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Symplectic surgeries from singularities

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Abstract

We describe a variety of symplectic surgeries (not a priori compatible with Kaehler structures) which are obtained by combining local Kaehler degenerations and resolutions of singularities. The effect of the surgeries is to replace configurations of Lagrangian spheres with symplectic submanifolds. We discuss several examples in detail, relating them to existence questions for symplectic manifolds with $c_1 > 0$, $c_1 = 0, c_1 < 0$ in four and six dimensions.

1. The local model

Given an isolated analytic hypersurface singularity $0 \in X_0 := \{f(z) = 0\} \subset \mathbb{C}^{n+1}$ one can form the *smoothing* $X_t = \{f(z) = t\}$ and the *resolution* $\widehat{X} \to X_0$, obtained by (repeatedly) blowing up the origin. These two associated spaces have the same link – that is, the intersection $\mathbb{S}^{2n+1} \cap f^{-1}(0)$ – and there is a smooth surgery which replaces the smoothing by the resolution, or vice-versa. Thinking of the smoothing as Kähler (and so symplectic) by restriction of the standard Kähler form on $\mathbb{C}^{n+1}$, the smoothing also looks somewhat like a resolution by symplectic parallel transport, described below. The result is that there is a canonical map or 'Lagrangian blow up' $X_0 \leftarrow X_t$ whose 'exceptional locus' (the inverse image of the singular point $0 \in X_0$) is a *Lagrangian* cycle, in fact a collection of Lagrangian spheres [21]. So the surgery replaces configurations of Lagrangian spheres by complex (symplectic) subvarieties. Symplectic parallel transport also shows that the X_ts are all isomorphic *as symplectic manifolds*, so denoting any such symplectic manifold by X, we can denote this surgery by the diagram (motivated by smoothings and resolutions in algebraic geometry)

$$\begin{array}{l} \hat{X} \\ \downarrow \\ X_0 \leftarrow X. \end{array} \tag{1}$$

More general singularities, including complete intersections and some non-isolated singularities, can be treated similarly; for simplicity we will restrict attention to isolated hypersurface singularities.

We briefly remind the reader about symplectic parallel transport [21]. The total space $\mathcal{X} = \{X_t\}_{t\in\mathbb{C}} \xrightarrow{p} \mathbb{C}$ is naturally a smooth symplectic submanifold of $\mathbb{C}^{n+2}$: $\mathcal{X} = \{(z,t) \in$

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$\mathbb{C}^{n+1}\times\mathbb{C} : f(z)=t\} \subset \mathbb{C}^{n+2}$. Away from the origin, there is a natural connection on this family of X_ts, whose horizontal subspaces of the tangent bundle $T\mathcal{X}$ are the annihilators under the symplectic form of $Tp = \ker(dp)$, the fibrewise tangent bundles of the fibres TX_t. Parallel transport along this connection identifies fibres $X_t \cong X_s$ (for $s,t \neq 0$), and preserves the symplectic form on the fibres, so that the isomorphism is of symplectic manifolds. Following the transport as $t \to 0$ gives in the limit the map $X_t \to X_0$ above, collapsing the Lagrangian sphere vanishing cycles of the singularity. (That these cycles are spheres follows from Morsifying f before totally smoothing it to $f-t$, and invoking the situation for ordinary double points as described in [21].)

The resolution is also Kähler (and so symplectic) as the blow up of X_0 sits naturally in $\mathbb{C}^{n+1}\times\mathbb{P}^n$. (*Small* resolutions may not be naturally Kähler, however, or there may be choices involved, leading to obstructions in patching these choices in a global situation; more of this later.) So in a natural way the surgery (1) is symplectic. This can also be seen in a different way, as an instance of "gluing along convex boundaries". Removing a small tubular neighbourhood of the Lagrangian vanishing cycles, the boundary of the resulting manifold is ω-convex [7] and (by parallel transport) contactomorphic to the ω-convex link of the exceptional divisor in the blow-up. The Lagrangian and complex fillings of this link can then be exchanged by a symplectic surgery [7]. In many cases of interest (for instance singularities arising from weighted homogeneous polynomials) the link is fibred by circles – leaves of the characteristic foliation – on which the symplectic form is degenerate. Quotienting out by these circles gives a space with a symplectic form; this is the blow up of the singularity. (If it is singular we can blow up again.) The choice of the size of the neighbourhood of the vanishing cycles goes over to the size of the symplectic form on the $\mathbb{P}^n$ factor of $\mathbb{C}^{n+1}\times\mathbb{P}^n \supset \widehat{X}$.

This paper describes work still in progress; we discuss several surgeries which fit into the above framework, but the examples to which they give rise need further study. For now, we will motivate the idea that they should be useful in addressing various existence questions for symplectic structures. After clarifying the global symplectic geometry of the surgeries below, the subsequent sections deal with surgeries relevant to symplectic manifolds with $c_1 > 0$ (Fano), $c_1 = 0$ (Calabi-Yau) and $c_1 < 0$ (general type) respectively.

2. The global model

Suppose we have a symplectic manifold X containing a configuration $\mathcal{C}$ of Lagrangians, with a neighbourhood $U(\mathcal{C})$ isomorphic to the neighbourhood of the vanishing cycles of a Kähler manifold which has a Kähler degeneration collapsing $\mathcal{C}$ to an isolated hypersurface singularity. (We discuss below conditions which can ensure this; see Proposition 7.3 for instance.) More generally, suppose X contains finitely many disjoint such configurations, all locally the vanishing cycles of (possibly different) Kähler degenerations.

Collapsing these Lagrangians gives a singular space with a neighbourhood of each singular point p isomorphic to a neighbourhood $V(p) \subset \mathbb{C}^{n+1}$ of the above singularity.

Moreover this isomorphism takes the symplectic form ω_X on X to the restriction of $\omega_{\mathbb{C}^{n+1}}$ to $V(p)$. Pulling back ω_X to the blow-up $\widehat{X}$ of X at the singular points gives a 2-form ω degenerate only along the exceptional locus E.

Assume for simplicity this blow-up $\widehat{X}$ is smooth; if not we can iterate the following process. Let σ be a closed 2-form on $\widehat{X}$ Poincaré dual to $-[E]$ supported on the neighbourhood of E that is the pull back of the Kähler neighbourhood of the singularity above. On this neighbourhood σ is cohomologous to the Kähler form used in the local model above (restricted from $\mathbb{C}^{n+1} \times \mathbb{P}^n$), and so can be taken to be equal to this Kähler form in a (smaller) neighbourhood U of E. Then we claim that for $0 < \varepsilon \ll 1$, $\omega + \varepsilon\sigma$ is symplectic globally on $\widehat{X}$.

Since nondegeneracy is an open condition, this is clear on $\widehat{X}\backslash U$ for small enough ε, and on E itself. On $U\backslash E$ it follows from the fact that both σ and ω are compatible with the local complex structure inherited from the local Kähler model; therefore any convex linear combination of the two forms is also symplectic by an observation of Gromov.

The size of ε is determined by the areas of curves inside the exceptional divisor E, which in turn is related to the volume of the neighbourhood $U(\mathcal{C})$. This phenomenon is well-known from symplectic blowing up at smooth points [12], cf. (3.1) below.

3. The ordinary double point

Let (X, ω) be a symplectic manifold and $L \subset X$ a Lagrangian sphere. According to a theorem of Weinstein, a neighbourhood of L in X is symplectomorphic to a neighbourhood of the zero-section L_0 in the cotangent bundle T^*L, equipped with its canonical symplectic structure. Slightly less well known is the existence of a symplectomorphism

$$\left(\left\{\sum_{i=1}^{n+1} z_i^2 = 0\right\}\backslash\{0\}, \frac{i}{2}\sum dz_j \wedge d\overline{z}_j\right) \;\cong\; (T^*\mathbb{S}^n\backslash\mathbb{S}^n_0, dp \wedge dq) \tag{2}$$

although this can be given in a straightforward manner in co-ordinates, for instance by taking $(z_j = a_j + ib_j)_j \mapsto (a_j/|a|, -|a|b_j)_j$. Here we have used the round metric to identify $T^*\mathbb{S}^n$ and $T\mathbb{S}^n$. The same map defines a global isomorphism from $T^*\mathbb{S}^n$ to $\{\sum z_j^2 = t\}$ when t is real and positive, explicitly exhibiting the cotangent bundle as a smoothing of the singularity (and there is a similar map for all $t \in \mathbb{C}^*$). The space W on the left hand side of (2) is a punctured neighbourhood of the n-fold *ordinary double point* or *node*. This admits a holomorphic desingularisation by blowing up the origin, which gives an exceptional divisor a complex quadric $Q_{n-1} \subset \mathbb{P}^n$. The total space of the normal bundle $\mathcal{L}$ to this quadric in the resolution admits two natural maps: the projection to Q_{n-1}, and a map to $\mathbb{C}^{n+1}$ (whose image lies inside the singular space W), which we label

$$Q_{n-1} \xleftarrow{\;p\;} \mathcal{L} \xrightarrow{\;\pi\;} \mathbb{C}^{n+1}.$$

We can define a model family of symplectic forms on the neighbourhood of the exceptional divisor in the resolution by setting

$$\rho_\lambda = \pi^* \omega_{\mathbb{C}^{n+1}} + \lambda^2 p^* \omega_{Q_{n-1}}$$

where the form on Q_{n-1} is the restriction of the Fubini-Study form[1] to $Q_{n-1} \subset \mathbb{P}^n$. The form ρ_λ gives the generators of $H_2(Q_{n-1})$ equal size $\pi\lambda^2$ (the generator is unique unless $n = 3$). Moreover let $B(\delta)$ denote the ball of radius δ in $\mathbb{C}^{n+1}$.

Lemma 3.1. *There is a symplectomorphism between* $[\pi^{-1}B(\delta)\backslash Q_{n-1}, \rho_\lambda]$ *and the "shell"* $[(B(\sqrt{\lambda^2+\delta^2})\backslash B(\lambda)) \cap W, \omega_{\mathbb{C}^{n+1}})]$.

Proof. There is a diagram $\mathbb{P}^n \leftarrow \tilde{\mathcal{L}} \rightarrow \mathbb{C}^{n+1}$ arising from blowing up the origin in $\mathbb{C}^{n+1}$, with $\tilde{\mathcal{L}}$ the total space of the $\mathcal{O}(-1)$ line bundle over $\mathbb{P}^n$. Denote by $\tilde{\rho}_\lambda$ the form $\tilde{\pi}^*\omega_{\mathbb{C}^{n+1}} + \lambda^2 \tilde{p}^* \omega_{\mathbb{P}^n}$, in an obvious notation. Then according to ([12], Lemma 7.11 or Lemma 6.40 in the 1st edition) there is a symplectomorphism $z \mapsto F \circ \tilde{\pi}(z)$ between $(\tilde{\pi}^{-1}B(\delta)\backslash\mathbb{P}^n, \tilde{\rho}_\lambda)$ and $(B(\sqrt{\lambda^2+\delta^2})\backslash B(\lambda), \omega_{\mathbb{C}^{n+1}})$, where F is the radial map $z \mapsto (\sqrt{|z|^2+\lambda^2}/|z|)z$. The map F preserves the quadric W, so there is an induced map $F \circ \pi$ between the spaces given in the Lemma. Since the symplectic structures on these are induced from the ambient $\mathbb{P}^n \times \mathbb{C}^{n+1}$ by restriction, this is a symplectomorphism. □

This formalises the way in which we can perform the obvious surgery, by removing $B(\sqrt{\lambda^2+\delta^2})$ and gluing back $\pi^{-1}B(\delta)$, in a manner compatible with symplectic forms. It is worth emphasising that although the blow-up of a singular projective variety will always remain projective (hence Kähler), in general there may be more symplectic degenerations of a projective variety than exist holomorphically; we can perform the symplectic surgery above starting with any Lagrangian sphere, and not just a vanishing cycle for a complex degeneration. Of course, the difference between these two classes (if any) is largely mysterious.

The formalism above motivates the question of determining the *maximum possible* value of λ that one can take in (3.1). This is analogous to "symplectic packing" questions, only looking not for symplectically standard balls but for symplectically standard (Lagrangian) disc bundles over Lagrangian submanifolds. One reason to focus on this variant of a packing number is given by the following (itself a variant of ideas of symplectic inflation). Fix once and for all a model of $T^*\mathbb{S}^n$ as $\{(u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |u| = 1, \langle u, v\rangle = 0\}$, and regard all Lagrangian spheres as parametrised by the zero-section in this model equipped with the standard form $du \wedge dv$. Let us say a symplectic manifold is a *symplectic Fano* if $[\omega] = c_1(TX, \omega) \in H^2(X, \mathbb{Z})$. These are essentially the well-known "monotone" symplectic manifolds, although to fix the constants in the next Lemma it is important that $[\omega]$ and c_1 co-incide and are not just positively proportional.

[1] Our convention throughout the paper is that the (Fubini-Study) symplectic form on $\mathbb{P}^n$ is given by quotienting the Hopf circles in the unit sphere in $\mathbb{C}^{n+1}$ and descending the standard form $\sum dx_j \wedge dy_j$, and this gives a line $\mathbb{P}^1 \subset \mathbb{P}^n$ area π.

Lemma 3.2. *Let (X, ω) be a symplectic Fano 6-manifold and $L \subset X$ a Lagrangian sphere. If $\{v \in T^*\mathbb{S}^3 \mid |v| \leq \mu\}$ symplectically embeds inside X for any $\mu > \frac{1}{2\pi}$ then the manifold Y obtained by surgery along L admits symplectic structures in the cohomology class $c_1(Y)$.*

Proof. The first Chern class of X is zero on the Lagrangian L, so lifts to a class $c_1(X) \in H^2(X, L) \cong H^2(Y, E) \to H^2(Y)$, with $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ the exceptional divisor. By the adjunction formula its image satisfies $c_1(Y) = c_1(X) - [E]$. On the other hand, the cohomology class of the form ρ_λ given above is given by $[\omega_X] - \pi\lambda^2 E$. (We again identify $[\omega_X]$ with an element of $H^2(Y)$, and the factor of π enters because a line in projective space, equipped with the Fubini-Study form as per our conventions, has area π.) We would therefore like to take $\lambda = 1/\sqrt{\pi}$ in performing the surgery. On the other hand, using the local model (2), one can check that the ball $(B(\sqrt{1/\pi + \delta^2}), \omega_{\mathbb{C}^3})$ symplectically embeds inside $\{v \in T^*\mathbb{S}^3 \mid |v| \leq 1/2\pi + \delta^2/2\}$. This is just because $|z|^2 \leq R \Rightarrow |v| = |\Re(z)||\Im(z)| \leq R/2$. The result follows. $\square$

In four dimensions, it is known that every symplectic Fano is in fact a del Pezzo surface [11], and so Kähler. (The above surgery would not be relevant in 4 dimensions, as it produces a symplectic -2-curve on which c_1 is zero, not positive.) In higher dimensions, there is no analogous result, nor is any counterexample known. Lemma (3.2) provides a symplectic surgery that preserves the class of symplectic Fanos. It can be applied in two directions. On the one hand, there is a classification of Fano 3-folds [14] and one can look for symplectic non-Kähler Fanos; on the other, restrictions on symplectic Fanos will translate into packing-type bounds for neighbourhoods of Lagrangian spheres. For instance, there is a Lagrangian sphere inside $(\mathbb{P}^2 \times \mathbb{P}^1, \frac{1}{\pi}(-3\omega_{FS} \oplus 2\omega_{FS}))$ given as follows. Embed a ball $B(\sqrt{2/\pi}) \subset \mathbb{P}^2$ with the standard Euclidean symplectic form on the left and $3/\pi$ times the Fubini-Study form on the right. The boundary $\mathbb{S}^3$ of this ball maps into $\mathbb{P}^2 \times \mathbb{P}^1$ via the graph of the Hopf map, and is Lagrangian with respect to the chosen form (which induces the usual orientation on each factor and is normalised so that $c_1 = [\omega]$). Alternatively, we can remove the minus sign, yielding a symplectic form on $\mathbb{P}^2 \times \mathbb{P}^1$ deformation equivalent to the usual Kähler form, if we compose the Hopf map with the antipodal map of $\mathbb{P}^1$ in the definition of the Lagrangian sphere.

Question 3.1. Is there a symplectic embedding of the μ-disc bundle of $T^*\mathbb{S}^3$ into $(\mathbb{P}^2 \times \mathbb{P}^1, \frac{1}{\pi}(-3\omega_{FS} \oplus 2\omega_{FS}))$ for any $\mu > 1/2\pi$?

If the answer to this question is yes, then there are symplectic Fano manifolds which are not Kähler. For according to the previous Lemma, the transition of $\mathbb{P}^2 \times \mathbb{P}^1$ in the Lagrangian would be a symplectic Fano. This manifold would have $b_2 = 4$, with two new H_2 classes coming from the rulings of the exceptional divisor (and classes in H_4 coming from the divisor, and the lift of a 4-chain bounded by the Lagrangian). However, the almost complex structure underlying this symplectic structure has $c_1^3 = 52$, which is not realised by *any* Fano 3-fold with $b_2 = 4$, according to the classification lists of [14][2].

On the other hand, if the question has a negative answer, this gives a new kind of packing obstruction: for the volume of the $(1/2\pi)$-disc bundle is strictly less than the volume of $\mathbb{P}^2 \times \mathbb{P}^1$ with the Fano symplectic form.

Incidentally, the result of the surgery above again contains a symplectic two-sphere with trivial normal bundle (coming from the diagonal curve inside $\mathbb{P}^1 \times \mathbb{P}^1$ viewed inside the total space of $\mathcal{O}(-1,-1)$). However, a few moment's reflection with Gromov's non-squeezing theorem shows that the symplectically trivial neighbourhood of this curve is not large enough to (necessarily) contain a Lagrangian sphere by the prescription above, so we cannot expect to iterate the surgery symplectically.

4. Small resolutions

The ordinary double point has special features in six real dimensions, where the exceptional divisor Q_2 has two dimensional second homology. In fact, $Q_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and there are *small resolutions* of the 3-fold node in which either of the two rulings of Q_2 are contracted (i.e. replace the singular point by just a rational curve). The two possible small resolutions differ by a flop, as described extensively in [22]. Explicitly, writing the node as $\{xy = zw\} \subset \mathbb{C}^4$, the small resolutions are given by taking the graphs of two distinct maps to $\mathbb{P}^1$, namely $(x/z = w/y)$ and $(x/w = z/y)$.

Now in contrast to blow-ups, small resolutions are not operations within the projective or Kähler category, but they have another special feature: they preserve the first Chern class of the manifold. This gives a route to searching not for exotic Fano manifolds, as above, but exotic Calabi-Yaus, at least under appropriate conditions for the surgery to exist symplectically at all. These were investigated in [22], where the surgeries given by replacing a Lagrangian 3-sphere by a symplectic two-sphere were called "conifold transitions".

Theorem 4.1. *Let X be a symplectic six-manifold containing disjoint Lagrangian spheres $L_1, \dots, L_n$ for which $\sum \lambda_i [L_i] = 0 \in H_3(X, \mathbb{Z})$ (with all $\lambda_i \neq 0$). Then at least one of the conifold transitions of X in the $\{L_i\}$ admits a symplectic structure.*

Sketch. We regard the existence of the homology relation as given by a four-chain with boundary the union of the Lagrangian spheres. The boundary is collapsed in the nodal space, giving a closed four-cycle which lifts to any small resolution. Flopping gives a cycle D hitting each resolving $\mathbb{P}^1$ positively (with intersection number $|\lambda_i|$). This cycle now plays the role taken by the exceptional divisors of blow-ups in the argument given in "The Global Model"; it is at least cohomologically positive on the exceptional curves. Some local analysis then shows that we can choose a two-form σ Poincaré dual to D such that the form $\omega_X + \varepsilon\sigma$ is globally symplectic for all $0 < \varepsilon \ll 1$: see [22] for details. □

[2] Recently, Mori has announced a gap in [14]; there is an additional Fano 3-fold with $b_2 = 4$, but with $c_1^3 = 26$, given by the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a $(1,1,3)$ curve.

As explained in [22], this theorem is related via mirror symmetry to an operation on complex 3-folds studied by Clemens, Friedman and Tian – this operation being the obvious inverse process, in which rational curves are contracted to give ordinary double points, and one looks for sufficient conditions to have a complex smoothing of the resulting nodal variety. It is not hard to find examples in which the theorem can be applied; for instance, the Lagrangian 3-sphere exhibited as the graph of a Hopf map, embedded in $X \times \mathbb{P}^1$ for any symplectic four-manifold $X \supset \mathbb{S}^3$ and an appropriate product symplectic form $\omega_X \oplus (-\omega_{FS})$, certainly satisfies $[L] = 0 \in H_3$. In general, the resulting manifolds can be Kähler; for instance, if the L_i are the vanishing cycles of a Kähler degeneration and satisfy a relation as in (4.1) the result reduces to an old theorem of Werner [25].

Here is an illustrative (and suggestive) example. Cohomology classes of Kähler forms always have special properties [9]. For instance, if ω is a Kähler form on a six-manifold then the pairing $H_4 \times H_4 \to \mathbb{R}$ given by $(A, B) \mapsto A \cap B \cap \mathrm{PD}[\omega]$ is non-degenerate (Hard Lefschetz theorem) and of signature $(1 + 2h^{2,0}, h^{1,1} - 1)$ (Hodge-Riemann bilinear relations). If $b_2(X) = 3$ the latter condition implies that the matrix of the bilinear form $\cap[\omega]$ has positive determinant, having an odd number of positive eigenvalues. Fix a split Kähler structure $\beta\omega_{\mathbb{P}^2} \oplus \alpha\omega_{\mathbb{P}^1}$ on $\mathbb{P}^2 \times \mathbb{P}^1$, with $\alpha, \beta > 0$. The manifold contains a Lagrangian sphere L, the graph of the composition of the Hopf map and the antipodal map on $\mathbb{P}^1$, precisely when $\beta > \alpha$. (To see this, note that the volume $\beta^2\pi^2/2$ of $\mathbb{P}^2$ is exactly filled by a symplectic ball of radius $\sqrt{\beta}$ in $\mathbb{C}^2$, whilst the boundary of the ball of radius $\sqrt{\alpha}$ induces a form on $\mathbb{P}^1$ of volume $\pi\alpha$.) Both conifold transitions along L admit symplectic structures; if we flop the resolving sphere then change $D \mapsto -D$, in the notation of the proof of Theorem (4.1). The symplectic forms guaranteed by the theorem have the form $\Omega_\varepsilon = \pi^*\omega \pm \varepsilon\mathrm{PD}[D]$, for some small ε. By computing Chern numbers, one can see that the conifold transitions are not given by (say) blowing up a rational curve inside $\mathbb{P}^1 \times \mathbb{P}^2$, so they seem to have no obvious Kähler construction.

Proposition 4.2. *The determinant of the matrix given by cap product with Ω_ε is positive for small ε if and only if $\beta > \alpha$.*

Proof. We will work in the standard complex orientations on both factors. However, to simplify some formulae and remove the antipodal maps, we will take the flipped symplectic structure $\alpha\omega_{\mathbb{P}^1} \oplus -\beta\omega_{\mathbb{P}^2}$. To describe the intersection form of the conifold transitions, we will first need to describe the cycles on these spaces. According to [1], the twistor space of $\mathbb{R}^4$ is isomorphic to the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{P}^1$, so taking duals we obtain an isomorphism between $\mathcal{O}(-1)^{\oplus 2}$ and $\mathbb{S}^2_J \times (\mathbb{R}^4)^*$. Here the first factor parametrises the complex structures on $(\mathbb{R}^4, \sum dx_i \otimes dx_i)$ and at a point $j \in \mathbb{S}^2_J$ the complex structure on $\{j\} \times (\mathbb{R}^4)^*$ is that induced from the natural isomorphism (over $\mathbb{R}$) between the complex dual of $(\mathbb{R}^4, j)$ and the real dual space $(\mathbb{R}^4)^*$. Combining with (2) and bearing in mind the discussion of ([22], Appendix) we obtain a map

$$\mathbb{S}^2_J \times (\mathbb{R}^4 \backslash \{0\}) \;\to\; T^*\mathbb{S}^3; \qquad (j, v) \mapsto (v/|v|, jv). \tag{3}$$

This is a diffeomorphism off zero-sections, and we use this map to exhibit a small resolution; in other words, we replace L by $\mathbb{S}^2_J$ via this map.

Consider the 3-sphere $U = \{([w], w) \in \mathbb{P}^1 \times \mathbb{S}^3\}$ where $\mathbb{S}^3 = \partial B \subset \mathbb{P}^2$ is the boundary of a standard embedded Euclidean ball B^4 of appropriate radius. Fix a vector $v \in \mathbb{C}^2$ of length 1. U intersects $[v] \times \mathbb{P}^2$ transversely along a Hopf circle $\mathbb{S}^1 = \{([v], e^{i\theta v})\}$, and locally about this $\mathbb{S}^1$ the copy of $\mathbb{P}^2$ can be smoothly identified with

$$T^*\mathbb{S}^3|_{\mathbb{S}^1} \cong \left\{(e^{i\theta v}, \lambda j e^{i\theta v}) \,|\, \lambda \in (0,\infty), j \in \mathbb{S}^2_J\right\} \subset T^*\mathbb{S}^3.$$

Via the map (3) this corresponds to a (non-complex) real rank two bundle

$$\mathbb{S}^2_J \times \mathbb{R}\langle v, iv\rangle \subset \mathbb{S}^2_J \times \mathbb{R}^4. \tag{4}$$

Let $|\mathbb{P}^2|$ denote the proper transform of the chain $[v] \times \mathbb{P}^2$ in the small resolution, given by taking the closure of its image under (3) when we include back the zero-section on the LHS. Since $[v] \times \mathbb{P}^2$ and $[w] \times \mathbb{P}^2$ are disjoint for $[v] \neq [w]$ the proper transforms meet only along the exceptional curve $\mathbb{S}^2_J$, and hence the triple intersection $(|\mathbb{P}^2|)^3$ is given by the Euler class of the normal bundle of $\mathbb{S}^2_J$ inside $|\mathbb{P}^2|$. However, we have argued that locally near the resolving sphere $|\mathbb{P}^2|$ looks like (4), and hence this Euler class is trivial: $|\mathbb{P}^2|^3 = 0$.

Now let R denote the chain $\{([w], w) \in \mathbb{P}^1 \times \mathbb{C}^2 \,|\, |w| \leq 1\}$ with boundary U. Abstractly, this is isomorphic to the disc bundle of the $\mathcal{O}(-1)$ line bundle over $\mathbb{P}^1$, or the blow-up of the disc D^4 at the origin. Write $|R|$ for its proper transform in the small resolution. We claim that $|R|^2 \cdot |\mathbb{P}^2| = -1$. First, we try to move R off itself. Let Φ_t be the flow $w \mapsto w\cos(t) + jw\sin(t)$ giving

$$R' = \left\{([w], \Phi_{1-|w|}(w)) \in \mathbb{P}^1 \times B^4 \,|\, |w| \leq 1\right\}.$$

This flow fixes the $\mathbb{P}^1$ at the origin, the $U = \mathbb{S}^3$ at the boundary, and nothing else. Hence $|R'| \cap |R|$ comprises the $\mathbb{P}^1$ at the origin, together with a contribution from $\mathbb{S}^2_J$; moreover, locally about the $\mathbb{P}^1$ both $|R|$ and $|R'|$ look like $\mathcal{O}(-1)$. We have already observed that $|\mathbb{P}^2| \cdot \mathbb{S}^2_J = 0$ and hence $|R|^2 \cdot |\mathbb{P}^2|$ can be computed inside $\mathbb{P}^2 \times \mathbb{P}^1$ as $([v] \times \mathbb{P}^2) \cdot [-\mathbb{P}^1] = -1$. To justify the sign, argue as follows. If $[w] \in \mathbb{P}^1$ is fixed, the cycle R is given by a fibre $\langle w, iw\rangle$ of $\mathcal{O}(-1)$, which is a complex line, whilst Φ is a perturbation in the direction $\langle jw, jiw = -kw\rangle$, where these span an anticomplex line. It follows that R and R' meet transversely but negatively along the zero-section.

From these two intersection calculations, we can deduce the result. Take as ordered basis for H_4 of the conifold transition the cycles $|\mathbb{P}^2|, |\mathbb{P}^1 \times \mathbb{P}^1|$ and $|R|$. Thinking of the $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^2$ as having a line at infinity in the second factor, it is immediate that $|R| \cap |\mathbb{P}^1 \times \mathbb{P}^1| = \emptyset$, from which we deduce:

$$|\mathbb{P}^1 \times \mathbb{P}^1| \cdot |\mathbb{P}^2|^2 = 0;\ |\mathbb{P}^1 \times \mathbb{P}^1|^2 \cdot |\mathbb{P}^2| = 1;\ |\mathbb{P}^1 \times \mathbb{P}^1| \cdot |R|^2 = |\mathbb{P}^1 \times \mathbb{P}^1|^2 \cdot |R| = 0.$$

Denote the remaining triple intersection numbers $|\mathbb{P}^2|^2 \cdot |R| = A$ and $|R|^3 = B$ for integers A, B. From ([22], Proof of Theorem 2.9) it follows that $A = \pm 1$, since near U our four-chain R is exactly a collar neighbourhood as described in (op.cit). (Locally, then, R

defines a complex surface in the small resolution which either meets the exceptional $\mathbb{P}^1$ transversely in a point or contains it with normal bundle $\mathcal{O}(-1)$.) The symplectic forms on the two conifold transitions have Poincaré dual homology classes

$$\mathrm{PD}[\Omega_\varepsilon] \ = \ \alpha[|\mathbb{P}^2|] - \beta[|\mathbb{P}^1 \times \mathbb{P}^1|] \pm \varepsilon[|R|]$$

with α, β both strictly positive. (Note that here the second term is $-\beta$ because with respect to $-\omega_{\mathbb{P}^2}$ we reverse the sign of the generator of $H_2(\mathbb{P}^2)$. However, the only non-trivial triple product involving the four-cycle $\mathbb{P}^1 \times \mathbb{P}^1$ is $|\mathbb{P}^1 \times \mathbb{P}^1|^2 \cdot |\mathbb{P}^2|$, where it occurs as a square and the sign is irrelevant.) In the given ordered basis, the matrix which encodes the bilinear form $\cap[\Omega_\varepsilon]$ on H_4 is the following:

$$\begin{pmatrix} \pm\varepsilon A & -\beta & \alpha A \mp \varepsilon \\ -\beta & \alpha & 0 \\ \alpha A \mp \varepsilon & 0 & -\alpha \pm B\varepsilon \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & -\beta & \alpha A \\ -\beta & \alpha & 0 \\ \alpha A & 0 & -\alpha \end{pmatrix} + O(\varepsilon).$$

The determinant of this matrix is $\beta^2\alpha - \alpha^3 A^2 + O(\varepsilon) = \alpha(\beta^2 - \alpha^2) + O(\varepsilon)$, which is positive if $\beta > \alpha$, and this completes the proof. □

The possibility of finding non-Kähler examples of the surgery is made more interesting by a question of Donaldson [5], who asked if every Lagrangian sphere in a complex algebraic variety arises as the vanishing cycle for some Kähler degeneration. Given a symplectic six-manifold X containing a Lagrangian sphere which is trivial in homology, we can form the conifold transition and the blow-up of the nodal space, and by Theorem 4.1, resp. Section 2, each has a distinguished deformation class of symplectic forms (at least once we fix the four-chain bounding the Lagrangian).

Proposition 4.3. *In the situation above, suppose that X is Kähler with $h^{2,0} = 0$, and that the Lagrangian can be collapsed in a Kähler degeneration. Then the above symplectic forms on the conifold transition are Kähler.*

Proof. $h^{2,0}$ is constant in the Kähler degeneration (by the constancy of $h^{2,0} + h^{1,1} + h^{0,2}$ and the upper semicontinuity of each term), and by standard technology (e.g. mixed Hodge structures) $h^{2,0}$ remains zero on the big resolution. Also using $h^{2,0} = 0$ we see that X and the big resolution are in fact *projective*, with hyperplane class $[H]$ arbitrarily close to a high multiple of any given symplectic form in the distinguished family. So we may as well choose the original symplectic form on X to be projective.

Since $[L] = 0 \in H_3(X)$ the proper transform of a chosen bounding four-chain gives a cycle on the big resolution which hits one $\mathbb{P}^1$ in the exceptional divisor and not the other ([22] Theorem 2.0); $h^{2,0} = 0$ implies that this cycle can be taken to be a divisor D.

Since D intersects one ruling of the exceptional divisor but on the other, we can find a linear combination of D and H which is ample except that it evaluates to zero on precisely one of the two rulings. Taking the map to $\mathbb{P}^N$ associated to the linear system of a high multiple of such a class will contract (only) this ruling and yield the small resolution. By varying the linear combination we get all (high multiples of) the symplectic forms $\omega_X + \varepsilon\sigma$

(using the notation of the proof of Theorem 4.1) with ε rational. The others follow by continuity. $\square$

Hence, if the (distinguished deformation class of forms on the) conifold transition of a projective variety with $h^{2,0} = 0$ (e.g. $\mathbb{P}^1 \times \mathbb{P}^2$) in a single Lagrangian L is non-Kähler, this Lagrangian is not a vanishing cycle for any Kähler degeneration. (More general conifold transitions presumably give rise to configurations of Lagrangians that are not simultaneously realised as vanishing cycles.) This opens another approach to proving certain Lagrangian spheres are not vanishing cycles of Kähler degenerations: employ Lemma 3.2 to see that the full blow-up is Fano, and then invoke the Mori-Mukai classification of Fano 3-folds. (We would like to use Fano-ness since there is no general classification of 3-folds and, as above, it seems hard to violate other topological constraints such as the Hard Lefschetz theorem via conifold transitions.)

The following is also relevant to Donaldson's question, tackling it from a Calabi-Yau, rather than Fano, perspective. Recall that a Calabi-Yau 3-fold is *rigid* if $b_3 = 2$; equivalently $h^{2,1} = 0 = h^1(TX)$, i.e. it has no complex deformations.

Lemma 4.4. *An essential Lagrangian sphere in a rigid Calabi-Yau 3-fold Z is never a vanishing cycle.*

Proof. If we have a flat family of varieties over the disc, with one fibre Z and central fibre nodal, then we may pick a fibrewise holomorphic 3-form varying holomorphically and not tending to zero at the central fibre (the pushdown of the relative canonical sheaf is torsion free rank 1, and so a line bundle, so we may pick a nowhere zero section). Thus the period $\int_L \Omega$ does not tend to zero and L is not collapsed. $\square$

If we could find sufficiently "degenerate" Lagrangian spheres, we would certainly have non-algebraic surgeries.

Lemma 4.5. *If $L \subset X$ is a Lagrangian three-sphere which bounds a smoothly embedded four-ball in X, the conifold transition of X in L is not homotopy Kähler.*

Proof. On collapsing the $\mathbb{S}^3$, the D^4 becomes an $\mathbb{S}^4$ that lifts to one of the small resolutions [22] (and $\overline{\mathbb{P}^2}$ in the other). Its H_4 class is nonzero (it intersects the exceptional $\mathbb{P}^1$ in $+1$) but has intersection zero with all other H_4 classes (as the intersection factors through $H_2(\mathbb{S}^4) = 0$). This would contradict the Hard Lefschetz theorem, so the manifold is not Kähler, though it is symplectic by (4.1). $\square$

It is not clear if such "contractible" Lagrangian 3-spheres can ever exist in closed symplectic six-manifolds, although the existence of such spheres in fake symplectic $\mathbb{R}^6$s [15] indicates that any obstruction would be global. In the Calabi-Yau context, there are no known examples even of homologically trivial Lagrangian 3-spheres. Nonetheless, the existence of the symplectic conifold transition makes it very plausible that there are families of symplectic non-Kähler Calabi-Yaus. For instance, the quintic hypersurface $Q \subset \mathbb{P}^4$ is a Calabi-Yau with $b_3(Q) = 204$. There are well-known examples of nodal quintics containing as many as 130 nodes [23], whose projective small resolutions are rigid.

Question 4.1. Is there a rigid Calabi-Yau 3-fold Q containing a Lagrangian 3-sphere ?

As in the previous section, all answers to the question are interesting. Suppose first the answer is positive, and that there are two disjoint essential Lagrangian spheres which (necessarily) satisfy a relation as in (4.1). Then the small resolution given by the Theorem above would be a symplectic Calabi-Yau 3-fold with *trivial* third Betti number. This could not be Kähler, since the cohomology class of a holomorphic volume form on a Calabi-Yau is necessarily non-zero. If an essential sphere exists but satisfies no good homology relation, we obtain examples of Lagrangian spheres in algebraic varieties that can never arise as vanishing cycles for complex degenerations, via (4.4). Finally, as remarked above, there are no known examples of homologically trivial 3-spheres in Calabi-Yaus, so even the existence of an inessential sphere in Q would be novel.

In a slightly different direction, one can show that the existence of the surgery (4.1) implies that either there are quintic 3-folds with j nodes for every large $j \leq 130$ – itself rather surprising – or there are non-Kähler symplectic Calabi-Yaus, given by conifold transitions on the examples from [23].

5. Fibre products and triple points

Let us consider one more complicated instance of passing from a configuration of Lagrangian vanishing cycles to a new symplectic manifold containing a symplectic exceptional divisor. This will show that these more complicated surgeries can indeed be amenable to explicit computation and construction. Again motivated by a desire to find new symplectic manifolds with $c_1 = 0$, we pass from double points to 3-fold *triple points*, that is isolated singularities of the form

$$R = \{x^3 + y^3 + z^3 + w^3 = 0\}.$$

The singularity at the origin can again be resolved by a single blow-up, and now the exceptional divisor E is a cubic surface in $\mathbb{P}^3$ (abstractly diffeomorphic to the six-fold blow up of $\mathbb{P}^2$, so $b_2(E) = 7$). Its normal bundle in the resolution is $\mathcal{O}_{\mathbb{P}^3}(-1)|_E = K_E$, so by the adjunction formula the blow-up has trivial canonical bundle over E; in particular the transition preserves the Calabi-Yau condition.

Before describing the smoothing of the singularity, it will be helpful to introduce a pretty construction of Lagrangian spheres (hence of degenerations, or Lagrangian blow-downs), often appropriate to the Calabi-Yau setting. Given a pair of smooth surfaces S_i fibred over a curve C, we can form their fibre product $S_1 \times_C S_2$, as used so effectively by [18], for example. To analyse its singularities, we look at the local model

$$S_i = \{f_i(x_i, y_i) = t)\} \subset \mathbb{C}^2 \times \mathbb{C},$$

fibred over $\mathbb{C}$ by the t variable. The fibre product, then, is locally the threefold

$$S_1 \times_C S_2 = \{f_1(x_1, y_1) = f_2(x_2, y_2)\} \subset \mathbb{C}^2 \times \mathbb{C}^2.$$

So we see that there are only singularities (x_1, y_1, x_2, y_2) if *both* points (x_i, y_i) lie on singular fibres above the *same* point t of C, at the singular points. So for instance if both

f_i define double point singularities in the curve fibres over $t = 0$, $f_i(x_i, y_i) = x_i y_i$, say, then the threefold also has a double point $x_1 y_1 = x_2 y_2$. More generally degree n curve singularities $f_i = x_i^n + y_i^n$ give degree n threefold singularities $x_1^n + y_1^n = x_2^n + y_2^n$. We discuss the $n = 3$ triple point case presently.

To smooth such singularities we can first Morsify the f_i to reduce to double points, then move one fibration by an automorphism of the base to move its singular fibres away from those of the other surface. The Lagrangian $\mathbb{S}^3$ vanishing cycle of this latter smoothing is easily described. Take a path γ in the base from the image of the singular fibre of S_1 to that of S_2. Over this, by symplectic parallel transport, lies a fibration by $\mathbb{S}^1$ vanishing cycles for the curve double point in the curve fibres of S_i. Taking the fibre product of these two fibrations over γ gives a $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$-fibration with the property that one $\mathbb{S}^1$ factor collapses at one end of γ and the other at the other end. See Figure 1. Thus over each half of γ we get a solid torus handlebody, glued together to form an $\mathbb{S}^3$. This is easily seen to be Lagrangian by putting the fibre product into the full product, equipped with the product symplectic form.

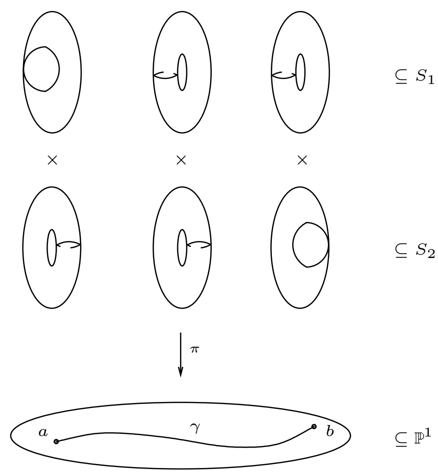

Figure 1. Fibred Lagrangian three-spheres.

With this background, we can study the smoothing of the 3-fold triple point singularity. A helpful picture of the configuration of vanishing cycles, similar to a description given already by Ebeling [3], is given in the following (note that the singularity has Milnor number 16):

Lemma 5.1. *Let R' be the smoothing of a triple point singularity. Then R' contains a configuration of 16 Lagrangian spheres with intersections as indicated in Figure 2.*

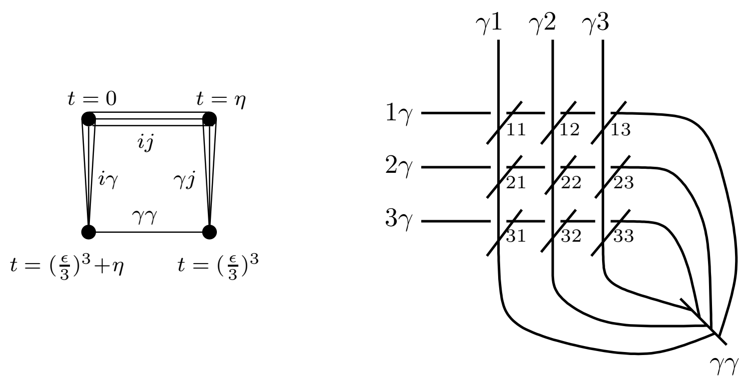

Figure 2. The vanishing cycles of the 3-fold triple point, $\mathbb{T}^2$-fibred over the paths in $\mathbb{P}^1$ on the left

Proof. We take as model for the triple point a fibre product of two elliptic fibrations $E_i \to \mathbb{P}^1$ each of which contain a two-dimensional triple point $\{xy(x+y)=0\}$ inside a torus fibre. So locally the surfaces look like $xy(x+y)=t$ and $uv(u+v)=t$; the threefold fibre product then is locally $xy(x+y)=uv(u+v)$ – a triple point.

To smooth this, we move the fibre products apart by changing t to $t-\eta$ in the second fibration, and then Morsify them both by replacing $xy(x+y)=t$ by $xy(x+y+\varepsilon)=t$. (So the threefold is now locally $xy(x+y+\varepsilon)=uv(u+v+\varepsilon)+\eta$.) This new surface has only nodes in its fibres, at

$$(x,y,t) \in \Big\{(0,0,0),\,(0,-\varepsilon,0),\,(-\varepsilon,0,0),\,\Big(-\frac{\varepsilon}{3},-\frac{\varepsilon}{3},\big(\frac{\varepsilon}{3}\big)^3\Big)\Big\}.$$

That is, the fibre over $t=0$ is 3 $\mathbb{P}^1$s, with the vanishing cycle of the node over $t=\big(\frac{\varepsilon}{3}\big)^3$ being the essential loop ("triangle") in this triangle of $\mathbb{P}^1$s; this is contracted to the original curve triple point (3 coincident lines) as $\varepsilon \to 0$. (In Kodaira's notation [2], these triangles of rational curves are called I_3 singular fibres.)

Labelling the 3 nodes over $t=0$ (and their $\mathbb{S}^1$ vanishing cycles in nearby fibres) by $1,2,3$, and the other vanishing cycle by γ, we get 16 Lagrangian $\mathbb{S}^3$ vanishing cycles by taking any one of these four in the first surface and any one of them in the second, and taking their fibre product to give a $\mathbb{T}^2$-fibration over the path between the points in $\mathbb{P}^1$ at which they collapse, as above (Figure 1). Since γ intersects each of $1,2,3$ in one point and there are no other intersections in the fibres, the intersection pattern of the resulting $\mathbb{S}^3$ is easily determined to be as in Figure 2. □

Now given two elliptic fibrations, we can often detect these triple point degenerations by hand, and hence give examples of the triple point transition. One example is given by studying one of Schoen's rigid Calabi-Yau fibre products from [18]. There is a rational elliptic surface $\pi : E(1) \to \mathbb{P}^1$ with four I_3 singular fibres. This can be obtained from the pencil of cubic curves $\{x^3 + y^3 + z^3 + 3txyz = 0\}_{t \in \mathbb{P}^1}$, which has an I_3 fibre when $t \in \{e^{i\pi/3}, e^{5i\pi/3}, -1, \infty\}$. Let Z denote the fibre product $E(1) \times_\pi E(1)$, which has 3-fold nodes at the 36 points (N, N') with N, N' nodes in some fibre $\pi^{-1}(t)$ of π. There is a small resolution $\tilde{Z}$ of Z at these 36 points which is a projective rigid Calabi-Yau 3-fold.

Lemma 5.2. *In a suitable basis, the monodromy of the elliptic surface $E(1)$ is given by*

$$t(a)^3 \cdot t(b)^3 \cdot t(a+b)^3 \cdot t(2a+b)^3 = 1$$

where a, b are the standard meridian / longitude curves on $\mathbb{T}^2$ and $t(C)$ denotes the Dehn twist in a simple closed curve in the homology class C. (Here the generating loops are ordered clockwise around the base-point.)

Proof. It is well-known that one can construct a holomorphic elliptic surface from any appropriate monodromy representation, and moreover that there is a *unique* elliptic surface with exactly four I_3 singular fibres (see [13] and [16]). Hence it is enough to see that the product of the four cubes of Dehn twists above is indeed the identity. In standard conventions one has

$$t(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \; t(b) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix};$$

$$t(a+b) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}; \; t(2a+b) = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$$

from which the result follows by a direct computation. □

It is important to notice that the a and b vanishing cycles are *adjacent* in the monodromy representation; if we perturb the $I_3(b)$ fibre into an I_1 and an I_2 fibre, the $I_3(a)$ and the $I_1(b)$ give the right configuration of cycles to degenerate to an elliptic surface with a fibre containing a triple point, as in the proof of Lemma (5.1).

This perturbation is certainly possible symplectically, so separate the $I_3(b)$ singular fibre into an I_1 and I_2. We can re-degenerate $E(1)$ to an elliptic surface with a triple point singularity. The new fibre product has a 3-fold triple point, which can be blown up preserving $c_1 = 0$. In this case, the resulting manifold is apparently Kähler; the degenerations can all be realised holomorphically, since monodromy data is enough to guarantee the existence of rational elliptic surfaces, according to the theory of Miranda and Persson [13, 16]. However, this is not to say that the surgery is uninteresting. The manifold we obtain is a new rigid Calabi-Yau which is not itself the small resolution of a fibre product of elliptic fibrations, and hence not on Schoen's list. (Blowing up a triple point singularity increases the Euler characteristic by 24, from which one can compute its Betti numbers.) In general constructions of rigid Kähler Calabi-Yaus are not plentiful, and it is likely that other examples can be found by arguments such as above. In any case, the fibred nature of these three-folds makes it very easy to find large collections of Lagrangian spheres (and compute their intersections from intersections of plane curves), something that is of considerable importance in the general programme of mirror symmetry.

6. A four-dimensional interlude

So far we have always passed from a smoothing to a resolution. If we go in the other direction, we can try (as in the minimal model programme) to eliminate curves on which the canonical class is negative, and find some consequences for symplectic manifolds which are neither Fano nor Calabi-Yau but of "general type"; that is, for which the canonical class of a given almost complex structure itself contains symplectic forms. In general, the difficulty here is that one needs a family of symplectic structures which degenerate (as forms, and not just in volume) along the locus which one wants to contract; as yet, there is no good theory of "symplectic extremal rays" in this general sense. However, this situation is often provided by algebro-geometric arguments. The canonical class of a complex surface cannot contain a Kähler form if the surface contains a holomorphic -2-curve. By contrast:

Proposition 6.1. *If X is a complex surface of general type, then X has symplectic general type.*

Sketch. The multicanonical linear system is an embedding away from ADE trees of rational curves which are contracted to isolated singularities. These have local smoothings in which the complex curves become Lagrangian two-spheres. By replacing one model with the other, in the vein of the first section, and noticing that the symplectic form is cohomologically unchanged since exact in a neighbourhood of the vanishing cycles, one quickly arrives at the result above. For details see [22] (the result was independently proven by Catanese in [4], although the former proof is more relevant for the discussion here). □

The small resolution of 3-fold double points goes over to the *simultaneous resolution* of surface double points, due to Brieskorn. This has been notably exploited by Seidel [19] and Kronheimer [10]. There is a family of symplectic manifolds over the disc D which for every $t \neq 0$ contains a Lagrangian two-sphere, and which at $t = 0$ contains a rational -2-curve (on which the symplectic structure is completely degenerate) – the resolution of the singular point fits into a family of smoothings (after passing to a double cover of the base). The total space of this family, containing the rational curve in the central fibre, is exactly the small resolution of a 3-fold ordinary double point which arises from the base change $\{x^2 + y^2 + z^2 = t^2\} \mapsto \{x^2 + y^2 + z^2 = t\}$. The smooth monodromy of the family $\{\sum_{j=1}^{3} x_j^2 = \varepsilon\}$ over the disc has order two, but the symplectic monodromy – given by a generalised Dehn twist in the Lagrangian vanishing cycle – has infinite order [19].

The resolution and smoothing are diffeomorphic precisely for simple singularities, so if we move beyond these then we obtain surgeries which have a non-trivial topological effect. For instance, there are examples analogous to that above in which the isolated complex curve has higher genus: a degree d surface singularity has a resolution with a degree d curve in $\mathbb{P}^2$ as exceptional set. This leads to a surgery in which configurations of Lagrangian spheres in symplectic four-manifolds can be blown down and replaced by symplectic surfaces of high genus and negative square. In particular cases, these surgeries (or more properly their inverses) can be related to familiar operations on symplectic four-manifolds. We give the following proposition in the case $d = 3$, parallel to the discussion of triple points in 3-folds in the previous section, but it is not hard to generalise to arbitrary degree.

The resolution of the surface triple point $\{x_1^3 + x_2^3 + x_3^3 = 0\}$ is a genus one curve with normal bundle having first Chern class -3. Introduce the notation $\mathcal{C}_3$ for an open neighbourhood of the configuration of vanishing cycles of the smoothing, which we can assume is diffeomorphic to an affine cubic. The singularity has Milnor number 8, but – as with all non-simple surface singularities – the intersection matrix of the vanishing cycles is not negative definite. Recall also that we can define the *proper transform* of a symplectic surface $C^2 \subset X^4$ inside the blow-up of X along C by making C J-holomorphic for some compatible almost complex structure J on X integrable near C, and then taking the usual Kähler blow-up and the holomorphic proper transform.

Proposition 6.2. *Let X be a symplectic four-manifold which contains a symplectic torus $\mathbb{T}^2$ of square zero. The following two operations are smoothly equivalent. (i) Blow up $\mathbb{T}^2$ three times, and then replace a tubular neighbourhood of its proper transform (a square -3 torus) by $\mathcal{C}_3$. (ii) Fibre sum X with a rational elliptic surface $E(1)$ along $\mathbb{T}^2$ and a fibre respectively.*

Proof. We regard the total space of the smoothing $\{x_1^3 + x_2^3 + x_3^3 = t\}$, for an appropriate polynomial t, as the complement of the hyperplane at infinity in a cubic surface $Z = \{\sum_{j=1}^3 x_j^3 = tx_4^3\} \subset \mathbb{P}^3$. Topologically Z is the six-fold blow-up of $\mathbb{P}^2$, and the hypersurface $\{x^3 + y^3 + z^3 = 0\}$ is a torus of square 3. The complement $Z \backslash \mathbb{T}^2$ of the torus – which has $b_2 = 8$ – can be identified with the complement of a fibre and three sections in the rational elliptic surface $\mathbb{P}^2 \# 9\overline{\mathbb{P}}^2 \cong E(1) \to \mathbb{P}^1$. The result now follows from the following equivalence: fibre summing the square 3 torus given by blowing up six base-points in a pencil of cubics with a -3 torus given by blowing up a square zero torus three times, is the same as fibre summing $E(1)$ along the original square zero torus. This equality in turn follows from considering Gompf's pairwise version of the symplectic sum, which enables one to perform the first fibre sum so the three -1-curves transverse to the -3 torus are glued onto the three base points of the linear system of square three tori. □

Thus, in the situation of the Proposition, the "degenerate-resolve" surgery is smoothly equivalent to "de-fibre-summing" with a copy of $E(1)$ and then blowing up three times. If $d > 3$ there is a similar interpretation; the "contract-deform" direction of the surgery – contracting a curve $C \subset X$ of genus $g = (d-1)(d-2)/2$ and square $-d$ which was obtained as a d-fold blow-up of a square zero curve – is smoothly equivalent to a fibre sum of X along a particular genus g Lefschetz fibration. The fibre-sum with $E(1)$ has been an extremely useful surgery [8] and it is encouraging that it can be recovered from this point of view.

7. Tree-like configurations

After isolated Lagrangians, the simplest situation to consider is that of contracting linear chains of spheres. In four real dimensions the A_n singularities have diffeomorphic resolutions and smoothings, but this is no longer true in six dimensions. These surgeries, however, still don't produce any more (diffeomorphism types of) symplectic manifolds than those one can obtain from the ordinary double point surgery, at least when working with small resolutions. One can see this as follows.

Lemma 7.1. *The link of an A_n chain of spheres is diffeomorphic to $\mathbb{S}^5$ if n is even and $\mathbb{S}^2 \times \mathbb{S}^3$ if n is odd.*

Proof. The link is a series of connect sums of $\mathbb{S}^3 \times \mathbb{S}^2$s, across $\mathbb{S}^2 \times \mathbb{S}^2$s (given by removing a ball in $\mathbb{S}^3$ to give $\mathbb{S}^2$ boundary, and timesing everything by $\mathbb{S}^2$). So inductively it is enough to prove that

$$(\mathbb{S}^3 \times \mathbb{S}^2)\#_{\mathbb{S}^2\times\mathbb{S}^2}(\mathbb{S}^2 \times \mathbb{S}^3) \cong \mathbb{S}^5.$$

Here the ordering of the factors is meant to indicate that the first factor in the gluing locus $\mathbb{S}^2 \times \mathbb{S}^2$ is the boundary of a ball in the first $\mathbb{S}^3$ factor, but is the full $\mathbb{S}^2$ factor in the second $\mathbb{S}^3 \times \mathbb{S}^2$, and the opposite for the second $\mathbb{S}^2$ factor in the gluing locus. In the same notation, the connect sum is

$$(D^3 \times \mathbb{S}^2) \cup_{\mathbb{S}^2\times\mathbb{S}^2} (\mathbb{S}^2 \times D^3),$$

since $\mathbb{S}^3$ minus a ball is D^3. But this is isomorphic to taking $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^5 \subset \mathbb{S}^5$ and filling it inside $\mathbb{R}^5$ to give the first $D^3 \times \mathbb{S}^2$ in the union, and filling it 'outside' in the $\mathbb{S}^5$ to give the other. (This is analogous to realising $\mathbb{S}^3$ as a union of two genus one handlebodies. The embedding $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{S}^5$ comes from $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 = \frac{1}{2} = |z_2|^2 + |z_3|^2\}$.) $\square$

We employ this as follows. It is well-known that the singularity given by contracting an A_n-chain of Lagrangian 3-spheres has a small resolution if and only if $n = 2k + 1$ is odd. This resolution is given by a k-fold cover of the ordinary double point resolution, lifting the obvious cover $\{\sum x_i^2 = t^{2k}\} \to \{\sum x_i^2 = t^2\}$ given by $t \mapsto t^k$. The exceptional locus is a k-times thickened $\mathbb{P}^1$. In this case, we can talk about conifold transitions in the entire A_n-chain, but:

Lemma 7.2. *The conifold transition of X in the A_n-chain ($n = 2k + 1$ odd) is diffeomorphic to a manifold obtained by performing conifold transitions in isolated Lagrangian spheres.*

Proof. Algebraically, we can collapse alternate Lagrangians in the chain by using the partial smoothing $\sum_{i=1}^3 x_i^2 = p_k(t^2)$ of the full degeneration $\sum x_i^2 = t^{2k}$; here p_k is a degree k polynomial with simple zeros. Taking conifold transitions along alternate spheres in the A_n-chain is equivalent to taking small resolutions of the double points in this partial smoothing. The interpolating 3-spheres acquire boundary, lifting to $\mathbb{S}^2\times[0, 1]$ homotopies between the exceptional loci of the small resolutions. The small resolution of the A_n singularity described earlier arises in the limit $p_k(t^2) \to t^{2k}$, and the k-times thickened $\mathbb{P}^1$ is then the limit of bringing the k $\mathbb{P}^1$s from the ODP conifold transitions together.

Alternatively, the conifold transition in the A_n-chain is given by dividing out a suitable circle action (determined by k) on the $\mathbb{S}^2 \times \mathbb{S}^3$ boundary of (7.1), and then collapsing one factor of the resulting $\mathbb{P}^1 \times \mathbb{P}^1$. This clearly gives a filling of the link $\mathbb{S}^2 \times \mathbb{S}^3$ with one-dimensional H_2. $\square$

As Eliashberg pointed out in [6], these chains and trees of Lagrangians are nonetheless not without interest. The contact structures induced on $\mathbb{S}^5$ and $\mathbb{S}^2 \times \mathbb{S}^3$ are, for instance, distinct as n varies. By looking at boundaries of more complicated trees, one can obtain distinct contact structures on 5-manifolds – distinguished by the topologies of their fillings (or by contact homology?) – in cases where the underlying classical invariants of the plane fields are equal.

With the triple point above, we dealt with configurations of Lagrangian spheres that arose from some projective degeneration, and the question of the modulus of the symplectic structure in a neighbourhood of the vanishing cycles did not really enter into the question of whether a surgery exists. For the ADE contractions on surfaces, the Kähler form can be assumed standard in a neighbourhood of the contracted spheres ([22] using a result from [21]). More generally, one ingredient in running a symplectic surgery via a transition in some collection of Lagrangians is the following (well-known folk-theorem).

Proposition 7.3. *Let $\{L_1, \ldots, L_n\}$ be a collection of Lagrangian spheres in X for which all intersections are transverse. Suppose the associated intersection graph is a tree (has no loops). Then the symplectic structure in a neighbourhood of the Lagrangians is unique to symplectomorphism.*

Proof. The proof is a "plumbed" version of Weinstein's theorem [24]. His argument shows that for any Lagrangian immersion $\phi : L \to X$ there is an immersion Φ from a neighbourhood of the zero-section in T^*L to X such that $\Phi^*\omega_X = dp \wedge dq$. To deduce the result, recall that symplectomorphisms act transitively on pairs of transverse Lagrangian subspaces of a symplectic vector space, so we can assume at each intersection point (say of L_i and L_j) Φ pulls back L_i to a fibre in the cotangent bundle of L_j and vice-versa. Fix an intersection point P, giving two preimages $\phi^{-1}(P)$ lying in L_i and L_j. We can define two box neighbourhoods $(D^n \times D^n)_{i,j}$ of these preimages in $\amalg_{j=1}^n T^*L_j$, the total spaces of embedded symplectic balls lying in fixed Darboux charts, and such that the first factor in the i-box describes fibres of T^*L_i and the second factor in the j-box describes fibres of T^*L_j. By construction, Φ is assumed to be the immersion (quotient map) which identifies $(D^n \times \{t\})_i$ with $(\{t\} \times D^n)_j$ (and is an embedding away from the boxes). Given two forms $\omega_X, \tilde{\omega}_X$ both making the same L_j Lagrangian, we obtain two such models comprising a collection of copies of $T^*\mathbb{S}^n$ with the standard symplectic structure and a quotienting relation describing the immersion on the boxes around intersection points. These models can be identified symplectically by lifting a diffeomorphism $\amalg L_i \to \amalg L_i$ whose differential matches the appropriate box neighbourhoods. This descends to give a diffeomorphism of the open neighbourhoods of $\phi(\amalg L_i)$ inside X which intertwines the two symplectic structures. $\square$

At least for tree-like configurations, this says that if a symplectic manifold contains a graph of Lagrangian spheres that is combinatorially the same as the graph obtained by smoothing some complex singularity, then there is an associated symplectic surgery which collapses the spheres and blows up the (locally analytic) singular point that results. There are well known lists and techniques for identifying these configurations coming from singularity theory.

In fact, it seems that a generalisation of the above Proposition is not much harder. For any configuration of Lagrangian spheres, there is a homological invariant – the relative intersection numbers for a choice of orientations on all the Lagrangians – but the argument above suggests this is the unique invariant. (Maslov classes play no role, indeed are not defined, since the loops in a configuration of Lagrangians are not locally spanned by discs.) Such more general configurations of Lagrangians can be produced from simple configurations by Dehn twist automorphisms *à la* Seidel [19].

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