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Galois symmetry on Floer cohomology

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Abstract

In this article, we define an action of $\hat{\mathbb{Z}}$ on the A_∞ category whose objects are rational Lagrangian submanifolds together with some other data. The group $\hat{\mathbb{Z}}$ is a Galois group of the (universal) Novikov ring (field) over Laurent polynomial field. By mirror symmetry it will corresponds to the algebraic fundamental group of a punctured disk which parametrize the maximal degenerate family of the mirror complex manifold.

1. Introduction

The Floer cohomology of a Lagrangian submanifold defined in [Fl, Oh, FOOO] is in general a module over a kind of formal power series ring, which is called the Novikov ring [No]. In [FOOO, Fu2] we used the universal Novikov ring $\Lambda_{\mathbb{C}}$, which consists of a formal sum

$$\sum_i a_i T^{\lambda_i} \tag{1}$$

where $a_i \in \mathbb{C}$ and $\lambda_i \in \mathbb{R}$ such that $\lambda_i < \lambda_{i+1}$, $\lim_{i\to\infty} \lambda_i = +\infty$ and T is a formal parameter.

For a rational Lagrangian submanifold (we define it later in Definition 2.2), we take a smaller ring $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ which consists of elements (1) satisfying in addition $\lambda_i \in \mathbb{Q}$.

Both of them are complete non-Archimedean valued fields whose norm is defined by

$$\left\| \sum_i a_i T^{\lambda_i} \right\| = \exp\left(-\min\{\lambda_i | a_i \neq 0\}\right). \tag{2}$$

The continuous Galois group of $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ over $\mathbb{C}[[T]][T^{-1}]$, the Laurent power series ring, is $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. In fact, the (topological) generator $\rho \in \hat{\mathbb{Z}}$ acts by

$$\sum_i a_i T^{\lambda_i} \mapsto \sum_i a_i e^{2\pi\sqrt{-1}\lambda_i} T^{\lambda_i} \tag{3}$$

We denote the map (3) by $x \mapsto x^\rho$. The rationality of λ_i implies that the $\mathbb{Z}$ action defined by (3) can be extended to a $\hat{\mathbb{Z}}$ action.

In the same way, $\mathbb{Z}$ acts as automorphisms on $\Lambda_{\mathbb{C}}/\mathbb{C}[[T]][T^{-1}]$ by the formula (3).

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In this article, we discuss an action of $\hat{\mathbb{Z}}$ on Floer cohomologies which are compatible with the Galois action on Novikov rings.

In section 2, we state our main result (Theorem 2.4) which establishes a $\hat{\mathbb{Z}}$ action on the filtered A_∞ category defined in [Fu2]. (See [Fu2] also for the definition of filtered A_∞ category.) In section 3 we prove it. In section 4 we discuss briefly how the $\hat{\mathbb{Z}}$ action constructed in this article is related to mirror symmetry. Sections 3 and 4 are independent of each other.

The reader will easily notice that there is influence of the paper [KS] on this article. Compare also [Fu3] §3.5, some of the argument of which are improved in this article.

2. Statement of main result

Let (M, ω) be a real $2n$-dimensional symplectic manifold. We assume $c_1(M) = 0$. (Here $c_1(M)$ is the first Chern class of the tangent bundle of M. We remark that a symplectic structure on M determines an almost complex structure on M uniquely up to homotopy. Hence the first Chern class of the tangent bundle of M is well defined.) We also assume that the cohomology class $[\omega]$ is contained in the image $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$.

Definition 2.1. A pair (ξ, ∇^ξ) of a complex line bundle on M and a unitary connection on it is called a *prequantum bundle* if $F_{\nabla^\xi} = 2\pi\sqrt{-1}\omega$ as differential 2-forms. Here F_{∇^ξ} is the curvature of ∇^ξ.

By our assumption $[\omega] \in H^2(M;\mathbb{Z})$, a prequantum bundle (ξ, ∇^ξ) exists. We choose and fix it throughout this article. Let L be a Lagrangian submanifold of M. Namely, L is an n-dimensional submanifold of M such that $\omega|_L = 0$. By definition, the restriction of (ξ, ∇^ξ) to L is flat. Hence it determines a representation $hol_\xi : \pi_1 L \to U(1)$.

Definition 2.2. L is called a *rational* Lagrangian submanifold if the image of hol_ξ : $\pi_1 L \to U(1)$ is a finite group.

We remark that L is called a Bohr-Sommerfeld orbit if $hol_\xi : \pi_1 L \to U(1)$ is trivial.

In [Fu2], the author defined a filtered A_∞ category whose objects are system $(L, \mathcal{L}, \tilde{s}, b)$ consisting of a Lagrangian submanifold L, a flat $U(1)$ bundle $\mathcal{L}$ on it, together with some additional data $(\tilde{s}, b)$. In this article, we consider the case when L is rational. Let us very briefly review the other data. See [Fu2] for details.

We first assume that L is oriented. We also fix st $\in H^2(M;\mathbb{Z}_2)$. We can then define the notion that L is relatively spin in M as in [FOOO] or [Fu2] Definition 2.4. We assume that L is relatively spin and fix a relative spin structure.

We next define the grading (see [Se, Fu2]). Let $Lag(M) \to M$ be the bundle on M whose fiber at $x \in M$ is the set of all Lagrangian linear subspaces of T_xM. Note that the fundamental group of Lagrangian Grassmanian (= the fiber of $Lag(M) \to M$) is $\mathbb{Z}$. It is proved in [Se] that, under our assumption $c_1(M) = 0$, there exists a $\mathbb{Z}$ cover $\widetilde{Lag}(M)$ of $Lag(M)$ such that its restriction to each fiber of $Lag(M) \to M$ is the universal covering. Let $x \in L$. We put $s(x) = T_xL \in Lag(M)$. We say that L is *graded* if there exists a lift $\tilde{s} : L \to \widetilde{Lag}(M)$ of s, and call $\tilde{s}$ a *grading* of L.

Now an object of our A_∞ category is $(L, \mathcal{L}, \tilde{s}, b)$. Here:

(1) L is a rational Lagrangian submanifold equipped with an orientation and relative spin structure.
(2) $\mathcal{L}$ is a flat line bundle on L.
(3) L is assumed to be graded and $\tilde{s}$ is its grading.

To explain b, we need to review the main result of [FOOO]. In [FOOO], we developed an obstruction theory to the well-definedness of the Floer cohomology of a Lagrangian submanifold. We also found, even in the case the obstruction vanishes, the Floer cohomology is not uniquely determined but depends on some extra parameter b. In the case when L is graded, b is represented by a chain

$$b = \sum T^{\lambda_i} b_i$$

where b_i is a singular $(n-1)$-chain which we regard as a 1-cochain. More precisely, b is a solution of some formal equation, which we describe below. (See [FOOO] Theorem D, more precisely.) We consider the cohomology group $H^k(L; \Lambda_{\mathbb{C}})$ or $H^k(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$. In [FOOO] Theorem D, we constructed a "map"

$$Q : H^1(L; \Lambda_{\mathbb{C}}) \to H^2(L; \Lambda_{\mathbb{C}}) \tag{4}$$

which is a kind of formal power series on T in the following sense.

$$Q = \sum_{i=1}^{\infty} Q_i \otimes T^{\lambda_i} \tag{5}$$

here

$$Q_i : H^1(L; \mathbb{C}) \to H^2(L; \mathbb{C})$$

is a formal power series, and $\lambda_i \in \mathbb{R}$, with $\lim_{i\to\infty} \lambda_i = +\infty$.

Q and Q_i depend on L,$\mathcal{L}$,$\tilde{s}$ and λ_i depends on L,$\tilde{s}$. Hereafter we write $Q_{L,\mathcal{L}}$ and $Q_{i,L,\mathcal{L}}$ in place of Q, Q_i, in case we need to specify L, $\mathcal{L}$.

We put "map" in quotes since $Q(b)$ is not well defined in general for two reasons. One is that Q_i is a formal power series whose convergence (in the usual topology of $\mathbb{C}$ vector space) is not known. The other is that the sum (5) is an infinite sum, whose convergence is not yet proved. However $Q(b)$ is well defined if $b \equiv 0 \mod \Lambda_{\mathbb{C}}^{+}$, namely if $b = \sum b_i T^{\lambda_i}$ with all $\lambda_i > 0$.

Now we put

$$\mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}) = \{ b \in H^1(L; \Lambda_{\mathbb{C}}) \mid b \equiv 0 \mod \Lambda_{\mathbb{C}}^{+},\ Q(b) = 0 \}. \tag{6}$$

We also put

$$\mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) = \mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}) \cap H^1(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}).$$

We call a 1-cochain representing an element of $\mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ a *bounding cochain.* The fourth data b consisting the objects of our category is a bounding cochain.

We thus described an object $(L, \mathcal{L}, \tilde{s}, b)$ of our filtered A_∞ category.

Remark 2.1. In [FOOO] our Novikov ring has one additional formal parameter e of degree 2. In the case of graded Lagrangian submanifolds, we do not need it. In general, a bounding cochain has various degrees. In our case of graded Lagrangian submanifold, the bounding cochain is of degree 1.

Remark 2.2. We introduced b as an element of a cohomology group satisfying the equation $Q(b) = 0$. In order to do so, we first need to work on the chain level, introduce gauge equivalence and then identify the gauge equivalence class with an element of the first cohomology. This is performed in [FOOO]. (In its revised version we will present it more algebraically. Compare also [Fu3].) We omit it here.

Actually the chain complex $CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ in Proposition 2.1 defining Floer cohomology does depend on the representative of b in each gauge equivalence class. However it is well defined up to chain homotopy. So we choose a representative in each gauge equivalence class, and fix it.

The set of morphisms of our A_∞ category is Floer cohomology. In [FOOO], we defined Floer cohomology

$$HF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}). \tag{7}$$

The Floer cohomology (7) is a module over $\Lambda_{\mathbb{C}}$ (here $b_i \in \mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}})$).

Proposition 2.1. *If L_i is rational and if $b_i \in \mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ then Floer cohomology is defined over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. Namely we have a cochain complex $CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ which defines $HF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}})$.*

Off course, the Floer cohomology in Proposition 2.1 will become (7) when tensored with $\Lambda_{\mathbb{C}}$.

We prove Proposition 2.1 in §3.

In the case of rational Lagrangian submanifolds, the operations $\mathfrak{m}_k$ (that is, the (higher) composition homomorphisms of our A_∞ category) are also defined over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. More precisely, we can prove the following Proposition 2.2. We take a countable set $\mathcal{LA}$ of rational Lagrangian submanifolds equipped with relative spin structure, and will consider only the Lagrangian submanifolds contained in this set. We assume that each pair of elements of $\mathcal{LA}$ is transversal.

Proposition 2.2. *There exists a filtered A_∞ category $\mathcal{LAG}$ whose objects are $(L, \mathcal{L}, \tilde{s}, b)$ where $L \in \mathcal{LA}$, $\mathcal{L}$ is a flat bundle on it, $\tilde{s}$ is a grading of L and $b \in \mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$. The set of morphisms between two objects is $CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ in Proposition 2.1.*

The A_∞ operations $\mathfrak{m}_k$ are defined in a similar way as [FOOO, Fu2]. We prove Proposition 2.2 in §3. We remark that Proposition 2.2 implies that the operations $\mathfrak{m}_k$ are multilinear over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$.

The main result of this article asserts that the Galois group $\hat{\mathbb{Z}}$ acts as automorphisms of $\mathcal{LAG}$. To state it precisely we first describe the action of $\hat{\mathbb{Z}}$ on the set of objects. We put

$$\mathcal{L}^\rho = \mathcal{L} \otimes \xi|_L. \tag{8}$$

The rationality of L implies that there exists m such that

$$\mathcal{L}^{\rho^m} = \mathcal{L}. \tag{9}$$

(9) implies that our action of ρ induces a $\hat{\mathbb{Z}}$ action.

The Galois action does not move L, $\tilde{s}$.

We next discuss the $\hat{\mathbb{Z}}$ action on the set of bounding cochains b. If the defining equation (4) of our moduli space $\mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ was defined over $\mathbb{C}[[T]][T^{-1}]$ then $\hat{\mathbb{Z}}$ would act on $\mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$ and (4) would be defined over $\mathbb{C}[[T]][T^{-1}]$ if λ_i in (5) were integers. However this is not the case.

Proposition 2.3. *The following diagram commutes.*

$$\begin{array}{ccc} H^1(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) & \xrightarrow{Q_{L,\mathcal{L}}} & H^2(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \\ {\scriptstyle 1\otimes\rho}\downarrow & & {\scriptstyle 1\otimes\rho}\downarrow \\ H^1(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) & \xrightarrow{Q_{L,\mathcal{L}^\rho}} & H^2(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \end{array}$$

Here the vertical arrow is $1 \otimes \rho : H^k(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \to H^k(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}})$, where $H^k(L; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) = H^k(L; \mathbb{C}) \otimes \Lambda_{\mathbb{C}}^{\mathbb{Q}}$.

Proposition 2.3 implies that there exists a map

$$\rho : \mathcal{M}(L, \mathcal{L}; \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \to \mathcal{M}(L, \mathcal{L}^\rho; \Lambda_{\mathbb{C}}^{\mathbb{Q}}).$$

We write $\rho(b) = b^\rho$.

Now we state the main result of this article.

Theorem 2.4. *There exists a $\hat{\mathbb{Z}}$ action on $\mathcal{LAG}$, which is compatible with the $\hat{\mathbb{Z}}$ action on $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ and whose action on objects is given by*

$$\rho(L, \mathcal{L}, \tilde{s}, b) = (L, \mathcal{L}^\rho, \tilde{s}, b^\rho).$$

In place of defining the notion "$\hat{\mathbb{Z}}$ action on $\mathcal{LAG}$ compatible with $\hat{\mathbb{Z}}$ action on $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$" in general, we explain it in the case of Theorem 2.4.

We have already defined a $\hat{\mathbb{Z}}$ action on the set of objects. We recall that the set of morphisms of $\mathcal{LAG}$ is Floer cohomology (more precisely a chain complex defining Floer cohomology). Existence of the $\hat{\mathbb{Z}}$ action on the set of morphisms means that there exists a (canonical) isomorphism:

$$\begin{aligned} \rho : HF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1),&(L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \\ &\cong HF((L_1, \mathcal{L}_1^\rho, \tilde{s}_1, b_1^\rho), (L_2, \mathcal{L}_2^\rho, \tilde{s}_2, b_2^\rho); \Lambda_{\mathbb{C}}^{\mathbb{Q}}). \end{aligned} \tag{10}$$

Here ρ is linear over $\mathbb{C}[[T]][T^{-1}]$ and satisfies

$$\rho(xv) = x^{\rho}\rho(v) \tag{11}$$

for $x \in \Lambda_{\mathbb{C}}^{\mathbb{Q}}$, $v \in HF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}})$.

Note that in the case when L_1 and L_2 are different from each other (and hence transversal to each other) we will define (see §3)

$$CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) = \bigoplus_{p \in L_1 \cap L_2} \Lambda_{\mathbb{C}}^{\mathbb{Q}}.$$

Hence the isomorphism

$$CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \cong CF((L_1, \mathcal{L}_1^{\rho}, \tilde{s}_1, b_1^{\rho}), (L_2, \mathcal{L}_2^{\rho}, \tilde{s}_2, b_2^{\rho}); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \tag{12}$$

satisfying (11) obviously exists. Thus the essential point to check is that (12) commutes with the boundary operator $\mathfrak{m}_1$. We will prove it in §3.

As we mentioned above, $\mathfrak{m}_1$ commutes with the isomorphism (11). The other part of the statement of Theorem 2.4 is that (12) also commutes with higher compositions $\mathfrak{m}_k$. The operator $\mathfrak{m}_k$ is a $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ multi-linear map

$$\begin{aligned} \mathfrak{m}_k : \prod_{\ell=0}^{k-1} CF((L_\ell, \mathcal{L}_\ell, \tilde{s}_\ell, b_\ell), &(L_{\ell+1}, \mathcal{L}_{\ell+1}, \tilde{s}_{\ell+1}, b_{\ell+1}); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) \\ &\to CF((L_0, \mathcal{L}_0, \tilde{s}_0, b_0), (L_k, \mathcal{L}_k, \tilde{s}_k, b_k); \Lambda_{\mathbb{C}}^{\mathbb{Q}}). \end{aligned} \tag{13}$$

Theorem 2.4 implies that

$$\mathfrak{m}_k \circ (\rho \otimes \cdots \otimes \rho) = \rho \circ \mathfrak{m}_k. \tag{14}$$

We have thus described the statement of our main theorem. The proof is given in the next section. Actually the proof is not so difficult. We only need to check the consistency of the action. The main idea of this article is contained in the statement of the theorem and the definitions in this section.

3. Proofs

We start with the proof of Proposition 2.3. During the proof we review briefly the definition of $Q_{L,\mathcal{L}}$. Let $\beta \in \pi_2(M, L)$. We put

$$E(\beta) = \int_{D^2} \varphi^* \omega \tag{15}$$

where $\varphi : D^2 \to M$ is a map representing the homotopy class β. We also put

$$H(\beta; \mathcal{L}) = hol_{\mathcal{L}}(\varphi(S^1)) \in U(1) \tag{16}$$

where the right hand side is the holonomy of the flat bundle $\mathcal{L}$ along the curve $\varphi(S^1)$. The key observations in the proof of Proposition 2.3 are the following Lemmata 3.1, 3.2.

Lemma 3.1. *If L is rational then $E(\beta) \in \mathbb{Q}$.*

Proof. By the definition of prequantum bundle we have

$$2\pi\sqrt{-1}E(\beta) = \int_{D^2} F_{\nabla_\xi}. \tag{17}$$

Hence

$$\exp(2\pi\sqrt{-1}E(\beta)) = hol_\xi(\partial\beta), \tag{18}$$

where the right hand side is the holonomy of the prequantum bundle along the curve $\varphi(S^1)$. By the rationality of L there exists m such that $hol_\xi(\partial\beta)^m = 1$. Hence $E(\beta) \in \mathbb{Q}$ as required. □

Lemma 3.2. $\exp(2\pi\sqrt{-1}E(\beta))H(\beta;\mathcal{L}) = H(\beta;\mathcal{L}^\sigma)$.

Proof. Immediate from (18) and the definition. □

We next need a moduli space $\mathcal{M}_{k+1}(L;\beta)$. We fix a compatible almost complex structure J on M. $\mathcal{M}_k(L;\beta)$ is the moduli space of pseudoholomorphic maps $\varphi : (D^2,S^1) \to (M,L)$ of homotopy class β, together with k marked points on the boundary $\partial D^2 = S^1$. Namely:

Definition 3.1. $\widetilde{\mathcal{M}}_{k+1}(L;\beta)$ is the set of $(\varphi;\vec{z})$. Here $\varphi : (D^2,S^1) \to (M,L)$ is J holomorphic and of homotopy class β, and $\vec{z} = (z_1,\cdots,z_{k+1}) \in (\partial D^2)^{k+1}$, such that the z_i are mutually distinct and respect the cyclic order on ∂D^2.

$Aut(D^2,J_{D^2}) \cong PSL(2;\mathbb{R})$ acts on $\widetilde{\mathcal{M}}_{k+1}(L;\beta)$ by

$$u\cdot(\varphi;z_1,\cdots,z_{k+1}) = (\varphi\circ u^{-1};u(z_1),\cdots,u(z_{k+1})).$$

We denote the quotient space by $\mathcal{M}_{k+1}(L;\beta)$.

$\mathcal{M}_{k+1}(L;\beta)$ can be compactified by including stable maps. (See [FOOO] §3). The compactification $\mathcal{CM}_{k+1}(L;\beta)$ is a space with Kuranishi structure with corners of dimension $n+k-2$. We remark that the dimension is independent of β. This is because we assumed that $c_1(M) = 0$ and L is graded. (We refer to [FO] for the definition of Kuranishi structure with corners and to [FOOO] for the proof of this statement.) Moreover, since we fixed a relative spin structure on L, it follows that $\mathcal{CM}_{k+1}(L;\beta)$ is oriented in the sense of Kuranishi structure. (See [FOOO] Chapter 6.) We remark that there exists an evaluation map

$$ev : \mathcal{CM}_{k+1}(L;\beta) \to L^{k+1}. \tag{19}$$

Namely

$$ev(\varphi,\vec{z}) = (\varphi(z_0),\cdots,\varphi(z_k)). \tag{20}$$

We put $ev = (ev_0,\cdots,ev_k)$.

Using the machinery developed in [FO], we can use $\mathcal{CM}_{k+1}(L;\beta)$ to define a $\mathbb{Q}$ cycle on L^{k+1}. We will use $\mathcal{CM}_{k+1}(L;\beta)$ to define Q. (To justify the construction below we need to work out carefully the transversality issue. We omit it since it is discussed in detail in [FOOO].)

By Gromov compactness, there exist $\beta_1, \beta_2, \cdots \in \pi_2(M, L)$, with the following properties: $E(\beta_i) \le E(\beta_{i+1})$: for each C there exist only a finite number of indices i with $E(\beta_i) \le C$: if $\beta \notin \{\beta_1, \beta_2, \cdots\}$ then $\mathcal{CM}_{k+1}(L;\beta)$ is empty. (See [FOOO] Proposition 5.8 for detail.)

Let $\overline{S}^k(L;\mathbb{C})$ be the set of all distribution valued k-forms on L which are represented by a singular $(n-k)$-chain. (See [FOOO] §A1 for the details of this argument.) We choose a countably generated subcomplex of it satisfying appropriate transversality conditions. We denote it by $C^k(L;\mathbb{C})$. The module $C^k(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}})$ is the completion of $C^k(L;\mathbb{C}) \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}^{\mathbb{Q}}$ with respect to the norm (2). The map Q is constructed by using

$$\mathfrak{m}_{k,\beta_i} : (C^1(L;\mathbb{C}))^{\otimes k} \to C^2(L;\mathbb{C}). \tag{21}$$

Here (21) is defined by

$$\mathfrak{m}_{k,\beta_i}(P_1, \cdots, P_k) = \pm ev_{0,*}\left(\mathcal{CM}_{k+1}(L;\beta)_{ev_1,\cdots,ev_k} \times_{L^k} (P_1 \times \cdots P_k)\right).$$

We omit the discussions on sign which is given in detail in [FOOO] Chapter 6. Using $\mathfrak{m}_{k,\beta_i}$ we define a formal map

$$Q'_{L,\mathcal{L}} : (C^1(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}}))^{\otimes k} \to C^2(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}})$$

by

$$Q'_{L,\mathcal{L}} = \sum_i Q'_{L;\beta_i} \otimes H(\beta_i;\mathcal{L}) T^{E(\beta_i)}. \tag{22}$$

Here

$$Q'_{L,\mathcal{L};\beta_i} : C^1(L;\mathbb{C}) \to C^2(L;\mathbb{C})$$

is a formal map defined by

$$Q'_{L,\mathcal{L};\beta_i}(b) = \sum_k \mathfrak{m}_{k,\beta_i}(b, \cdots, b).$$

In the case $b \equiv 0 \mod \Lambda_{\mathbb{C}}^{\mathbb{Q}}$, $Q'_{L,\mathcal{L}}(b)$ converges in the topology induced by the norm of $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. We also remark that Lemma 3.1 implies that $E(\beta_i)$ is a rational number. Hence the right hand side of (22) is contained in $C^2(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}})$.

Lemma 3.3. *The following diagram commutes.*

$$\begin{array}{ccc} C^1(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}}) & \xrightarrow{Q'_{L,\mathcal{L}}} & C^2(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}}) \\ {\scriptstyle 1\otimes\rho}\downarrow & & {\scriptstyle 1\otimes\rho}\downarrow \\ C^1(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}}) & \xrightarrow{Q'_{L,\mathcal{L}^\rho}} & C^2(L;\Lambda_{\mathbb{C}}^{\mathbb{Q}}) \end{array}$$

Proof. We remark that $Q'_{L;\beta_i}$ is defined over $\mathbb{Q}$ and is independent of $\mathcal{L}$. Hence by (22) and Lemma 3.2, we have

$$\begin{aligned}\rho(Q'_{L,\mathcal{L}}(b)) &= \sum_i Q'_{L;\beta_i}(b^\rho)\otimes\rho(H(\beta_i;\mathcal{L})T^{E(\beta_i)})\\ &= \sum_i Q'_{L;\beta_i}(b^\rho)\otimes\exp(2\pi\sqrt{-1}E(\beta_i))H(\beta_i;\mathcal{L})T^{E(\beta_i)}\\ &= \sum_i Q'_{L;\beta_i}(b^\rho)\otimes H(\beta_i;\mathcal{L}^\rho)T^{E(\beta_i)} = Q'_{L,\mathcal{L}^\rho}(b^\rho).\end{aligned}$$

The proof of the lemma is complete. □

The formal map $Q_{L,\mathcal{L}}$ is obtained from $Q'_{L,\mathcal{L}}$ by homological algebra. (See [FOOO] §A6.) Inspecting the construction there it is easy to see that Lemma 3.3 implies Proposition 2.3. The proof of Proposition 2.3 is now complete.

We next prove Proposition 2.1. We only consider the case $L_1 \ne L_2$. The case $L_1 = L_2$ is easier. We remark that then L_1 is transversal to L_2. We begin with the following lemma.

Lemma 3.4. *We assume that L_1 and L_2 are rational. If $\varphi : S^1\times[0,1]\to M$ is a smooth map such that $S^1\times\{0\}\subset L_1$, $S^1\times\{1\}\subset L_2$, then*

$$\int_{S^1\times[0,1]}\varphi^*\omega\in\mathbb{Q}.$$

The proof is similar to the proof of Lemma 3.1 and is omitted.

Now we recall that in [FOOO, Fu4], we defined Floer's chain complex by:

$$CF((L_1,\mathcal{L}_1,\tilde s_1,b_1),(L_2,\mathcal{L}_2,\tilde s_2,b_2);\Lambda_{\mathbb{C}}) = \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p})\otimes_{\mathbb{C}}\Lambda_{\mathbb{C}}. \tag{23}$$

(The case when $\mathcal{L}_i$ is trivial is written in [FOOO]. The case when $\mathcal{L}_i$ is nontrivial is in [Fu4].)

We need to modify boundary operator $\mathfrak{m}_1$ a bit so that it is defined over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. Let $p,q\in L_1\cap L_2$. We assume that the degree (Maslov-Viterbo index) of q is 1+ the degree of p.

Remark 3.1. We do not discuss the definition of degree. We need the grading $\tilde s$ for this purpose. See [Se, Fu2].

Hereafter we denote by $\mathfrak{m}'_1$ the boundary operator defined in [FOOO, Fu4] in order to distinguish it from one we use in this article. A component of the boundary operator $\mathfrak{m}'_1$ is

$$\mathfrak{m}_1'^{p,q} : Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p})\otimes_{\mathbb{C}}\Lambda_{\mathbb{C}}^{\mathbb{Q}}\to Hom(\mathcal{L}_{1,q},\mathcal{L}_{2,q})\otimes_{\mathbb{C}}\Lambda_{\mathbb{C}}^{\mathbb{Q}}. \tag{24}$$

It is defined by using the moduli space of pseudoholomophic disks, which we discuss now. Let us put

$$\partial_1 D^2 = \{z \in \partial D^2 \,|\, \operatorname{Im} z < 0\}, \qquad \partial_2 D^2 = \{z \in \partial D^2 \,|\, \operatorname{Im} z > 0\}.$$

Definition 3.2. We denote by $\pi_2(M; L_1, L_2; p, q)$ the set of all homotopy classes of maps $\varphi : D^2 \to M$ such that

(1) $\varphi(1) = q$, $\varphi(-1) = p$.
(2) $\varphi(\partial_1 D^2) \subset L_1$, $\varphi(\partial_2 D^2) \subset L_2$.

For $\beta \in \pi_2(M; L_1, L_2; p, q)$, we put

$$E(\beta) = \int_{D^2} \varphi^*\omega,$$

where φ is a map in the homotopy class of β. (Note that the right hand side depends only on the homotopy class of φ since L_i are Lagrangian submanifolds.) Lemma 3.4 implies that

$$E(\beta) - E(\beta') \in \mathbb{Q} \tag{25}$$

for any $\beta, \beta' \in \pi_2(M; L_1, L_2; p, q)$. We next define

$$H(\beta; \mathcal{L}_1, \mathcal{L}_2) \in Hom(Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p}), Hom(\mathcal{L}_{1,q}, \mathcal{L}_{2,q})),$$

by

$$H(\beta; \mathcal{L}_1, \mathcal{L}_2)(v) = \mathrm{Pal}_{\mathcal{L}_2}(\varphi(\partial_2 D^2)) \circ v \circ (\mathrm{Pal}_{\mathcal{L}_1}(\varphi(\partial_1 D^2)))^{-1}. \tag{26}$$

Here $\mathrm{Pal}_{\mathcal{L}_1}(\varphi(\partial_1 D^2)) : \mathcal{L}_{1,p} \to \mathcal{L}_{1,q}$ is the parallel transport of the flat bundle $\mathcal{L}_1$ along the path $\varphi(\partial_1 D^2)$.

We next define the moduli space we use. Let $\beta \in \pi_2(M; L_1, L_2; p, q)$. We consider the set of maps $\varphi : D^2 \to M$ which are J holomorphic and in the homotopy class β. We denote the set of all such φ by $\widetilde{\mathcal{M}}(L_1, L_2; p, q; \beta)$. The group of automorphisms of D^2 which preserve ± 1 is isomorphic to $\mathbb{R}$. It acts on $\widetilde{\mathcal{M}}(L_1, L_2; p, q; \beta)$. We denote by $\mathcal{M}(L_1, L_2; p, q; \beta)$ the quotient space. Using stable maps it has a compactification which we denote by $\mathcal{CM}(L_1, L_2; p, q; \beta)$. (See [FOOO] §3.) If the difference of the degrees of p and q is 1, then $\mathcal{CM}(L_1, L_2; p, q; \beta)$ has a Kuranishi structure of dimension 0 with corners. (See [FOOO] Chapter 5.) Hence we obtain a rational number $\sharp\mathcal{CM}(L_1, L_2; p, q; \beta)$.

We now review the boundary operator $\mathfrak{m}_1'^{p,q}$ defined in [FOOO, Fu4]. Its leading term $\mathfrak{m}_{1;0}'^{p,q}$ is defined as follows.

$$\mathfrak{m}_{1;0}'^{p,q} = \sum \sharp\mathcal{CM}(L_1, L_2; p, q; \beta_i) H(\beta_i; \mathcal{L}_1, \mathcal{L}_2) \otimes T^{E(\beta_i)}. \tag{27}$$

Here β_i are elements of $\pi_2(M; L_1, L_2; p, q)$ such that $\mathcal{CM}(L_1, L_2; p, q; \beta_i)$ is nonempty. By using Gromov compactness we can prove that the right hand side converges in the norm of $\Lambda_{\mathbb{C}}$.

In general, we need correction terms to define $\mathfrak{m}_1'^{p,q}$, satisfying $\mathfrak{m}_1' \circ \mathfrak{m}_1' = 0$. However, in the case $b_1 = b_2 = 0$, we have $\mathfrak{m}_1'^{p,q} = \mathfrak{m}_{1;0}'^{p,q}$. See [FOOO] for the definition of correction terms. For simplicity, we only consider the case $b_1 = b_2 = 0$ here. (In general $Q(0) \neq 0$. Hence we do need to include correction terms.)

We remark that $E(\beta_i)$ is not necessary a rational number even in the case when L_1, L_2 are rational. So the homomorphism $\mathfrak{m}_{1;0}'^{p,q}$ in (30) is not defined on $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. We need to modify $\mathfrak{m}_{1;0}'^{p,q}$ so that it is defined over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$.

First we remark that there is an obvious map

$$\pi_2(M; L_1, L_2; p, q) \times \pi_2(M; L_1, L_2; q, r) \to \pi_2(M; L_1, L_2; p, r),$$

which we write $(\beta, \beta') \mapsto \beta\sharp\beta'$. Clearly

$$E(\beta\sharp\beta') = E(\beta) + E(\beta'), \quad H(\beta\sharp\beta'; \mathcal{L}_1, \mathcal{L}_2) = H(\beta'; \mathcal{L}_1, \mathcal{L}_2) \circ H(\beta; \mathcal{L}_1, \mathcal{L}_2). \tag{28}$$

Next we fix $p_0 \in L_1 \cap L_2$. For each $p \in L_1 \cap L_2$ we fix $\beta^p \in \pi_2(M; L_1, L_2; p_0, p)$.

Definition 3.3. Let $p, q \in L_1 \cap L_2$, $\beta \in \pi_2(M; L_1, L_2; p, q)$. We define

$$E'(\beta) = E(\beta) - E(\beta^p) + E(\beta^q),$$
$$H'(\beta; \mathcal{L}_1, \mathcal{L}_2) = H(\beta^q; \mathcal{L}_1, \mathcal{L}_2)^{-1} \circ H(\beta; \mathcal{L}_1, \mathcal{L}_2) \circ H(\beta^p; \mathcal{L}_1, \mathcal{L}_2).$$

We remark that

$$H'(\beta; \mathcal{L}_1, \mathcal{L}_2) \in Hom(\mathcal{L}_{1,p_0}, \mathcal{L}_{2,p_0}), Hom(\mathcal{L}_{1,p_0}, \mathcal{L}_{2,p_0})) \cong \mathbb{C}.$$

Moreover $H'(\beta; \mathcal{L}_1, \mathcal{L}_2) \in U(1)$.

We remark that Lemma 3.4 implies that $E'(\beta) \in \mathbb{Q}$.

Now we put

$$CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) = \bigoplus_{p \in L_1 \oplus L_2} \Lambda_{\mathbb{C}}^{\mathbb{Q}}[p]. \tag{29}$$

Remark 3.2. More precisely we put

$$CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}}) = \bigoplus_{p \in L_1 \cap L_2} \Lambda_{\mathbb{C}}^{\mathbb{Q}}[p] \otimes_{\mathbb{C}} Hom((\mathcal{L}_1)_{p_0}, (\mathcal{L}_2)_{p_0}).$$

where we fixed $p_0 \in L_1 \cap L_2$ as before. But, since $Hom((\mathcal{L}_1)_{p_0}, (\mathcal{L}_2)_{p_0})$ is independent of p, we omit this factor.

We define

$$\mathfrak{m}_{1;0}^{p,q} = \sum_{\beta_i \in \pi_2(M; L_1, L_2; p, q)} \sharp\mathcal{CM}(L_1, L_2; p, q; \beta_i) H'(\beta_i; \mathcal{L}_1, \mathcal{L}_2) T^{E'(\beta_i)} \in \Lambda_{\mathbb{C}}^{\mathbb{Q}}, \tag{30}$$

and then

$$\mathfrak{m}_{1,0}([p]) = \sum_q \mathfrak{m}_{1;0}^{p,q}[q].$$

In the case when $b_1 = b_2 = 0$ we can prove $\mathfrak{m}_{1,0} \circ \mathfrak{m}_{1,0} = 0$. In general we need to add correction terms which we do not discuss here. (See [FOOO].) We put $\mathfrak{m}_1 = \mathfrak{m}_{1,0} + \cdots$ where $\cdots$ are correction terms we do not define. We have thus defined the chain complex $(CF((L_1, \mathcal{L}_1, \tilde{s}_1, b_1), (L_2, \mathcal{L}_2, \tilde{s}_2, b_2); \Lambda_{\mathbb{C}}^{\mathbb{Q}}), \mathfrak{m}_1)$. We prove the following:

Lemma 3.5. *The chain complex* $(CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}^{\mathbb{Q}}),\mathfrak{m}_1)\hat{\otimes}_{\Lambda_{\mathbb{C}}^{\mathbb{Q}}}\Lambda_{\mathbb{C}}$ *is isomorphic to the chain complex* $(CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}),\mathfrak{m}_1')$.

Here $\hat{\otimes}$ is the completion of the algebraic tensor product.

Proof. We recall

$$CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}) = \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p})\otimes_{\mathbb{C}}\Lambda_{\mathbb{C}}.$$

We first define an isomorphism

$$\begin{aligned} I : CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),&(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}^{\mathbb{Q}})\hat{\otimes}_{\Lambda_{\mathbb{C}}^{\mathbb{Q}}}\Lambda_{\mathbb{C}} \\ &\cong CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}). \end{aligned} \tag{31}$$

Let $p\in L_1\cap L_2$. We take $\varphi : D^2\to M$ which is in the homotopy class β^p. We then put:

$$I_0([p]) = \mathrm{Pal}_{\mathcal{L}_2}(\varphi(\partial_2 D^2))\circ(\mathrm{Pal}_{\mathcal{L}_1}(\varphi(\partial_1 D^2)))^{-1}\in Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p})$$

where the notation is as in (26) and

$$I([p]) = I_0([p])\otimes T^{E(\beta^p)}.$$

It is easy to see that I induces an isomorphism (31). It is easy to check that I is a chain map. □

The proof of Proposition 2.1 is now complete.

The proof of Proposition 2.2 is similar to the proof of Proposition 2.1. Namely we need to modify the operations $\mathfrak{m}_k$ so that they are defined over $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. We can do it in the same way as the proof of Proposition 2.1. We leave the details to the reader.

Now we are in the position to prove Theorem 2.4. We already constructed an action on the set of objects. We next define the action on the set of morphisms. Namely we define:

$$\rho : CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}^{\mathbb{Q}})\to CF((L_1,\mathcal{L}_1^{\rho},\tilde{s}_1,b_1^{\rho}),(L_2,\mathcal{L}_2^{\rho},\tilde{s}_2,b_2^{\rho});\Lambda_{\mathbb{C}}^{\mathbb{Q}}).$$

Actually $CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}^{\mathbb{Q}})$ is a free $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$ module with basis $[p]$, $p\in L_1\cap L_2$ and is independent of $\mathcal{L}_i$, b_i by (29). We define

$$\rho(\sum x_i[p_i]) = x_i^{\rho}[p_i].$$

We will prove that ρ is compatible with $\mathfrak{m}_k$, $k=1,2,\cdots$. We prove it only in the case $k=1$. The other case is similar. Namely we prove the following:

Proposition 3.6. *The following diagram commutes.*

$$\begin{array}{ccc} CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}^{\mathbb{Q}}) & \xrightarrow{\mathfrak{m}_1} & CF((L_1,\mathcal{L}_1,\tilde{s}_1,b_1),(L_2,\mathcal{L}_2,\tilde{s}_2,b_2);\Lambda_{\mathbb{C}}^{\mathbb{Q}}) \\ \rho\big\downarrow & & \rho\big\downarrow \\ CF((L_1,\mathcal{L}_1^{\rho},\tilde{s}_1,b_1^{\rho}),(L_2,\mathcal{L}_2^{\rho},\tilde{s}_2,b_2^{\rho});\Lambda_{\mathbb{C}}^{\mathbb{Q}}) & \xrightarrow{\mathfrak{m}_1} & CF((L_1,\mathcal{L}_1^{\rho},\tilde{s}_1,b_1^{\rho}),(L_2,\mathcal{L}_2^{\rho},\tilde{s}_2,b_2^{\rho});\Lambda_{\mathbb{C}}^{\mathbb{Q}}) \end{array}$$

Proof. We need the following:

Lemma 3.7. *Let $\varphi : S^1 \times [0,1] \to M$ be a smooth map such that $S^1 \times \{0\} \subset L_1$, $S^1 \times \{1\} \subset L_2$; then we have*

$$\exp\left(2\pi\sqrt{-1}\int_{S^1\times[0,1]} \varphi^*\omega\right) hol_{\mathcal{L}_1}(\varphi(S^1\times\{0\}))^{-1} hol_{\mathcal{L}_2}(\varphi(S^1\times\{1\})) = hol_{\mathcal{L}_1^\rho}(\varphi(S^1\times\{0\}))^{-1} hol_{\mathcal{L}_2^\rho}(\varphi(S^1\times\{1\})).$$

Proof. It is easy to see from the definition of prequantum bundle that

$$\exp\left(2\pi\sqrt{-1}\int_{S^1\times[0,1]} \varphi^*\omega\right) = hol_\xi(\varphi(S^1\times\{0\}))^{-1} hol_\xi(\varphi(S^1\times\{1\})). \tag{32}$$

Lemma 3.7 follows easily from (32) and the definitions. □

Let $\beta \in \pi_2(M; L_1, L_2; p, q)$. Then Lemma 3.7 and the definition implies

$$\exp(E'(2\pi\sqrt{-1}\beta))H'(\beta; \mathcal{L}_1, \mathcal{L}_2) = H'(\beta; \mathcal{L}_1^\rho, \mathcal{L}_2^\rho).$$

Proposition 3.6 in the case when $b_i = 0$ then follows immediately. The proof of the general case is similar. □

The proof of Theorem 2.4 is now complete.

4. Relation to Mirror symmetry

In this section, we describe what the $\hat{\mathbb{Z}}$ action of Theorem 2.4 corresponds to, in the mirror, the complex manifold. The argument of this section is rather sketchy since the construction here is not actually new. The author describes it in a way so that it will be the direct analogue of the construction of §2 in the complex category.

Before starting the construction we review several points which are widely believed (though not proved) among the workers of mirror symmetry. Let us consider a symplectic manifold (M, ω) such that $c_1(M) = 0$ and $[\omega] \in H^2(M; \mathbb{Z})$. Let us assume that there exists a mirror $M^\vee$ of M. The manifold $M^\vee$ is a complex manifold whose complex structure depends on ω. So we write it as $(M^\vee, J_\omega)$. First, it is believed that the complex structure of the mirror manifold depends on the complexified symplectic structure. Namely, for $\Omega = \omega + \sqrt{-1}B$ where B is a closed 2-form and ω is a symplectic structure on M, the mirror $M^\vee$ has a corresponding complex structure J which we denote by J_Ω. Ω is called a complexified symplectic structure.

Moreover it is believed that if $[\Omega] - [\Omega'] \in \sqrt{-1}H^2(M; \mathbb{Z})$ then the two complex manifolds $(M^\vee, J_\Omega)$ and $(M^\vee, J_{\Omega'})$ are isomorphic to each other.

Now we consider a family of complexified symplectic structures $(M, -\sqrt{-1}\tau\omega)$ where (M, ω) is as above and $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$. We denote its mirror by $(M^\vee, J_\tau)$. (Namely we write J_τ in place of $J_{-\sqrt{-1}\omega}$.) Since $-\sqrt{-1}\tau\omega - \sqrt{-1}(\tau+1)\omega \in \sqrt{-1}H^2(M;\mathbb{Z})$ we expect that there exists a biholomorphic map

$$\psi_\tau : (M^\vee, J_\tau) \cong (M^\vee, J_{\tau+1}).$$

We put

$$q = \exp(2\pi\sqrt{-1}\tau) \in D^2_*.$$

Here we write $D^2_* = D^2 - \{0\}$. Since J_τ depends only on q we write J_q instead of J_τ. Using ψ as a monodromy we have a holomorphic map

$$\pi : \mathfrak{M}^\circ \to D^2_*$$

whose fiber at q is isomorphic to $(M^\vee, J_q)$. Thus when we start with a single symplectic manifold, its mirror is expected to be a family of complex manifold is parametrized by D^2_*. It is also believed that the family can be compactified to

$$\pi : \mathfrak{M} \to D^2 \tag{33}$$

and the fiber of 0 ($= \pi^{-1}(0)$) is singular. Also it is expected that $\pi^{-1}(0)$ is of maximal degeneration. (See [LTY] for its definition.)

Actually what I have been describing in this section so far are conjectures which are not yet proved in the general case. So, logically speaking, we start from here, and the above arguments are regarded as a motivation. More precisely, we start with the following situation: There exists a family of complex manifolds (33) such that the fiber $\pi^{-1}(q)$ is smooth except for $q = 0$, and $\pi^{-1}(0)$ is of maximal degeneration.

Now we consider the category whose objects are families of sheaves on the fibers of (33). More precisely we proceed as follows. Let $\mathcal{P}_m : D^2 \to D^2$ be the map $\mathcal{P}_m(z) = z^m$. We put

$$\mathfrak{M}_m = \mathfrak{M} \ {}_\pi\!\times_{\mathcal{P}_m} D^2.$$

There exist projections $\pi_m : \mathfrak{M}_m \to D^2$ and $\mathcal{P}_{m,m'} : \mathfrak{M}_{mm'} \to \mathfrak{M}_m$.

We take a neighborhood U of 0 in D^2 and consider a coherent sheaf $\mathfrak{F}$ on $\pi^{-1}\mathcal{P}_m^{-1}(U) \subseteq \mathfrak{M}_m$. Let us consider the set of all such $(\mathfrak{F}, m, U)$ and define an equivalence relation on it as follows.

Let $(\mathfrak{F}, m, U)$,$(\mathfrak{G}, \ell, V)$ be in this set. We say $(\mathfrak{F}, m, U) \sim (\mathfrak{G}, \ell, V)$ if there exists a neighborhood W of 0 with $W \subset U \cap V$ and if there exists a positive integer d and an isomorphism

$$\Psi : (\mathcal{P}^*_{m,d\ell}\mathfrak{F})|_{\pi^{-1}\mathcal{P}^{-1}_{dm\ell}(W-\{0\})} \cong (\mathcal{P}^*_{\ell,dm}\mathfrak{G})|_{\pi^{-1}\mathcal{P}^{-1}_{dm\ell}(W-\{0\})}$$

such that $q^k\Psi$ extends to a morphism : $(\mathcal{P}^*_{m,d\ell}\mathfrak{F})|_{\pi^{-1}\mathcal{P}^{-1}_{dm\ell}(W)} \to (\mathcal{P}^*_{\ell,dm}\mathfrak{G})|_{\pi^{-1}\mathcal{P}^{-1}_{dm\ell}(W)}$ and that $q^k(\Psi^{-1})$ extends to a morphism : $(\mathcal{P}^*_{\ell,dm}\mathfrak{G})|_{\pi^{-1}\mathcal{P}^{-1}_{dm\ell}(W)} \to (\mathcal{P}^*_{m,d\ell}\mathfrak{F})|_{\pi^{-1}\mathcal{P}^{-1}_{dm\ell}(W)}$, for some k. It is easy to see that $\sim$ is an equivalence relation.

An object of our category $\mathcal{SH}(\mathfrak{M})$ is a $\sim$ equivalence class of such triples $(\mathfrak{F}, m, U)$. We denote this set by $\mathfrak{OB}(\mathcal{SH}(\mathfrak{M}))$.

Now, for each pair of objects $[\mathfrak{F}, m, U], [\mathfrak{G}, \ell, V] \in \mathfrak{OB}(\mathcal{SH}(\mathfrak{M}))$, we are going to construct a chain complex of $\Lambda^{\mathbb{Q}}_{\mathbb{C}}$ modules as follows.

For each object of $\mathcal{SH}(\mathfrak{M})$, we fix its representative $(\mathfrak{F}, m, U)$ and its injective resolution $\mathfrak{F} \to \mathbf{R}\mathfrak{F}^*$.

Let $[\mathfrak{F}, m, U], [\mathfrak{G}, \ell, V] \in \mathcal{SH}(\mathfrak{M})$ and $W \subseteq U \cap V$ with $0 \in W$. We consider a complex

$$Hom\left(\mathcal{P}^*_{\ell,m}\mathbf{R}\mathfrak{F}^*|_{\pi^{-1}\mathcal{P}^{-1}_{m\ell}(W)}, \mathcal{P}^*_{m,\ell}\mathbf{R}\mathfrak{G}|_{\pi^{-1}\mathcal{P}^{-1}_{m\ell}(W)}\right) \tag{34}$$

of coherent sheaves on $\pi^{-1}\mathcal{P}^{-1}_{m\ell}(W)$. Let $\mathfrak{O}_{\mathcal{P}^{-1}_{m\ell}(W)}$ be the sheaf of holomorphic function on $\mathcal{P}^{-1}_{m\ell}(W)$ $(\subset D^2)$. We then obtain a complex of $\mathfrak{O}_{\mathcal{P}^{-1}_{m\ell}(W)}$ module sheaves whose stalk at $q \in \mathcal{P}^{-1}_{m\ell}(W)$ is

$$\Gamma\left(\pi^{-1}_{\ell m}(q); Hom\left(\mathcal{P}^*_{\ell,m}\mathbf{R}\mathfrak{F}^*, \mathcal{P}^*_{m,\ell}\mathbf{R}\mathfrak{G}\right)\right).$$

We denote this complex of sheaves by $C^*_W(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G})$.

For any open neighborhood W' of 0 contained in W, we consider the set of sections s of $C^*_W(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G})$ on $\mathcal{P}^{-1}_{\ell m}(W') - \{0\}$ such that $q^k s$ extends to $\mathcal{P}^{-1}_{\ell m}(W')$ for some k. We denote this set by $\Gamma(\mathcal{P}^{-1}_{\ell m}(W'_*); C^*_W(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G}))$.

We put:

$$C^*_0(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G}) = \varprojlim \Gamma(\mathcal{P}^{-1}_{\ell m}(W'_*); C^*_W(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G})) \tag{35}$$

Here the right hand side is the projective limit as $W' \to \{0\}$.

We denote by $\mathbb{C}\langle\langle q^{1/\ell m}\rangle\rangle[q^{-1}]$ the set of all germs at $0 \in \pi^{-1}_{\ell m}(W) \subseteq D^2$ of meromorphic functions. $C^*_0(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G})$ is a complex of $\mathbb{C}\langle\langle q^{1/\ell m}\rangle\rangle[q^{-1}]$ modules.

We may identify $\mathbb{C}\langle\langle q^{1/\ell m}\rangle\rangle[q^{-1}]$ with a subring of $\Lambda^{\mathbb{Q}}_{\mathbb{C}}$ by sending $q^{1/mm'}$ to $T^{1/mm'}$. Now we put

$$C^*([\mathfrak{F}, m, U], [\mathfrak{G}, \ell, V]) = C^*_0(\mathbf{R}\mathfrak{F}, \mathbf{R}\mathfrak{G}) \ \hat{\otimes}_{\mathbb{C}\langle\langle q^{1/\ell m}\rangle\rangle[q^{-1}]} \Lambda^{\mathbb{Q}}_{\mathbb{C}}. \tag{36}$$

Here $\hat{\otimes}$ is the completion of the algebraic tensor product.

It is straightforward to check that the right hand side is independent of the representative $(\mathfrak{F}, m, U), (\mathfrak{G}, \ell, V)$ and of the resolution up to chain homotopy. (However, since we are going to construct a filtered A_∞ category, we need to work at the chain level. This is the reason we fixed the representative and the resolution. The resulting filtered A_∞ category is independent of the choice of them up to homotopy equivalence of filtered A_∞ categories.)

Proposition 4.1. *There exists a filtered A_∞ category $\mathcal{SH}(\mathfrak{M})$ such that the set of its objects is $\mathfrak{OB}(\mathcal{SH}(\mathfrak{M}))$, and the modules of morphisms are $C^*([\mathfrak{F}, m, U], [\mathfrak{G}, \ell, V])$.*

Moreover there exists a $\hat{\mathbb{Z}}$ action on $\mathcal{SH}(\mathfrak{M})$ compatible with its action on $\Lambda^{\mathbb{Q}}_{\mathbb{C}}$.

Proof. The boundary operator $\mathfrak{m}_1$ is the boundary operator of the complex $C(\mathfrak{F}, \mathfrak{G})$. $\mathfrak{m}_2$ is induced by the composition of homomorphisms. We put $\mathfrak{m}_k = 0$ for $k \geq 3$. It is easy to check the axiom of filtered A_∞ category.

We next construct a $\hat{\mathbb{Z}}$ action on $\mathcal{SH}(\mathfrak{M})$. We note that $\hat{\mathbb{Z}}$ is the algebraic fundamental group of D^2_*. The $\hat{\mathbb{Z}}$ action on $\mathcal{SH}(\mathfrak{M})$ is induced from this fact. More precisely we proceed as follows.

We first construct an action on the set of objects. Let $\rho \in \hat{\mathbb{Z}}$ be the topological generator. We let it act as the generator of the deck transformation group of $\mathcal{P}_{1,m} : \mathfrak{M}_m \to \mathfrak{M}$. Let $[\mathfrak{F}, m, U] \in \mathcal{SH}(\mathfrak{M})$. Then $[\rho^*\mathfrak{F}, m, U]$ is also an object of $\mathcal{SH}(\mathfrak{M})$. So we put

$$\rho[\mathfrak{F}, m, U] = [\rho^*\mathfrak{F}, m, U].$$

It is easy to see that this gives an action of $\hat{\mathbb{Z}}$ on the set of objects.

We choose a resolution so that if $\mathfrak{F} \to \mathbf{R}\mathfrak{F}^*$ is the resolution chosen for $\mathfrak{F}$ then $\rho^*\mathfrak{F} \to \rho^*\mathbf{R}\mathfrak{F}^*$ is the resolution chosen for $\rho^*\mathfrak{F}$.

Then it is easy to construct a homomorphism

$$\rho : C^*([\mathfrak{F}, m, U], [\mathfrak{G}, \ell, V]) \to C^*([\rho^*\mathfrak{F}, m, U], [\rho^*\mathfrak{G}, \ell, V])$$

which satisfies (11) and which is a chain map. It is also easy to see that ρ commutes with compositions. The proof of Proposition 4.1 is now complete. □

We have thus described a filtered A_∞ category on which $\hat{\mathbb{Z}}$ acts. Using it and one constructed in §2,3, we can formulate the homological Mirror symmetry conjecture of M.Kontsevich at the level of formal power series. (In [Fu1], for example, it was stated assuming the convergence of $\mathfrak{m}_k$.) Namely:

Conjecture 4.2. *Let M be a Calabi-Yau manifold M with Kähler form ω. We assume $[\omega] \in H^2(M; \mathbb{Z})$. Let us assume that there exists a mirror family $\mathfrak{M} \to D^2_*$ as above.*

Then there exists a filtered A_∞ functor $\mathrm{Mir} : \mathcal{LAG} \to \mathcal{SH}(\mathfrak{M})$ *which preserves $\hat{\mathbb{Z}}$ actions. Moreover the induced homomorphisms* $\mathrm{Mir} : HF((L_1, \mathcal{L}_1, b_1), (L_2, \mathcal{L}_2, b_2)) \to H(C^*(\mathrm{Mir}(L_1, \mathcal{L}_1, b_1), \mathrm{Mir}(L_2, \mathcal{L}_2, b_2)), \mathfrak{m}_1)$ *on cohomologies are isomorphisms.*

We also remark that the $\hat{\mathbb{Z}}$ action described in this section is closely related to the mixed Hodge structure which plays an important role in Mirror symmetry.

We next exhibit our construction in the case of an elliptic curve T^2. In this case, the isomorphism of categories was established in [PZ, Ko] for $\mathfrak{m}_1, \mathfrak{m}_2$ and in [Fu1] for $\mathfrak{m}_k$, $k \ge 3$. Let us calculate the $\hat{\mathbb{Z}}$ action in this case.

The prequantum bundle ξ is $(T^2 \times \mathbb{C}, \pi\sqrt{-1}(xdy - ydx))/\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by $((x, y), v) \mapsto ((x+1, y), e^{\pi\sqrt{-1}y}v)$, $((x, y), v) \mapsto ((x, y+1), e^{-\pi\sqrt{-1}x}v)$.

We here apply the construction of [Fu1] to the case of elliptic curves. Let us consider Lagrangian submanifolds $L_{\mathrm{pt}}(a) = \{(a, y) | y \in S^1\}$, $L_{\mathrm{st}}(b) = \{(x, b) | x \in S^1\}$, $L = \{(x, x) | x \in S^1\}$.

It is easy to see that $L_{\mathrm{pt}}(a)$, $L_{\mathrm{st}}(b)$ are rational if $a, b \in \mathbb{Q}$. Moreover L is rational and $\xi|_L^{\otimes 2}$ is trivial. ($\xi|_L$ is nontrivial, however, since we identify $(x, y; v)$ with $(x+1, y+1; -v)$.)

So the object corresponding to $(L, \mathbb{C})$ is fixed by ρ^2. (Here $\mathbb{C}$ stands for the trivial line bundle.) The object corresponding to $(L_{\mathrm{st}}(0), \mathbb{C})$ is fixed by ρ. We remark that $(L_{\mathrm{st}}(0), \mathbb{C})$ will become a structure sheaf on the mirror and $(L, \mathbb{C})$ will become a polarization E, that is, a line bundle of degree 1. For (x, y^*) we consider the object $L_{x,y^*} = (L_{\mathrm{pt}}(x), \mathcal{L}(y^*))$. Here $\mathcal{L}(y^*)$ stands for the complex line bundle on $S^1 \cong L_{\mathrm{pt}}(x)$ with monodromy $e^{2\pi\sqrt{-1}y^*}$. L_{x,y^*} corresponds to the skyscraper sheaf at $(x, y^*) \in T^2_\tau$ by mirror symmetry. Here we regard $T^2_\tau = \mathbb{R}^2/\mathbb{Z}^2$ such that $(x, y^*) \mapsto \tau x - y^*$ is a complex isomorphism. (So if we take $z = z(x, y^*) = \tau x - y^*$ as a complex coordinate, we have $T^2_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$.)

Remark 4.1. Let us put $\tau = a + \sqrt{-1}b$. Then the complexified symplectic structure is $(b - \sqrt{-1}a)\omega$. We defined a complex structure in [Fu1] so that $(b - \sqrt{-1}a)x + \sqrt{-1}y^*$ is a complex coordinate. It is easy to see that $(b - \sqrt{-1}a)x + \sqrt{-1}y^* = \sqrt{-1}z$.

Since the holonomy of the flat bundle $\xi|_{L_{\rm pt}(x)}$ is

$$\exp\left(\int_0^1 \pi\sqrt{-1}x dy\right) \exp\left(\pi\sqrt{-1}x\right),$$

where the first factor is the integration of the connection form and the second factor is induced by the action $((x, y), v) \mapsto ((x, y + 1), e^{-\pi\sqrt{-1}x}v)$, it follows that

$$\rho(L_{x,y^*}) = L_{x,y^*+x}. \tag{37}$$

Let $\mathbb{C}$ be the trivial line bundle on T^2_τ, that is, the structure sheaf. Let $\mathfrak{F}_z$ be the skyscraper sheaf supported at $z \in T^2_\tau$. We consider the operation (Yoneda product)

$$\mathfrak{m}_2 : \mathrm{Ext}^0(T^2_\tau; \mathbb{C}, E) \otimes \mathrm{Ext}^1(T^2_\tau; E, \mathfrak{F}_z) \to \mathrm{Ext}^1(T^2_\tau; \mathbb{C}, \mathfrak{F}_z). \tag{38}$$

It is easy to see that $\mathrm{Ext}^1(T^2_\tau; E, \mathfrak{F}_z) \cong E^*_z$, the dual to the fiber of E at z. On the other hand, $\mathrm{Ext}^1(T^2_\tau; \mathbb{C}, \mathfrak{F}_z)$ is canonically isomorphic to $\mathbb{C}$. Moreover $\mathrm{Ext}^0(T^2_\tau; \mathbb{C}, E)$ is one dimensional. We find that the map (38) is described as follows. Let $s \in \mathrm{Ext}^0(T^2_\tau; \mathbb{C}, E)$ which is a global holomorphic section of E. Let $v \in E^*_z \cong \mathrm{Ext}^1(T^2_\tau; E, \mathfrak{F}_z)$. Then

$$\mathfrak{m}_2(s, v) = v(s(z)) \in \mathbb{C} \cong \mathrm{Ext}^1(T^2_\tau; \mathbb{C}, \mathfrak{F}_z).$$

The mirror of (38) is

$$\begin{aligned}\mathfrak{m}_2 : HF^0((L_{\rm st}(0), \mathbb{C}), (L, \mathbb{C})) \otimes HF^1((L, \mathbb{C}), ((L_{\rm pt}(x), \mathcal{L}(y^*))) \\ \to HF^1((L_{\rm st}(0), \mathbb{C}), ((L_{\rm pt}(x), \mathcal{L}(y^*))).\end{aligned} \tag{39}$$

Remark 4.2. Here the coefficient ring of the Floer cohomology HF above is $\mathbb{C}$ and is not the Novikov ring. In the case of elliptic curve (or more generally affine Lagrangian submanifolds of a symplectic torus) the operator $\mathfrak{m}_k$ converges when we put $T = e^{-1}$, as was shown in [Fu1]. So Floer cohomology of $\mathbb{C}$ coefficient can be defined in those cases.

$HF^0((L_{\rm st}(0), \mathbb{C}), (L, \mathbb{C}))$ is canonically isomorphic to $\mathbb{C}$. We next remark that

$$HF^1((L, \mathbb{C}), ((L_{\rm pt}(x), \mathcal{L}(y^*))) \cong \mathcal{L}(y^*)|_{(x,0)}$$

(see Remark 3.2) is canonically isomorphism to $\mathbb{C}$. Finally

$$HF^1((L_{\rm st}(0), \mathbb{C}), ((L_{\rm pt}(x), \mathcal{L}(y^*))) \cong \mathcal{L}(y^*)|_{(x,x)}.$$

We can identify it with $\mathbb{C}$ by regarding $\mathcal{L}(y^*)$ as the trivial line bundle with connection $\sqrt{-1}y^* dy$. Here y is a coordinate of $L_{\rm pt}(x) = \{(x, y) | y \in \mathbb{R}\}/\mathbb{Z}$.

We thus identify three Floer cohomology groups with $\mathbb{C}$. Then (39) is

$$\Theta(\tau;(x,y^*)) = \sum_n \exp(\pi\sqrt{-1}\tau(x+n)^2 - 2\pi\sqrt{-1}(x+n)y^*) \tag{40}$$

(40) was first obtained by [Ko]. See also [PZ]. The above formulation is due to [Fu1].

We recall that $\rho^2(L,\mathbb{C}) = (L,\mathbb{C})$, $\rho(L_{\rm st},\mathbb{C}) = (L_{\rm st},\mathbb{C})$, $\rho(L_{x,y^*}) = L_{x,y^*+x}$.

By a direct calculation, it is easy to check

$$\Theta(\tau+2;x,y^*+2x) = \exp(-2\pi\sqrt{-1}x^2)\Theta(\tau;x,y^*). \tag{41}$$

We remark that the isomorphism

$$\xi|_{L_{\rm pt}(x)} \otimes (L_{\rm pt}(x)\times\mathbb{C}, 2\pi\sqrt{-1}y^*dy) \cong (L_{\rm pt}(x)\times\mathbb{C}, 2\pi\sqrt{-1}(y^*+x)dy)$$

is $((x,y),v) \mapsto (x, e^{-\pi\sqrt{-1}xy}v)$. Hence at (x,x) it is multiplication by $e^{-\pi\sqrt{-1}x^2}$. Thus we have a commutative diagram:

$$\begin{array}{ccc} HF^1((L,\mathbb{C}),((L_{\rm pt}(x),\mathcal{L}(y^*))) & \longrightarrow & \mathbb{C} \\ {\scriptstyle\rho^2}\big\downarrow & & \big\downarrow{\scriptstyle\exp(-2\pi\sqrt{-1}x^2)} \\ HF^1((L,\mathbb{C}),((L_{\rm pt}(x),\mathcal{L}(y^*+2\rho))) & \longrightarrow & \mathbb{C} \end{array}$$

Here the horizontal arrow is the isomorphism explained above.

Thus (41) is equivalent to the ρ^2 invariance of $\mathfrak{m}_2$, which is a part of the statement of Theorem 2.4.

In other words, we can proceed as follows. Let us discuss it at the mirror side. The canonical generator 1 of $HF^0((L_{\rm st}(0),\mathbb{C}),(L,\mathbb{C}))$ will turn out to be a global section of E which we write s. The isomorphism $HF^1((L,\mathbb{C}),((L_{\rm pt}(x),\mathcal{L}(y^*))) \cong \mathbb{C}$ explained above will give a unitary frame of E. We multiply it by $\exp(\pi\sqrt{-1}\tau x^2 - 2\pi\sqrt{-1}xy^*)$ to get a holomorphic frame $e_z : E_z \to \mathbb{C}$. Using it we obtain

$$\begin{aligned}\mathfrak{m}_2(s,e_z) = e_z(s(z)) &= \Theta(\tau;x,y)\exp(-\pi\sqrt{-1}\tau x^2 + 2\pi\sqrt{-1}xy^*) \\ &= \sum_n \exp(\pi\sqrt{-1}n^2\tau + 2\pi\sqrt{-1}nz).\end{aligned}$$

$(z = \tau x - y^*)$. We denote by $\vartheta(\tau,z)$ the right hand side. We have

$$\vartheta(\tau,z) = \vartheta(\tau+2,z),$$

which is a standard identity of theta functions.

We remark that there is another (more interesting) identity

$$\vartheta(-1/\tau, z/\tau) = e^{-\pi\sqrt{-1}/4}\tau^{1/2}\vartheta(\tau,z) \tag{42}$$

of theta functions. These two symmetries generate an index two subgroup of $PSL(2;\mathbb{Z})$. It is harder to see (42) from symplectic geometry side of the story. In fact, the action $T^\tau \mapsto T^{-1/\tau}$ is not well-defined on $\Lambda_{\mathbb{C}}^{\mathbb{Q}}$. So, one needs to find a smaller field (whose Galois group over $\mathbb{C}(T)$ (the field of rational functions on $\mathbb{C}P^1$) might be a completion of $PSL(2;\mathbb{Z})$) over which the Floer cohomology and the operations $\mathfrak{m}_k$ are defined.

In the case of elliptic curve, it seems possible to work out such a construction by using the fact that there exists such a symmetry at the complex side and that mirror symmetry is proved in this case. However, to carry it out in detail, one needs to work out what happens for $\mathfrak{m}_k$, $k \geq 3$, at the complex side of the story. It is nontrivial to do so, since $\mathfrak{m}_3$ etc. are secondary operators, so in what sense it is symmetric with respect to $PSL(2;\mathbb{Z})$ is not so obvious.

We can describe the mirror family also as follows. Let us denote by $\mathfrak{h}$ the upper half plane. Ws consider the quotient of $\mathbb{C} \times \mathfrak{h}$ by the $\mathbb{Z}^2 \tilde{\times} PSL(2,\mathbb{Z})$ action defined by $(v,\sigma)(w,z) = (\sigma w + v, \sigma z)$, where σ acts on $\mathfrak{h}$ as a fractional linear transformation and on $\mathbb{C} \cong \mathbb{R}^2$ as a linear transformation. We can compactify it by adding an immersed $\mathbb{C}P^1$ which intersects with itself transversaly. (Type I singular fiber.) This is a family defined over $\mathfrak{h}/PSL(2;\mathbb{Z}) \cup \{[\infty]\}$. We restrict ourselves to a neighborhood of $[\infty]$ and take the double cover branched at $[\infty]$. Then we have a family of elliptic curves $\mathfrak{M} \to D^2$ whose fiber F_0 at $0 = [\infty]$ is the type I_2 singular fiber. Namely F_0 is a union of two embedded $\mathbb{C}P^1$'s say D_1 and D_2 such that $D_1 \cap D_2$ consists of two points, where D_1 and D_2 intersect transversally to each other. This is the double cover of the mirror family of our symplectic T^2. Let us denote by $F_{0,\mathrm{reg}}$ the regular part of F_0. Namely $F_{0,\mathrm{reg}} \cong \mathbb{C}_* \cup \mathbb{C}_*$. For each $p \in F_{0,\mathrm{reg}}$ we can find a section $s_p : D^2 \to \mathfrak{M}$ such that $s_p(0) = p$. (Such a section is not unique.)

Let us discuss the following puzzling point here[1]. We consider affine Lagrangian submanifolds of T^2 parallel to L_{pt}. Such a Lagrangian submanifold corresponds to a section $s_p : D^2 \to \mathfrak{M}$ if it defines an object fixed by ρ^2. (We remark that, in our situation, $\mathbb{C}[[T^2]]$ rational point of the mirror family is such a section.) It is easy to see that there are exactly two Lagrangian submanifolds parallel to L_{pt} on which $\xi^{2\otimes}$ is trivial. Namely they are given by $x = 0$ and by $x = 1/2$. Let us write them as $L_{\mathrm{pt}}(0), L_{\mathrm{pt}}(1/2)$. At first sight, these two are not enough to produce all of the sections s_p. (Here $p \in \mathbb{C}_* \cup \mathbb{C}_*$.)

To solve this puzzle, we recall that objects of our category $\mathcal{LAG}$ consist of $(L, \mathcal{L}, \tilde{s}, b)$. The extra parameter b plays an essential role here. In our case $L = L_{\mathrm{pt}}(0), L_{\mathrm{pt}}(1/2)$, the map $Q : H^1(L) \to H^2(L)$ defining b is trivial since there is no holomorphic disk bounding L. So the moduli space of b's are identified with $H^1(S^1;\mathbb{C}) \otimes_{\mathbb{C}} \Lambda^{\mathbb{Q}}_{\mathbb{C},+}$. Here $\Lambda^{\mathbb{Q}}_{\mathbb{C},+} = \Lambda_{\mathbb{C},+} \cap \Lambda^{\mathbb{Q}}_{\mathbb{C}}$. ($\Lambda_{\mathbb{C},+}$ is defined just before Formula (6).) Namely its element is a sum $\sum_i a_i T^{\lambda_i}$ with $\lambda_i > 0$ ($a_i \in H^1(S^1;\mathbb{C})$). We need to assume $\lambda_i > 0$ in the general situation since otherwise Q may not converge. In the case of elliptic curve, however, we know that all of the operations, especially Q, converge. This means that we may take $b = \sum_i a_i T^{\lambda_i}$ where $\lambda_0 = 0$ and $\lambda_i > 0$ for $i > 0$. Then b will contain a parameter $H^1(L;\mathbb{C})$. The imaginally part of the parameter $H^1(L;\mathbb{C})$ coincides with the parameter deforming the flat connection $\mathcal{L}$ and does not give new parameter. However its real part $H^1(L;\mathbb{R})$ is a new parameter. We also remark that the Galois action of $\hat{\mathbb{Z}}$ is trivial on this factor. If we include this parameter, then each of the Lagrangian submanifolds

[1] The author thanks to the referee who pointed out this interesting puzzle.

$L_{\rm pt}(0), L_{\rm pt}(1/2)$ produces objects parametrized by $\mathbb{C}_* = S^1 \times \mathbb{R}_+$. Here S^1 is a parameter of $\mathcal{L}$ and $\mathbb{R}_+ = \exp(\mathbb{R})$ is new parameter. Hence they give $\mathbb{C}_* \cup \mathbb{C}_*$ as expected.

We also remark that the parameter $H^1(S^1; \Lambda^{\mathbb{Q}}_{\mathbb{C},+})$ in the choice of b corresponds to the different choices of s_p. (Namely various choices of sections s such that $s(0) = p$.)

Remark 4.3. In the above argument, we considered the mirror family parametrized by the double cover of the disks. This is only because I want to find the most standard theta function on the mirror family. Note that $L' = \{[x, x+1/2] | x \in \mathbb{R}\}$ is a Bohr-Sommerfeld orbit. Hence, using L' instead of L, we obtain a family of line bundles parametrized by a neighborhood of $[\infty]$ in the compactification of $\mathfrak{h}/PSL(2;\mathbb{Z})$ (not by its double cover).

Remark 4.4. In the case of Novikov homology of closed 1-form on a finite dimensional manifold, the operator $\mathfrak{m}_1$ is proved to be a rational function (that is, a meromorphic function on $\mathbb{C}P^1$) by [Pa]. It seems likely to be true for higher $\mathfrak{m}_k$. In other words, the "mirror of finite dimensional Novikov homology" gives a family parametrized by $\mathbb{C}P^1$. In our case of infinite dimensional Floer theory, the story should be more involved even in the case of an elliptic curve.

It might be possible to generalize the story of elliptic curves mentioned above to the case of higher dimensional torus and affine Lagrangian submanifolds, by using [Fu1] (at least as far as $\mathfrak{m}_2$ is concerned).

However beyond that case, to establish a symmetry bigger than $\hat{\mathbb{Z}}$ symmetry on the symplectic side of the story is extremely difficult.

Let us consider, for example, the case of the quintic M. We have $H^2(M) = \mathbb{C}$, so the moduli space of complexified symplectic structures of M is one dimensional. Let $M^\vee$ be the mirror quintic. The moduli space of its complex structures is a quotient of the upper half plane by the symmetries $\tau \mapsto \tau + 1$ and $\tau \mapsto \frac{\tau\cos 2\pi/5 - \sin 2\pi/5}{\tau \sin 2\pi/5 + \cos 2\pi/5}$, an element of $PSL(2,\mathbb{R})$, of order 5 which fixes $\sqrt{-1}$. (See for example [COGP].) The first symmetry corresponds to the $\hat{\mathbb{Z}}$ symmetry which we now understand. However the second symmetry at the symplectic side is mysterious and is very interesting.

One may also try to study a similar symmetry in the case of K3 surface. Namely the action of the group $O(3,19;\mathbb{Z})$ are supposed to exist on the category of Lagrangian submanifolds. (In this case we need to move symplectic structure of K3 surface also to get full symmetry.)

The existence of such symmetries in the symplectic side seems to be one of the deepest consequences of Mirror symmetry.

Remark 4.5. We remark that there are several works comparing two symmetries by mirror symmetry. (See [ST, SK]). However in [ST, SK] the symmetry is induced by a symplectic diffeomorphism (generalized Dehn twist) in the symplectic side, and in the complex side, it is not induced by a biholomorphic map but is a Fourier-Mukai transform. In our situation, in the complex side the symmetry is induced by a biholomorphic map, and in the symplectic side it is not induced by a symplectic diffeomorphism.

We also remark that the $\hat{\mathbb{Z}}$ action on objects of $\mathcal{LAG}$ is easy to calculate. If we assume that the mirror is obtained by a dual torus fibration as in [SYZ], a point of the mirror manifold $M^\vee$ can be identified with an object of $\mathcal{LAG}$ of M. Therefore, Conjecture 4.2 gives a nice description of the monodromy $\psi_\tau : (M^\vee, J_\tau) \cong (M^\vee, J_{\tau+1})$. For example, the following will be proved if Conjecture 4.2 is true.

Conjecture 4.3. *In the situation of Conjecture 4.2, the monodromy $\psi_\tau : (M^\vee, J_\tau) \cong (M^\vee, J_{\tau+1})$. can be taken as a "completely integrable system".*

Namely the following holds. There exists a Baire set $M_0^\vee$ of $M^\vee$ such that if $x \in M_0^\vee$ then the closure of the orbit $\{\psi_\tau^k(x) | k \in \mathbb{Z}\}$ is an n dimensional Lagrangian torus. ($n = \dim_{\mathbb{C}} M^\vee$). Moreover, for any x, the closure of $\{\psi_\tau^k(x) | k \in \mathbb{Z}\}$ is a finite union of isotropic tori.

For example, in the case of the elliptic curve, the monodromy is $(x, y^*) \mapsto (x, x + y^*)$. Hence the closure of its orbit is S^1 if x is not rational.

Solving Conjecture 4.3 might give a way to find a Lagrangian torus fibration of a Calabi-Yau manifold near the maximal degeneration point.

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