D, which in this case is n (for example, see [11]). For a K3 surface this suffices, as it proves $\mathrm{h}^i(D)$ vanishes for all even $i > 0$; but as soon as $n \geq 2$ we still have some work to do. This is as much as we can expect to learn from general vanishing theorems, as we have used both the fact that D is nef and that it has numerical Iitaka dimension n. In the non-projective case, we even lack a full analogue of the Kawamata-Viehweg theorem at the moment (cf. [9]).

To make further progress we need to use some other property of X, and there is really only one possibility, the holomorphic symplectic form

$$\sigma \in \mathrm{H}^0(X, \Lambda^2 T^*) = \mathrm{H}^{2,0}(X).$$

Take its complex conjugate

$$\bar{\sigma} \in \mathrm{H}^{0,2}(X) = \mathrm{H}^2(X, \mathcal{O}_X).$$

Verbitsky [44] showed that tensoring with σ gives a map on cohomology, which together with the adjoint map generates a holomorphic Lefschetz action on the cohomology of a holomorphic symplectic manifold. (This is just part of a more elaborate $\mathrm{SO}(4,1)$ action; see [44].) Might it be possible to extend this action in some way to the cohomology of line bundles on X?

It is certainly possible to define a map on smooth forms with values in a sheaf $\mathcal{F}$

$$L : \mathrm{C}^\infty(\mathcal{E}^{p,q} \otimes \mathcal{F}) \to \mathrm{C}^\infty(\mathcal{E}^{p,q+2} \otimes \mathcal{F})$$

by multiplying with $\bar{\sigma}$; it also has an adjoint

$$L^* : \mathrm{C}^\infty(\mathcal{E}^{p,q+2} \otimes \mathcal{F}) \to \mathrm{C}^\infty(\mathcal{E}^{p,q} \otimes \mathcal{F}).$$

Moreover, $\bar{\sigma}$ is $\bar{\partial}$-closed (in fact, parallel) so we get induced maps on cohomology for which we will use the same notation.

Definition 4.1. Define the *sheaf Lefschetz action* to be the map on sheaf cohomology induced by multiplication by $\bar{\sigma}$

$$L : \mathrm{H}^q(X, \Omega^p \otimes \mathcal{F}) \to \mathrm{H}^{q+2}(X, \Omega^p \otimes \mathcal{F})$$

and the map induced by the adjoint to multiplication by $\bar{\sigma}$

$$L^* : \mathrm{H}^{q+2}(X, \Omega^p \otimes \mathcal{F}) \to \mathrm{H}^q(X, \Omega^p \otimes \mathcal{F}).$$

Remark 4.5. A priori, these maps may simply be trivial at the level of cohomology.

We are interested in the case $p = 0$ and $\mathcal{F} = \mathcal{O}(D)$ for a nef divisor D. The most desirable behaviour for our purposes is that L^* is injective. If this is the case, then we see immediately that $\mathrm{h}^0(D)$ must be at least as large as any other $\mathrm{h}^i(D)$ for i even. For X projective, the Kawamata-Viehweg vanishing theorem gives $\mathrm{h}^i(D) = 0$ for $i > n$. We really only need $\mathrm{h}^{2n}(D) = 0$, and in fact this holds for X non-projective too, provided it is compact and Kähler (see Demailly and Peternell [9]). We already know

$$\sum_{i \text{ even}} h^i(D) = n + 1 + \sum_{i \text{ odd}} h^i(D) \geq n + 1$$

and since $\mathrm{h}^{2n}(D) = 0$ there are at most n non-zero terms on the left hand side. Therefore $\mathrm{h}^0(D)$ is at least two, and the map ϕ_D has positive dimensional image. As mentioned earlier, provided ϕ_D is a genuine morphism, Matsushita Theorem 3.1 then implies that D has Iitaka dimension n.

4.3. Some calculations

Let us compare our 'hoped-for' behaviour of the previous subsection with what happens in practice. Let $\pi : X \to \mathbb{P}^n$ be an abelian fibred irreducible holomorphic symplectic manifold, and let D be the nef divisor with $q_X(D) = 0$ given by pulling-back a hyperplane. We will compute $\mathrm{h}^i(mD)$ using the Leray spectral sequence, which gives

$$\mathrm{H}^p(\mathbb{P}^n, R^q\pi_*\mathcal{O}(mD)) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{O}(mD)).$$

The left hand side equals

$$\mathrm{H}^p(\mathbb{P}^n, R^q\pi_*\pi^*\mathcal{O}_{\mathbb{P}^n}(m)) = \mathrm{H}^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m) \otimes R^q\pi_*\mathcal{O}_X).$$

For $q = 0$

$$R^0\pi_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^n}.$$

For $q = 1$, observe that the fibre of $R^1\pi_*\mathcal{O}_X$ over a point $x \in \mathbb{P}^n$ is

$$\mathrm{H}^1(\pi^{-1}(x), \mathcal{O}_{\pi^{-1}(x)}) \cong \overline{\mathrm{H}^0(\pi^{-1}(x), \Omega^1_{\pi^{-1}(x)})}.$$

The holomorphic symplectic form on X identifies the tangent T_X and cotangent Ω^1_X bundles. Restricting to $\pi^{-1}(x)$ these decompose into

$$\pi^*T_{\mathbb{P}^n} \oplus T_{\pi^{-1}(x)} \qquad \text{and} \qquad \pi^*\Omega^1_{\mathbb{P}^n} \oplus \Omega^1_{\pi^{-1}(x)}$$

and since the fibres are holomorphic Lagrangian, we can identify $\Omega^1_{\pi^{-1}(x)}$ with $\pi^*T_{\mathbb{P}^n}$. Thus

$$(R^1\pi_*\mathcal{O}_X)_x \cong \overline{\mathrm{H}^0(\pi^{-1}(x), \pi^*T_{\mathbb{P}^n})} = \overline{(T_{\mathbb{P}^n})_x}.$$

Finally, the Kähler form identifies the antiholomorphic tangent space with $(\Omega^1_{\mathbb{P}^n})_x$, and so

$$R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}_n}.$$

More generally, Matsushita [33] proved that

$$R^q\pi_*\mathcal{O}_X \cong \Omega^q_{\mathbb{P}_n}.$$

However, since it is more cumbersome to calculate the cohomology of these sheaves on $\mathbb{P}^n$ for $q \geq 2$, we will restrict to small values of n.

Example 4.6. When X is an elliptic K3 surface ($n = 1$) all terms in the spectral sequence vanish apart from $(p, q) = (0, 0)$ and $(0, 1)$, and we find

$$\mathrm{h}^i(mD) = \begin{cases} m + 1 & i = 0 \\ m - 1 & i = 1 \\ 0 & i = 2. \end{cases}$$

Example 4.7. When $n = 2$ we can calculate the $q = 2$ term

$$\begin{aligned} R^2\pi_*\mathcal{O}_X &\cong R^2\pi_*(\mathcal{K}_X \otimes \pi^*\mathcal{K}^\vee_{\mathbb{P}^2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-3)) \\ &\cong \mathcal{O}_{\mathbb{P}^2}(-3) \otimes R^2\pi_*\mathcal{K}_{X/\mathbb{P}^2} \\ &\cong \mathcal{O}_{\mathbb{P}^2}(-3) \otimes (R^0\pi_*\mathcal{O}_X)^\vee \\ &\cong \mathcal{O}_{\mathbb{P}^2}(-3) \end{aligned}$$

where we have used the triviality of the canonical bundle $\mathcal{K}_X$ and Serre duality. All terms in the spectral sequence vanish apart from $(p, q) = (0, 0)$, $(0, 1)$, and $(0, 2)$. We can calculate the dimensions of these remaining spaces using the Bott-Borel-Weil theorem (for example, see [10]) and hence we find

$$\mathrm{h}^i(mD) = \begin{cases} (m+2)(m+1)/2 & i = 0 \\ (m+1)(m-1) & i = 1 \\ (m-1)(m-2)/2 & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.8. When $n = 3$ the calculation is similar to the previous example. We find

$$\mathrm{h}^i(mD) = \begin{cases} (m+3)(m+2)(m+1)/6 & i = 0 \\ (m+2)(m+1)(m-1)/2 & i = 1 \\ (m+1)(m-1)(m-2)/2 & i = 2 \\ (m-1)(m-2)(m-3)/6 & i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.9. It is straight-forward to check that $\chi(\mathcal{O}(mD)) = n + 1$ in each of these examples. Moreover, for $m = 1$ we find $\mathrm{h}^0(D) = n + 1$ and $\mathrm{h}^i(D) = 0$ for $i > 0$, as expected.

Remark 4.10. For $n = 3$ and large m, we see that $\mathrm{h}^2(mD)$ is greater than $\mathrm{h}^0(mD)$. Therefore the map

$$L^* : \mathrm{H}^2(X, \mathcal{O}(mD)) \to \mathrm{H}^0(X, \mathcal{O}(mD))$$

cannot be injective. However, for $n = 2$ we find $\mathrm{h}^2(mD) < \mathrm{h}^0(mD)$ for all $m > 0$. Moreover, a more detailed analysis seems to indicate that L^* *is* injective in this case, suggesting the following conjecture.

Conjecture 4.4. *If E is a nef line bundle on an irreducible holomorphic symplectic 4-fold X, then the map*

$$L^* : \mathrm{H}^2(X, E) \to \mathrm{H}^0(X, E)$$

in the sheaf Lefschetz action is injective.

As described in the previous subsection, this would go a long way towards proving Conjecture 4.1 for 4-folds. In the six-dimensional case, it ought to be possible to analyze the failure of injectivity of L^*, and formulate a slightly weaker statement instead.

5. Classification of abelian fibrations which admit sections

In this section we assume that the irreducible holomorphic symplectic manifold X^{2n} is fibred by abelian varieties, and that this fibration has a section. Ultimately, our goal is to relate X to the Examples 3.1 and 3.2 due to Beauville [1].

5.1. Integrable systems

Since by Matsushita's Theorem 3.1 the fibres of X are Lagrangian with respect to the holomorphic symplectic form, X is really an *algebraic completely integrable Hamiltonian system*. The local geometry of such systems was studied by Hurtubise [18]. Under certain hypotheses, he associated to an integrable system, whose base is an open subset of $\mathbb{C}^n$, a complex surface Q which admits a holomorphic symplectic form (a priori, Q is incomplete, though in examples it often extends to an algebraic surface and the symplectic form to a global meromorphic form). Moreover, the integrable system is isomorphic (indeed, symplectomorphic) to the Hilbert scheme of points on Q, at least over a dense open set.

Ultimately, we would like a global version of Hurtubise's result in the compact case. Namely, we want to associate to our abelian fibration X a compact complex surface which admits a holomorphic symplectic form. Such a surface is necessarily a K3 surface or a torus, and therefore the Hilbert scheme of points on it essentially gives one of Beauville's Examples 3.1 or 3.2. Thus we wish to relate X to one of Beauville's examples; ideally they will be birational, and therefore deformation equivalent. However, we know this cannot be the case with O'Grady's examples, so the relationship must be more subtle.

Let us briefly review Hurtubise's results [18]. His first hypotheses is that the abelian fibres are Jacobians of curves. Later in joint work with Markman [19], he also dealt with integrable systems whose fibres are generalized Prym varieties (note that by a Theorem of Welters, stated as Corollary 2.4 in Chapter 12 of [5], every principally polarized abelian varieties occurs as a generalized Prym variety). Let $\mathbb{J}^{2g} \to U \subset \mathbb{C}^g$ be a local family of Jacobians of curves, where U is an open subset of $\mathbb{C}^g$, and let $\mathbb{S}^{g+1} \to U$ be the corresponding family of curves. Since U is open in $\mathbb{C}^g$, we can choose a section of $\mathbb{S} \to U$, so that each curve has a base-point. Then the Abel map gives an inclusion $I : \mathbb{S} \hookrightarrow \mathbb{J}$.

The next hypotheses is that

$$I^*\Omega \wedge I^*\Omega = 0$$

on $\mathbb{S}$, where Ω is the holomorphic symplectic form on $\mathbb{J}$ (note that $I^*\Omega$ is non-vanishing on $\mathbb{S}$ for dimensional reasons). This implies that $\mathbb{S}$ is a coisotropic submanifold of $\mathbb{J}$, and we can quotient by the null foliation on $\mathbb{S}$ to obtain a surface Q. The form $I^*\Omega$ then projects to a holomorphic symplectic form ω on Q. Hurtubise then proves

Theorem 5.1. *There is a birational map*

$$\Phi : Q^{[g]} \to \mathbb{J}$$

which preserves symplectic structures, ie. Φ takes the holomorphic symplectic form on $Q^{[g]}$ induced by ω to Ω on $\mathbb{J}$. Moreover, for a given curve S_t where $t \in U$, Φ takes the Hilbert scheme $S_t^{[g]} = \mathrm{Sym}^g S_t$ to the Jacobian J_t of S_t by the Abel map (these are both Lagrangian submanifolds).

Remark 5.1. The existence of the birational map Φ follows from Sklyanin's separation of variables, and can be viewed as finding Darboux coordinates on $\mathbb{J}$.

Hurtubise also studies the geometry of the surface Q. He proves that the curves S_t are embedded in Q and all belong to the same linear system $|S_t|$. Moreover, if they aren't hyperelliptic curves, then this linear system embeds Q into $\mathbb{P}^g$ (a similar statement is true in the hyperelliptic case). For our purposes, the important point is the following: if a family of Jacobians of curves $\mathbb{J}$ is holomorphic symplectic, then the base U is an open subset of a linear system of curves S_t which all lie in a *single* complex surface Q, and moreover $Q^{[g]}$ is birational to $\mathbb{J}$.

Example 5.2. An example where these local results extend to global statements is the Hitchin system [16], the moduli space $\mathcal{M}$ of rank r Higgs bundles over a Riemann surface Σ. Firstly, $\mathcal{M}$ is isomorphic to the cotangent bundle $T^*\mathcal{N}$ of the moduli space $\mathcal{N}$ of stable bundles over Σ. Since the latter is a complex manifold, $T^*\mathcal{N}$ has a canonical holomorphic symplectic structure. As an abelian fibration, the base of $\mathcal{M}$ is the space of *spectral curves*, r-to-one finite covers of Σ. If $\rho : \mathcal{K}_\Sigma \to \Sigma$ denotes the canonical bundle of Σ, then we can think of the spectral curves as belonging to the linear system $|\mathcal{O}(r\rho^*\mathcal{K}_\Sigma)|$ of curves sitting in the total space $\mathcal{K}_\Sigma$. In this example, the surface Q is $\mathcal{K}_\Sigma$, and there is a birational map between $\mathcal{M}$ and $\mathcal{K}_\Sigma^{[g]}$, where g is the genus of the spectral curves. These statements are global, though the manifolds are all non-compact.

Another global example comes from the Hilbert scheme of points on a K3 surface S.

Example 5.3. Recall from Example 3.6 that if S contains a smooth curve C of genus g then $S^{[g]}$ is birational to an abelian fibred moduli space Z with base the linear system $|C| \cong \mathbb{P}^g$ and fibres degree g Picard groups Pic^g of the curves in the linear system. Associated to the fibration $Z^{2g} \to \mathbb{P}^g$ is the family of curves $Y^{g+1} \to \mathbb{P}^g$. The total space Y can be defined by

$$Y := \{(p, D)| p \in D \in |C|\}$$

in which case the map to $\mathbb{P}^g$ is simply given by $(p, D) \mapsto D$. Now Y is also foliated, with space of leaves S and leaves

$$Y_q := \{(p, D) \in Y | p = q\}$$

for $q \in S$. In other words, Y_q is the linear system of curves D which pass through q, and hence is isomorphic to $\mathbb{P}^{g-1}$. Thus Y is both fibred over $\mathbb{P}^g$ and foliated by copies of $\mathbb{P}^{g-1}$.

If Y admits a section, then we can embed Y in Z by mapping

$$(p, D) \mapsto i_*\mathcal{O}_D(p + (g-1)p_0)$$

where $p_0 \in D$ is the value of the section at $D \in \mathbb{P}^g$, $\mathcal{O}_D(p + (g-1)p_0)$ is a degree g line bundle on D, and i denotes the inclusion $D \hookrightarrow S$. In fact, it is enough that Y admit a $(g-1)$-valued multisection. One checks that $\sigma^2|_Y = 0$, where σ is the holomorphic symplectic form on Z, and hence Y is coisotropic. The foliation described above is precisely the null foliation and quotienting by it allows us to reconstruct the surface S, the space of leaves. Moreover, Z is birational to $S^{[g]}$. This example is both global and compact.

Remark 5.4. In the local case $\mathbb{J} \to U$ automatically admits sections. In the global case, assuming the existence of a section gives us a base-point in each fibre and allows us to identify it *unambiguously* with the Jacobian, which is isomorphic to Pic^0. However, in Example 5.3 the fibres were actually degree g Picard groups Pic^g. Moreover, we actually needed a section of Y, or at least a $(g-1)$-valued multisection. Of course, a section of Y will induce a section of Z, and a multisection will induce a multisection. Precisely what kind of section or multisection we need on Z will therefore vary according to the dimension, and is a delicate question.

5.2. Fibrations and foliations

Returning to our general abelian fibration $X \to \mathbb{P}^n$, let us try to imitate the geometry of Example 5.3. Thus we want to regard the base $\mathbb{P}^n$ as being a linear system of curves on some compact complex surface. Let us assume that the fibres of $X \to \mathbb{P}^n$ are (isomorphic to) Jacobians of curves, and denote the family of curves by $Y^{n+1} \to \mathbb{P}^n$. Moreover, let us assume that Y can be embedded in X via the Abel map on each fibre (cf. Remark 5.4), and that $\sigma^2|_Y$ vanishes. Then Y is coisotropic and we obtain a surface Q by quotienting by the null foliation. Moreover, each curve Y_t, for $t \in \mathbb{P}^n$, should project isomorphically to a curve in Q, and these curves should all lie in the same linear system. For $q \in Q$, the leaf $Y_q \cong \mathbb{P}^{n-1}$ will then correspond to the linear subsystem of curves passing through q.

Thus Y is both fibred by curves over $\mathbb{P}^n$, and foliated by copies of $\mathbb{P}^{n-1}$ with space of leaves Q. The compact surface Q will admit a holomorphic symplectic form, and ideally we'd like to show it is a K3 surface.

Conjecture 5.2. *Suppose the irreducible holomorphic symplectic manifold X^{2n} is an abelian fibration, with fibres isomorphic to Jacobians of curves. If the fibration admits the appropriate multisection (cf. Remark 5.4) then X is birational to the Hilbert scheme $S^{[n]}$ of n points on a K3 surface S.*

Remark 5.5. When $n = 2$ the conjecture has already been proved by Markushevich [31], and we shall discuss his result in the next subsection. Note that in this dimension, the condition $\sigma^2|_Y = 0$ is automatic. When $n = 5$ O'Grady's space $\widetilde{\mathcal{M}}$ will possibly be a counter-example to the conjecture, but recall (Example 3.10) that we didn't establish a genuine fibration on $\widetilde{\mathcal{M}}$. Moreover, the fibration may not admit the appropriate multi-section. We will look more closely at the fibration on $\widetilde{\mathcal{M}}$ later, in Example 6.9.

We saw in Remark 3.9 that the generalized Kummer variety K_n can be deformed to an abelian fibration whose fibres have polarization of type $(1, \ldots, 1, n+1)$; presumably principally polarized fibres cannot occur. In the projective case, the following observation is apparently due to Mukai; the author is grateful to Kieran O'Grady for pointing this out.

Proposition 5.3. *Suppose the projective abelian fibration $X \to \mathbb{P}^n$ is a deformation of the generalized Kummer variety K_n. The induced polarization of the fibres of X cannot be principal.*

Proof. Let E be the hyperplane section of $X \subset \mathbb{P}^N$, and let D be the pull-back of a hyperplane from the base $\mathbb{P}^n$. Applying Fujiki's formula (1) to $E + tD$ gives

$$\begin{aligned} \int_X (E + tD)^{2n} &= c_X q_X (E + tD)^n \\ &= c_X (q_X(E) + 2t q_X(E, D))^n \end{aligned}$$

and taking the coefficient of t^n we find

$$\int_X E^n D^n = \frac{2^n (n!)^2}{(2n)!} c_X q_X(E, D)^n.$$

Since D^n is a fibre F, the left hand side is just $\int_F (E|_F)^n$. On the right hand side, the Fujiki constant c_X is a deformation invariant and for K_n is

$$c_{K_n} = \frac{(n+1)(2n)!}{2^n n!}$$

(this is computed in [40], for example). Therefore

$$\int_F (E|_F)^n = (n+1)! q_X(E, D)^n$$

where $q_X(E, D)^n \in \mathbb{Z}$, but if $E|_F$ were a principal polarization of F, the left hand side would equal $n!$. $\qquad\square$

Remark 5.6. The polarization of the fibres in Example 3.8 are therefore 'minimal' according to the formula above (ie. correspond to $q_X(E, D) = 1$). By comparison, the Fujiki constant for $S^{[n]}$ is

$$c_{S^{[n]}} = \frac{(2n)!}{2^n n!}$$

and hence the induced polarization of the fibres is principal when $q_X(E, D) = 1$.

The analogue of Conjecture 5.2 for generalized Kummer varieties would be the following.

Conjecture 5.4. *Suppose the irreducible holomorphic symplectic manifold X^{2n} is an abelian fibration, and the fibres have polarization $(1, \ldots, 1, n+1)$. If the fibration admits the appropriate multisection then X is birational to a generalized Kummer variety K_n.*

The goal would be to again show that the base $\mathbb{P}^n$ of the fibration is a linear system of curves on a surface (more precisely, a complex tori T), but a priori we don't even have a family of curves. In Example 3.8 the fibres were abelian subvarieties of Jacobians, and perhaps this is one way to approach the problem.

If our goal is to classify holomorphic symplectic manifolds we should investigate other polarizations too. Presumably there will be some related to $(1, \ldots, 1)$ and $(1, \ldots, 1, n+1)$ (for example, multiples) which will occur for $S^{[n]}$ and K_n respectively, but the other polarizations will not arise. The justification would be that if one can associate a holomorphic

symplectic surface to the abelian fibration X, then the K3 surface and complex tori already exhaust all possibilities.

5.3. Four-folds fibred by Jacobians

The following theorem is due to Markushevich [31].

Theorem 5.5. *Suppose the irreducible holomorphic symplectic four-fold $\pi : X \to \mathbb{P}^2$ is fibred by Jacobians of genus-two curves, and that the fibration admits a section. Then X is birational to $S^{[2]}$ for some K3 surface S.*

Proof. A careful proof may be found in [31]. Here let us just outline how the K3 surface S is constructed: it is actually the double cover of the dual plane $(\mathbb{P}^2)^\vee$ branched over a sextic B.

Let $Y^3 \to \mathbb{P}^2$ be the family of genus-two curves. Each curve Y_t, for $t \in \mathbb{P}^2$, is hyperelliptic, being a double cover of $\mathbb{P}^1_t := \mathbb{P}(\mathrm{H}^0(Y_t, \mathcal{K}_{Y_t}))$ branched over six points. We claim that each of these lines $\mathbb{P}^1_t$ is canonically embedded in the dual plane $(\mathbb{P}^2)^\vee$.

The fibre X_t is the Jacobian of Y_t, and therefore its tangent space $T_p X_t$ at any point $p \in X_t$ is $\mathrm{H}^0(Y_t, \mathcal{K}_{Y_t})$. Using the holomorphic symplectic form, and the fact that the fibres of X are Lagrangian, we can identify $T_p X_t$ with

$$(\pi^* \Omega^1_{\mathbb{P}^2})_p = (\Omega^1_{\mathbb{P}_2})_t.$$

The Euler sequence on $\mathbb{P}^2$ gives

$$0 \to \Omega^1_{\mathbb{P}^2} \to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^2} \to 0$$

and projectivizing we get the inclusion

$$\mathbb{P}(\Omega^1_{\mathbb{P}^2}) \hookrightarrow \mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(1))) = (\mathbb{P}^2)^\vee$$

where the right hand side is the trivial bundle over $\mathbb{P}^2$ with fibre $(\mathbb{P}^2)^\vee$. Taking the fibre over $t \in \mathbb{P}^2$ proves the claim.

The six branch points on $\mathbb{P}^1_t$ will vary holomorphically with t, so for a pencil of these lines in $(\mathbb{P}^2)^\vee$ it will cut out a sextic. A priori, different pencils could give different sextics; however, the six branch points on $\mathbb{P}^1_t$ will actually be the points of intersection of $\mathbb{P}^1_t$ with the curve B dual to the the degeneracy locus of the fibration X (ie. the fibres of X will degenerate over some curve $\Delta \in \mathbb{P}^2$, and $B \in (\mathbb{P}^2)^\vee$ will be dual to Δ). This establishes that B is a sextic, and the double cover of $(\mathbb{P}^2)^\vee$ branched over B is therefore a K3 surface S. Moreover, pulling-back the line $\mathbb{P}^1_t$ from $(\mathbb{P}^2)^\vee$ will give us a curve in S isomorphic to Y_t, as both curves are double covers of $\mathbb{P}^1_t$ branched over the same six points.

We have thus realized the base $\mathbb{P}^2$ as a linear system of curves on a K3 surface. Indeed X is isomorphic to the moduli space Z in Example 3.6, which moreover is birational to $S^{[2]}$. (One should be careful about singular fibres: see [31] for the details.) $\square$

Remark 5.7. The fact that the curves Y_t are hyperelliptic is no longer true for higher genus, in general. However, genus-three curves can be realized as plane quartics, and there is hope in the six-fold case of constructing a K3 surface as a quartic in $(\mathbb{P}^3)^\vee$. On

the other hand, a proof of Markushevich's theorem using Hurtubise's approach would be more likely to generalization to higher dimensions.

6. Abelian fibrations which don't admit sections

In the previous sections we have looked at the first two steps of our three-part programme for understanding holomorphic symplectic manifolds. Our discussion involved some conjectures, some ideas on how to prove them, and some supporting evidence. The ideas in this section will be more complete, based on the work of Căldăraru [7]; an example of their application will appear in the article [42] currently in preparation. Nevertheless, there are still some curious phenomena which remain unexplained (specifically, Example 6.9).

6.1. Gerbes and elliptic fibrations

Let X be an irreducible holomorphic symplectic manifold which is an abelian fibration; we don't assume that X admits a section. We wish to relate to X another abelian fibration which *does* admit a section. Ideally we'd like a one-parameter family of deformations connecting the two manifolds, as in the case of elliptic K3 surfaces. In this subsection we will describe an approach to the corresponding problem for elliptic fibrations, due to Căldăraru [7]; in the next we will investigate how to generalize to the case of higher dimensional (abelian) fibres.

Recall from Subsection 2.3 that the set of all elliptic K3 surfaces with relative Jacobian J was classified by the Brauer group $\mathrm{H}^2(J, \mathcal{O}_J^*)$. This group also classifies *holomorphic gerbes* on J (see Hitchin [17], for example).
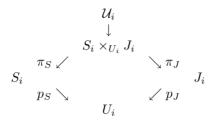
Definition 6.1. Let $\{U_i\}$ be a Čech cover for J and let $\{\mathcal{L}_{ij}\}$ be a collection of holomorphic line bundles over $U_{ij} := U_i \cap U_j$ which satisfy

1. $\mathcal{L}_{ji} \cong \mathcal{L}_{ij}^{-1}$ for all i and j,
2. $\mathcal{L}_{ijk} := \mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$ on U_{ijk} has a trivialization $\beta_{ijk} : U_{ijk} \to \mathcal{O}^*$ for all i, j, and k,
3. the trivializations $\{\beta_{ijk}\}$ of $\mathcal{L}_{ijk}$ define a Čech 2-cocycle, ie. $\delta\beta = 1$ (this means that $\mathcal{L}_{ijkl} := \mathcal{L}_{jkl} \otimes \mathcal{L}_{ikl}^{-1} \otimes \mathcal{L}_{ijl} \otimes \mathcal{L}_{ijk}^{-1}$ on U_{ijkl} is canonically trivial).

Then $\{\mathcal{L}_{ij}\}$ defines a *holomorphic gerbe* on J. Up to isomorphism (which we won't define here) the gerbe is classified by $\beta \in \mathrm{H}^2(J, \mathcal{O}_J^*)$.

Now suppose S is an elliptic K3 surface with relative Jacobian J. The latter is a moduli space of pure semi-stable sheaves on S (with a certain Hilbert polynomial); we can ask whether there exists a universal sheaf for this moduli problem. For a single fibre J_t, the Poincaré line bundle over $S_t \times J_t$ is a universal bundle, which moreover can be extended over a local fibration. Thus if $\{U_i\}$ is an affine cover of the base $\mathbb{P}^1$, and $S_i := p_S^{-1}(U_i)$ and $J_i := p_J^{-1}(U_i)$ the corresponding local fibrations, then there is a local universal bundle $\mathcal{U}_i$

over the fibre-product $S_i \times_{U_i} J_i$.

$$\mathcal{U}_i$$
$$\downarrow$$
$$S_i \times_{U_i} J_i$$

$$S_i \qquad {}^{\pi_S}\nearrow \qquad\qquad \searrow {}^{\pi_J} \qquad J_i$$

$$p_S \searrow \qquad\qquad \nearrow p_J$$

$$U_i$$

However, these local universal bundles may not patch together to give a global universal bundle. Above the overlap U_{ij} the local universal bundles $\mathcal{U}_i$ and $\mathcal{U}_j$ will differ by the pull-back of a line bundle $\mathcal{L}_{ij}$ from $J_{ij} := p_J^{-1}(U_{ij})$. One can check that the collection $\{\mathcal{L}_{ij}\}$ defines a gerbe on J, whose equivalence class we denote by $\alpha_S \in \mathrm{H}^2(J, \mathcal{O}_J^*)$. If α_S vanishes, the local universal bundles can be patched together to give a global bundle. Hence α_S is the obstruction to the existence of a global universal bundle.

To relate this to the classification of elliptic fibrations S we observe that the choice of a local section $s_i : U_i \to S_i$ gives a family of basepoints on the fibres S_t. This uniquely defines the Poincaré line bundle over each fibre, and hence uniquely defines the local universal bundle $\mathcal{U}_i$. In fact, choosing a local section is equivalent to choosing a local universal bundle, and thus our earlier analysis of S based on sections can be rephrased in terms of universal bundles.

Proposition 6.1. *The following are equivalent:*

1. *S is isomorphic to its relative Jacobian J,*
2. *S admits a section,*
3. *there exists a universal bundle over $S \times_{\mathbb{P}^1} J$.*

The obstruction to all of these existence problems is $\alpha_S \in \mathrm{H}^2(J, \mathcal{O}_J^*)$.

Remark 6.1. The approach via gerbes easily generalizes to higher dimensional elliptic fibrations [7]: if the elliptic fibration $S \to B$ has relative Jacobian $J \to B$, then S is classified by an element of

$$\mathrm{H}^2(J, \mathcal{O}_J^*)/\mathrm{H}^2(B, \mathcal{O}_B^*).$$

Note that the denominator did not appear in the K3 case as there are no gerbes on $\mathbb{P}^1$, or indeed on any curve; but $\mathrm{H}^2(B, \mathcal{O}_B^*)$ may be non-zero as soon as the dimension of B is at least two. In this case gerbes on B can be pulled-back to J, giving an inclusion

$$\mathrm{H}^2(B, \mathcal{O}_B^*) \hookrightarrow \mathrm{H}^2(J, \mathcal{O}_J^*).$$

See Căldăraru's thesis [7] for the details.

6.2. Gerbes and abelian fibrations

Now let us consider the case where the fibres themselves are higher dimensional, namely abelian varieties. We start with a generalization of the relative Jacobian.

Definition 6.2. Let $X \to \mathbb{P}^n$ be an abelian fibration. We define the *relative Picard* of X to be the moduli space P consisting of push-forwards of degree-zero rank-one torsion-free sheaves on fibres of X.

Remark 6.2. In general one has to be careful with singular fibres. Let $i : X_t \hookrightarrow X$ be the inclusion of a smooth fibre, and let $i_* L$ be the push-forward of a degree-zero line bundle on X_t. Then more precisely, P should be defined as the moduli space of pure stable sheaves on X with the same Hilbert polynomial as $i_* L$. If P is not already complete, then we must add strictly semi-stable sheaves (which could of course result in a singular space).

Remark 6.3. On smooth fibres P is given by replacing an abelian variety X_t with its degree-zero Picard group $P_t := \mathrm{Pic}^0(X_t)$, ie. the dual abelian variety. If X_t is principally polarized then it is isomorphic to P_t (this is automatically true for elliptic curves), and moreover X and P will be locally isomorphic as fibrations. For a general polarization this will no longer be true. A solution to this problem is to take the relative Picard of P: taking the double dual of an abelian variety returns the original variety.

As P is a moduli space of sheaves on X, we can investigate the existence of a universal bundle, like before. For a single (smooth) fibre X_t and its dual P_t, there is a universal bundle on $X_t \times P_t$. This can be extended to a local universal bundle, at least away from singular fibres. As in the K3 case, we get a holomorphic gerbe whose equivalence class $\beta_X \in \mathrm{H}^2(P, \mathcal{O}_P^*)$ is the obstruction to patching these local universal bundles together to give a global universal bundle. Note that there are no gerbes on the base $\mathbb{P}^n$, so we needn't worry about $\mathrm{H}^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^*)$.

Denote the relative Picard of P by X_0 (we use this notation because we will actually have $\beta_{X_0} = 1$, the trivial gerbe in $\mathrm{H}^2(P, \mathcal{O}_P^*)$). Then as a fibration, X is locally isomorphic to X_0, at least away from singular fibres, but stuck together in a different way. Notably, the relative Picard $P \to \mathbb{P}^n$ has a canonical section given by the flat family $\mathcal{O}_X$, thought of as the family of trivial line bundles $\mathcal{O}_{X_t}$ on each fibre of X. Likewise X_0 must also have a section s_0, as it is the relative Picard of P. Like in Subsection 2.3, choosing a local section of $X \to \mathbb{P}^n$ determines a local isomorphism with X_0, which takes the local section to s_0. So X is really a torsor over X_0, and is thus classified by an element of $\mathrm{H}^1(\mathbb{P}^n, \mathcal{B})$ where $\mathcal{B}$ is the sheaf of local holomorphic sections of $X_0 \to \mathbb{P}^n$, or equivalently, the sheaf of translations in fibres.

To relate this to the gerbes above, we observe that local sections of X are equivalent to local universal bundles. This is almost the same as the elliptic K3 case, except that a basepoint on an abelian variety is used to *translate* a theta divisor, whereas on an elliptic curve the basepoint *is* a theta divisor. This essentially proves the following conjecture: the only remaining concern being the singular fibres.

Conjecture 6.2. *The following are equivalent:*

1. *X is isomorphic to X_0,*
2. *X admits a section,*

3. *there exists a universal bundle over $X \times_{\mathbb{P}^n} P$.*

The obstruction to all of these existence problems is $\beta_X \in \mathrm{H}^2(P, \mathcal{O}_P^*)$.

Remark 6.4. In some examples, P is a smooth irreducible holomorphic symplectic manifold (in general only smoothness should be in doubt). The exponential long exact sequence

$$0 \to \mathrm{H}^1(P, \mathcal{O}_P^*) \to \mathrm{H}^2(P, \mathbb{Z}) \to \mathrm{H}^2(P, \mathcal{O}_P) \to \mathrm{H}^2(P, \mathcal{O}_P^*) \to \mathrm{H}^3(P, \mathbb{Z}) \to 0$$

and the fact that $\mathrm{H}^2(P, \mathcal{O}_P) \cong \mathbb{C}$ then tells us that $\mathrm{H}^2(P, \mathcal{O}_P^*)$ will be one-dimensional, with a strange topology. This is like the K3 case, except that now $\mathrm{H}^3(P, \mathbb{Z})$ could be non-trivial and thus $\mathrm{H}^2(P, \mathcal{O}_P^*)$ might not be connected.

Remark 6.5. The reason that singular fibres cause no problems when constructing the relative Jacobian J of an elliptic K3 surface S is that

1. they are isolated,
2. S and J are locally isomorphic away from the singular fibres.

For elliptic surfaces, the monodromy around the singular fibre determines it uniquely. For higher dimensional elliptic fibrations the behaviour of singular fibres is already more complicated, and for abelian fibrations it is even worse.

Remark 6.6. The relation between X_0 and P is an example of the Strominger-Yau-Zaslow mirror symmetry conjecture [43]. Moreover, the relation between X and P is an example of mirror symmetry with non-trivial B-field β_X. Verbitsky [45] showed that generic hyperkähler manifolds (those with Picard number zero) are self-mirror, but our abelian fibrations have positive Picard number, which is why our mirror manifolds are not isomorphic in general.

In fact, the main goal of Căldăraru's thesis [7] was to construct *twisted Fourier-Mukai transforms*, which combine homological mirror symmetry with the SYZ conjecture. Our Example 6.7 in the next subsection also leads to a twisted Fourier-Mukai transform, as will be proved elsewhere [42].

6.3. Two examples

This first example will be described in detail in the paper [42] currently in preparation. It has also been studied by Markushevich [31] from a slightly different point of view.

Example 6.7. If the K3 surface contains a smooth genus g curve C, then we saw in Example 3.6 that the Hilbert scheme $S^{[g]}$ is birational to a moduli space Z which is abelian fibred over the base $|C| \cong \mathbb{P}^g$. The fibre $\mathrm{Pic}^g D$ of Z over $D \in |C|$ is principally polarized, and hence Z is locally isomorphic to its relative Picard P, at least away from singular fibres.

In the case $g = 2$, S is the double cover of the dual plane $(\mathbb{P}^2)^\vee$, branched over a sextic; the curves D are pull-backs of lines in $(\mathbb{P}^2)^\vee$. Write Z_k for the fibration over $|C| \cong \mathbb{P}^2$ with fibres degree k Picard groups Pic^k. Then Z_k is just the Mukai moduli space of stable sheaves on S with Mukai vector $v = (0, [C], k-1)$. In the generic situation, v is primitive and hence Z_k is smooth and compact for all values of k. 'Generic' here means that the

sextic defining S does not admit a tritangent, for if it admitted a tritangent then some divisor in $|C|$ would split into two homologous parts; then v would not be primitive for k odd. It follows also that Z_k is deformation equivalent to $S^{[2]}$.

The fibrations Z_k are all locally isomorphic to each other. We also know that Z_0 and Z_2 admit (canonical) sections, given by taking $\mathcal{O}_D \in \mathrm{Pic}^0$ and $\mathcal{K}_D \in \mathrm{Pic}^2$, respectively, for each curve $D \in |C|$. Thus $Z = Z_2$ is isomorphic to Z_0.

We claim that Z_1 does not admit a section in the generic case. Thus Z_1 is not isomorphic to Z_0 and Z_2. In fact we prove this in the opposite order: using O'Grady's description of the weight-two Hodge structure of a moduli space of sheaves on a K3 surface [37] we can show that Z_1 is generically not isomorphic to Z_0 and Z_2. Therefore it cannot admit a section.

All of the fibrations Z_k have the same relative Picard P, which has fibre

$$\mathrm{Pic}^0(\mathrm{Pic}^k D) \cong \mathrm{Pic}^0 D$$

over $D \in |C|$ (the isomorphism is canonical). This shows that P is isomorphic to Z_0 away from the singular fibres, but then P can be completed by adding the same singular fibres as in Z_0, and hence is deformation equivalent to $S^{[2]}$. The exponential long exact sequence then shows that $\mathrm{H}^2(P, \mathcal{O}_P^*)$ is one-dimensional and connected. So although Z_0 and Z_1 are generically not isomorphic, they do sit in the same one-dimensional family of deformations (of course, they are both deformations of $S^{[2]}$). Furthermore, it can be shown that Z_1 admits a 2-valued multisection and hence the gerbe $\beta_1 \in \mathrm{H}^2(P, \mathcal{O}_P^*)$ classifying Z_1 is 2-torsion.

Remark 6.8. The moduli space of K3 surfaces S which are double covers of the plane is 19-dimensional. Therefore the manifold $Z = Z_0$, which is the Mukai flop of $S^{[2]}$, belongs to a 19-dimensional family of abelian fibrations, which all admit sections. Then $\mathrm{H}^2(P, \mathcal{O}_P^*)$ adds an extra dimension, giving us a 20-dimensional family of abelian fibrations, which sits inside the 21-dimensional moduli space of deformations of $S^{[2]}$. This is in agreement with Conjectures 4.1 and 4.2, which imply that being an abelian fibration is a codimension one property.

This 20-dimensional family of abelian fibrations also intersects transversely the set of projective varieties in the moduli space of $S^{[2]}$ (projectivity is also codimension one). This is because the projective varieties correspond to torsion points in $\mathrm{H}^2(P, \mathcal{O}_P^*)$, as in the K3 case (cf. Remark 2.5).

Example 6.7 exhibits the expected behaviour: namely, *twisting* the fibration Z_0 by the non-trivial gerbe β_1 gives a new fibration Z_1 which is not isomorphic to, but is nonetheless a deformation of Z_0. This is because the singular fibres were well controlled, which is not always the case. The most curious effects appear to be produced by *multiple fibres*. Recall that elliptic K3 surfaces cannot contain multiple fibres, so the behaviour in the next example is purely a higher dimensional phenomena.

Example 6.9. Assume again that the K3 surface S is the double cover of $\mathbb{P}^2$ branched over a generic sextic, but this time let C be the pull-back of a conic from $\mathbb{P}^2$. Then C

is a genus five curve. Denote by Z_5 and Z_6 the Mukai moduli spaces $\mathcal{M}^s(0, [C], 1)$ and $\mathcal{M}^{ss}(0, [C], 2)$ respectively. The latter has non-primitive Mukai vector, so we have added strictly semi-stable sheaves to obtain a singular space. Both Z_5 and Z_6 are abelian fibrations over $|C| \cong \mathbb{P}^5$, with fibres $\mathrm{Pic}^5 D$ and $\mathrm{Pic}^6 D$ over $D \in |C|$, respectively. Therefore over smooth curves D, Z_5 and Z_6 are locally isomorphic as fibrations; one can also show that Z_6 admits a section, whereas Z_5 generically does not.

We expect that Z_5 will be obtained from Z_6 by twisting, as in Example 6.7. However, this cannot be the case because the singular fibres of Z_5 and Z_6 are clearly different, and indeed Z_6 is not even smooth. In particular, the pull-back from $\mathbb{P}^2$ of a double line gives a non-reduced divisor $2E \in |C|$, and the fibres $F_5 \subset Z_5$ and $F_6 \subset Z_6$ over $2E$ are somewhat like multiple fibres in the theory of elliptic surfaces. In this case, F_5 is certainly not isomorphic to F_6 (over simpler kinds of non-smooth divisors in $|C|$, the singular fibres of Z_5 and Z_6 might still be isomorphic). If we write $U_5 \subset Z_5$ and $U_6 \subset Z_6$ for the unions of the smooth fibres, then we can perhaps regard U_5 as a twisting of U_6 (after extending the theory to incomplete manifolds).

The importance of this example comes from the fact that Z_5 is birational to $S^{[5]}$, and an open subset of Z_6 is birational to an open subset of O'Grady's moduli space $\widetilde{M}$, as we saw in Examples 3.6 and 3.10 respectively. Thus $S^{[5]}$ can be deformed to Z_5, and a symplectic desingularization $\hat{Z}_6$ of Z_6 (if it exists) can be deformed to $\widetilde{M}$; but $S^{[5]}$ and $\widetilde{M}$ are not deformation equivalent (they have different second Betti numbers). So the combination of twisting U_6 to get U_5, and gluing in different singular fibres, has produced a topological change. In fact, this is more likely to be due to the different singular fibres, as the twisting should belong to a one-dimensional family of deformations.

Remark 6.10. These two examples are related to a point raised by Beauville in [3]. He describes the birational correspondence between $S^{[g]}$ and $Z_g := Z$ (cf. Example 3.6), and remarks that not much seems to be known about the birational geometry of the other abelian fibrations Z_k, whose fibres over $D \in |C|$ are $\mathrm{Pic}^k D$.

In Example 6.7 we saw that Z_k is isomorphic to either Z_0 or Z_1 depending on whether k is even or odd. Moreover, Z_0 and Z_1 are deformation equivalent but generically not isomorphic. To prove they are not isomorphic, we used O'Grady's result [37] to show that they have different weight-two Hodge structures. This also implies they are not birational in the generic situation, as birational holomorphic symplectic manifolds represent non-separated points [22] in the moduli space and would therefore have isomorphic periods.

In Example 6.9 we actually find that Z_5 and Z_6 (after the appropriate desingularization) are not even deformation equivalent, and therefore certainly not birational.

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