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U(1)-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ and special Lagrangian fibrations

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Abstract

This is a survey of the author's series of three papers [8, 9, 10] on *special Lagrangian 3-folds* (*SL 3-folds*) in $\mathbb{C}^3$ invariant under the U(1)-action $(z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3)$, and their sequel [11] on special Lagrangian fibrations and the SYZ Conjecture.

We briefly present the main results of these four long papers, giving some explanation and motivation, but no proofs. The aim is to make the results and ideas accessible to String Theorists and others who have an interest in special Lagrangian 3-folds and fibrations, but have no desire to read pages of technical analysis.

Let N be an SL 3-fold in $\mathbb{C}^3$ invariant under the U(1)-action above. Then $|z_1|^2 - |z_2|^2 = 2a$ on N for some $a \in \mathbb{R}$. Locally, N can be written as a kind of graph of functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfying a nonlinear Cauchy–Riemann equation depending on a, so that $u + iv$ is like a holomorphic function of $x + iy$. When $a = 0$ the equations may have singular points where u, v are not differentiable, which leads to analytic difficulties.

We prove existence and uniqueness results for solutions u, v on domains S in $\mathbb{R}^2$ with boundary conditions, including singular solutions. We study their singularities, giving a rough classification by *multiplicity* and *type*. We prove the existence of large families of *fibrations* of open subsets of $\mathbb{C}^3$ by U(1)-invariant SL 3-folds, including singular fibres. Finally, we use these fibrations as local models to draw conclusions about the *SYZ Conjecture* on Mirror Symmetry of Calabi-Yau 3-folds.

1. Introduction

Special Lagrangian submanifolds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, which are calibrated with respect to the m-form $\mathrm{Re}(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m)$. They can also be defined in (almost) Calabi–Yau manifolds, are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry between Calabi–Yau 3-folds.

This paper surveys three papers [8, 9, 10] studying special Lagrangian 3-folds N in $\mathbb{C}^3$ invariant under the U(1)-action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3) \quad \text{for } e^{i\theta} \in \mathrm{U}(1), \tag{1}$$

and also the sequel [11], which applies their results to study the *SYZ Conjecture* about Mirror Symmetry of Calabi–Yau 3-folds [14].

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Locally we can write N in the form

$$N = \Big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : \mathrm{Im}(z_3) = u\big(\mathrm{Re}(z_3),\mathrm{Im}(z_1z_2)\big), \\ \mathrm{Re}(z_1z_2) = v\big(\mathrm{Re}(z_3),\mathrm{Im}(z_1z_2)\big), \quad |z_1|^2-|z_2|^2 = 2a\Big\}, \tag{2}$$

where $a\in\mathbb{R}$ and $u,v:\mathbb{R}^2\to\mathbb{R}$ are continuous functions. Then N is an SL 3-fold if and only if u,v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial v}{\partial x} = -2\big(v^2+y^2+a^2\big)^{1/2}\frac{\partial u}{\partial y}. \tag{3}$$

If $a\neq 0$ then (3) is *elliptic*, and so solutions u,v of (3) are automatically real analytic, and the corresponding SL 3-folds N are nonsingular.

However, if $a=0$ then at points $(x,0)$ with $v(x,0)=0$ the factor $(v^2+y^2+a^2)^{1/2}$ becomes zero, and (3) is no longer elliptic. Because of this, when $a=0$ the appropriate thing to do is consider *weak* solutions of (3), which may have *singular points* $(x,0)$ with $v(x,0)=0$. At such a point u,v may not be differentiable, and $\big(0,0,x+iu(x,0)\big)$ is a singular point of the SL 3-fold N.

Equation (3) is a *nonlinear Cauchy–Riemann equation*, so that if u,v is a solution then $u+iv$ is a bit like a holomorphic function of $x+iy$. Therefore we may use ideas and methods from complex analysis to study the solutions of (3), and the corresponding SL 3-folds N.

Section 2 introduces special Lagrangian geometry, and §3 recalls some analytic background. Section 4 begins the discussion of SL 3-folds of the form (2), looking at equation (3) and what properties we expect of singular solutions. Some examples of solutions u,v to (3) are given in §5.

Section 6 rewrites (3) in terms of a *potential* f with $\frac{\partial f}{\partial y}=u$ and $\frac{\partial f}{\partial x}=v$. Then f satisfies the equation

$$\Big(\Big(\frac{\partial f}{\partial x}\Big)^2+y^2+a^2\Big)^{-1/2}\frac{\partial^2 f}{\partial x^2}+2\,\frac{\partial^2 f}{\partial y^2}=0. \tag{4}$$

We can prove existence and uniqueness for the Dirichlet problem for (4) on a suitable class of convex domains in $\mathbb{R}^2$. This yields existence and uniqueness results for U(1)-invariant SL 3-folds N in $\mathbb{C}^3$ with boundary conditions, including singular SL 3-folds.

Section 7 studies zeroes of $(u_1,v_1)-(u_2,v_2)$ when u_j,v_j satisfy (3) for $j=1,2$. As $(u_1,v_1)-(u_2,v_2)$ acts like a holomorphic function, if $(u_1,v_1)\not\equiv(u_2,v_2)$ zeroes are isolated, and have a positive integer *multiplicity*. This is applied in §8 to singular solutions u,v of (3) with $a=0$. We find that either $u(x,-y)\equiv u(x,y)$ and $v(x,-y)\equiv -v(x,y)$, so that u,v is singular all along the x-axis, or else singularities are isolated, with a positive integer *multiplicity* and one of two *types*.

Section 9 uses the material of §6–§7 to construct large families of U(1)-invariant *special Lagrangian fibrations* of open subsets of $\mathbb{C}^3$. Finally, §10 discusses the *SYZ Conjecture*, and summarizes the conclusions of [11] on special Lagrangian fibrations of (almost) Calabi–Yau 3-folds.

2. Special Lagrangian geometry

We shall define special Lagrangian submanifolds first in $\mathbb{C}^m$ and then in *almost Calabi–Yau manifolds*, a generalization of Calabi–Yau manifolds. For introductions to special Lagrangian geometry, see Harvey and Lawson [5, §III] or the author [7].

2.1. Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrated submanifolds*, following Harvey and Lawson [5].

Definition 2.1. Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane V* on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [5, Th. II.4.2]. Here is the definition of SL m-folds in $\mathbb{C}^m$, taken from [5, §III].

Definition 2.2. Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$\begin{aligned} g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega &= \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega &= \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{aligned} \tag{5}$$

Then $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\mathrm{Re}\,\Omega$, in the sense of Definition 2.1.

Harvey and Lawson [5, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

Proposition 2.1. *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into a special Lagrangian submanifold of $\mathbb{C}^m$ if and only if $\omega|_L \equiv 0$ and $\mathrm{Im}\,\Omega|_L \equiv 0$.*

An m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\mathrm{Im}\,\Omega|_L \equiv 0$, which is how they get their name.

2.2. Almost Calabi–Yau m-folds and SL m-folds

Probably the best general context for special Lagrangian geometry is *almost Calabi–Yau manifolds*.

Definition 2.3. Let $m \geqslant 2$. An *almost Calabi–Yau m-fold* is a quadruple (X, J, ω, Ω) such that (X, J) is a compact m-dimensional complex manifold, ω the Kähler form of a Kähler metric g on X, and Ω a non-vanishing holomorphic $(m, 0)$-form on X.

We call (X, J, ω, Ω) a *Calabi–Yau m-fold* if in addition

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}.$$

Then for each $x \in X$ there exists an isomorphism $T_xX \cong \mathbb{C}^m$ that identifies g_x, ω_x and Ω_x with the flat versions g, ω, Ω on $\mathbb{C}^m$ in (5). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\mathrm{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it. Motivated by Proposition 2.1, we define *special Lagrangian submanifolds* of almost Calabi–Yau manifolds.

Definition 2.4. Let (X, J, ω, Ω) be an almost Calabi–Yau m-fold with metric g, and N a real m-dimensional submanifold of X. We call N a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N \equiv \mathrm{Im}\, \Omega|_N \equiv 0$.

The properties of SL m-folds in almost Calabi–Yau m-folds are discussed by the author in [7]. It turns out [7, §9.5] that SL m-folds in almost Calabi–Yau m-folds are also calibrated w.r.t. $\mathrm{Re}\, \Omega$, but using a conformally rescaled metric $\tilde{g} = f^2 g$.

The deformation and obstruction theory for *compact* SL m-folds in almost Calabi–Yau m-folds is well understood, and beautifully behaved. Locally, SL m-folds in almost Calabi–Yau m-folds are expected to behave like SL m-folds in $\mathbb{C}^m$, especially in their singular behaviour. Thus, by studying singular SL m-folds in $\mathbb{C}^m$, we learn about singular SL m-folds in almost Calabi–Yau m-folds.

3. Background material from analysis

Here are some definitions we will need to make sense of analytic results from [8, 9, 10]. A closed, bounded, contractible subset S in $\mathbb{R}^n$ will be called a *domain* if the *interior* S° of S is connected with $S = \overline{S^\circ}$, and the *boundary* $\partial S = S \setminus S^\circ$ is a compact embedded hypersurface in $\mathbb{R}^n$. A domain S in $\mathbb{R}^2$ is called *strictly convex* if S is convex and the curvature of ∂S is nonzero at every point.

Let S be a domain in $\mathbb{R}^n$. Define $C^k(S)$ for $k \geqslant 0$ to be the space of continuous functions $f : S \to \mathbb{R}$ with k continuous derivatives, and $C^\infty(S) = \bigcap_{k=0}^\infty C^k(S)$. For $k \geqslant 0$ and $\alpha \in (0, 1)$, define the *Hölder space* $C^{k,\alpha}(S)$ to be the subset of $f \in C^k(S)$ for which

$$[\partial^k f]_\alpha = \sup_{x \neq y \in S} \frac{\left|\partial^k f(x) - \partial^k f(y)\right|}{|x - y|^\alpha} \quad \text{is finite.}$$

A *second-order quasilinear operator* $Q : C^2(S) \to C^0(S)$ is an operator of the form

$$\big(Qu\big)(x) = \sum_{i,j=1}^{n} a^{ij}(x,u,\partial u)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + b(x,u,\partial u),$$

where a^{ij} and b are continuous maps $S \times \mathbb{R} \times (\mathbb{R}^n)^* \to \mathbb{R}$, and $a^{ij} = a^{ji}$ for all $i,j = 1,\ldots,n$. We call the functions a^{ij} and b the *coefficients* of Q. We call Q *elliptic* if the symmetric $n \times n$ matrix (a^{ij}) is positive definite at every point.

A second-order quasilinear operator Q is in *divergence form* if it is written

$$\big(Qu\big)(x) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(a^j(x,u,\partial u)\big) + b(x,u,\partial u)$$

for functions $a^j \in C^1\big(S\times\mathbb{R}\times(\mathbb{R}^n)^*\big)$ for $j = 1,\ldots,n$ and $b \in C^0\big(S\times\mathbb{R}\times(\mathbb{R}^n)^*\big)$. If Q is in divergence form, we say that integrable functions u, f are a *weak solution* of the equation $Qu = f$ if u is weakly differentiable with weak derivative ∂u, and $a^j(x,u,\partial u), b(x,u,\partial u)$ are integrable with

$$-\sum_{j=1}^{n} \int_S \frac{\partial \psi}{\partial x_j} \cdot a^j(x,u,\partial u)\mathrm{d}\mathbf{x} + \int_S \psi \cdot b(x,u,\partial u)\mathrm{d}\mathbf{x} = \int_S \psi \cdot f \,\mathrm{d}\mathbf{x}$$

for all $\psi \in C^1(S)$ with $\psi|_{\partial S} \equiv 0$.

If Q is a second-order quasilinear operator, we may interpret the equation $Qu = f$ in three different senses:

- We just say that $Qu = f$ if $u \in C^2(S)$, $f \in C^0(S)$ and $Qu = f$ in $C^0(S)$ in the usual way.
- We say that $Qu = f$ *holds with weak derivatives* if u is twice weakly differentiable and $Qu = f$ holds almost everywhere, defining Qu using weak derivatives.
- We say that $Qu = f$ *holds weakly* if Q is in divergence form and u is a weak solution of $Qu = f$. Note that this requires only that u be *once* weakly differentiable, and the second derivatives of u need not exist even weakly.

Clearly the first sense implies the second, which implies the third. If Q is *elliptic* and a^j, b, f are suitably regular, one can usually show that a weak solution to $Qu = f$ is a classical solution, so that the three senses are equivalent. But for singular equations that are not elliptic at every point, the three senses are distinct.

4. Finding the equations

Let N be a special Lagrangian 3-fold in $\mathbb{C}^3$ invariant under the U(1)-action

$$\mathrm{e}^{i\theta} : (z_1,z_2,z_3) \mapsto (\mathrm{e}^{i\theta}z_1, \mathrm{e}^{-i\theta}z_2, z_3) \quad \text{for } \mathrm{e}^{i\theta} \in \mathrm{U}(1). \tag{6}$$

Locally we can write N in the form

$$N = \big\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x,y) + iy, \quad z_3 = x + iu(x,y), \\ |z_1|^2 - |z_2|^2 = 2a, \quad (x,y) \in S\big\}, \tag{7}$$

where S is a domain in $\mathbb{R}^2$, $a \in \mathbb{R}$ and $u, v : S \to \mathbb{R}$ are continuous.

Here $|z_1|^2 - |z_2|^2$ is the *moment map* of the U(1)-action (6), and so $|z_1|^2 - |z_2|^2$ is constant on any U(1)-invariant Lagrangian 3-fold in $\mathbb{C}^3$. We choose the constant to be $2a$. Effectively (7) just means that we choose $x = \mathrm{Re}(z_3)$ and $y = \mathrm{Im}(z_1 z_2)$ as local coordinates on the 2-manifold $N/\mathrm{U}(1)$. Then we find [8, Prop. 4.1]:

Proposition 4.1. *Let S, a, u, v and N be as above. Then*

(a) *If $a = 0$, then N is a (possibly singular) special Lagrangian 3-fold in $\mathbb{C}^3$ if u, v are differentiable and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2\big(v^2 + y^2\big)^{1/2}\frac{\partial u}{\partial y}, \tag{8}$$

except at points $(x, 0)$ in S with $v(x, 0) = 0$, where u, v need not be differentiable. The singular points of N are those of the form $(0, 0, z_3)$, where $z_3 = x + iu(x, 0)$ for $(x, 0) \in S$ with $v(x, 0) = 0$.

(b) *If $a \neq 0$, then N is a nonsingular special Lagrangian 3-fold in $\mathbb{C}^3$ if and only if u, v are differentiable in S and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2\big(v^2 + y^2 + a^2\big)^{1/2}\frac{\partial u}{\partial y}. \tag{9}$$

The proof is elementary: at each point $\mathbf{z} \in N$ we calculate the tangent space $T_{\mathbf{z}}N$ in terms of $\partial u, \partial v$, and use Proposition 2.1 to find the conditions for $T_{\mathbf{z}}N$ to be a special Lagrangian $\mathbb{R}^3$ in $\mathbb{C}^3$. If $\mathbf{z} = (0, 0, z_3)$ then $\mathrm{d}\big(|z_1|^2 - |z_2|^2\big) = 0$ at $\mathbf{z}$, so $\mathbf{z}$ is a singular point of N, and $T_{\mathbf{z}}N$ does not exist.

Using (9) to write $\frac{\partial}{\partial y}\big(\frac{\partial u}{\partial x}\big)$ and $\frac{\partial}{\partial x}\big(\frac{\partial u}{\partial y}\big)$ in terms of v and setting $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$, we easily prove [8, Prop. 8.1]:

Proposition 4.2. *Let S be a domain in $\mathbb{R}^2$ and $u, v \in C^2(S)$ satisfy (9) for $a \neq 0$. Then*

$$\frac{\partial}{\partial x}\Big[\big(v^2 + y^2 + a^2\big)^{-1/2}\frac{\partial v}{\partial x}\Big] + 2\,\frac{\partial^2 v}{\partial y^2} = 0. \tag{10}$$

Conversely, if $v \in C^2(S)$ satisfies (10) then there exists $u \in C^2(S)$, unique up to addition of a constant $u \mapsto u + c$, such that u, v satisfy (9).

Now (10) is a second order quasilinear elliptic equation, in divergence form. Thus we can consider *weak solutions* of (10) when $a = 0$, which need be only once weakly differentiable. We shall be interested in solutions of (8) with singularities, and the corresponding SL 3-folds N. It will be helpful to define a class of *singular solutions* of (8).

Definition 4.1. Let S be a domain in $\mathbb{R}^2$ and $u, v \in C^0(S)$. We say that (u, v) is a *singular solution* of (8) if

(i) u, v are weakly differentiable, and their weak derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ satisfy (8).

(ii) v is a *weak solution* of (10) with $a = 0$, as in §3.

(iii) Define the *singular points* of u, v to be the $(x, 0) \in S$ with $v(x, 0) = 0$. Then except at singular points, u, v are C^2 in S and real analytic in S°, and satisfy (8) in the classical sense.
(iv) For $a \in (0, 1]$ there exist $u_a, v_a \in C^2(S)$ satisfying (9) such that $u_a \to u$ and $v_a \to v$ in $C^0(S)$ as $a \to 0_+$.

This list of properties is somewhat arbitrary. The point is that [9, §8–§9] gives powerful existence and uniqueness results for solutions u, v of (8) satisfying conditions (i)–(iv) and various boundary conditions on ∂S, and all of (i)–(iv) are useful in different contexts.

5. Examples

Here are four examples of SL 3-folds N in the form (7), taken from [8, §5].

Example 5.1. Let $a \geqslant 0$, and define

$$N_a = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 2a = |z_2|^2 = |z_3|^2, \\ \mathrm{Im}\big(z_1 z_2 z_3\big) = 0, \quad \mathrm{Re}\big(z_1 z_2 z_3\big) \geqslant 0\Big\}.$$

Then N_a is a nonsingular SL 3-fold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$ when $a > 0$, and N_0 is an SL T^2-cone with one singular point at $(0, 0, 0)$. The N_a are invariant under the $\mathrm{U}(1)^2$-action

$$(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}) : (z_1, z_2, z_3) \longmapsto (\mathrm{e}^{i\theta_1} z_1, \mathrm{e}^{i\theta_2} z_2, \mathrm{e}^{-i\theta_1 - i\theta_2} z_3),$$

which includes the U(1)-action (6), and are part of a family of explicit $\mathrm{U}(1)^2$-invariant SL 3-folds written down by Harvey and Lawson [5, §III.3.A]. By [8, Th. 5.1], these SL 3-folds can be written in the form (7), using functions $u_a, v_a : \mathbb{R}^2 \to \mathbb{R}$.

Example 5.2. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and define $u(x, y) = \alpha x + \beta$ and $v(x, y) = \alpha y + \gamma$. Then u, v satisfy (9) for any value of a.

Example 5.3. Define $u(x, y) = y \tanh x$ and $v(x, y) = \frac{1}{2} y^2 \mathrm{sech}^2 x - \frac{1}{2} \cosh^2 x$. Then u and v satisfy (8). Equation (7) with $a = 0$ defines an explicit nonsingular SL 3-fold N in $\mathbb{C}^3$. It arises from Harvey and Lawson's 'austere submanifold' construction [5, §III.3.C] of SL m-folds in $\mathbb{C}^m$, as the normal bundle of a catenoid in $\mathbb{R}^3$.

Example 5.4. Define $u(x, y) = |y| - \frac{1}{2} \cosh 2x$ and $v(x, y) = -y \sinh 2x$. Then u, v satisfy (8), except that $\frac{\partial u}{\partial y}$ is not well-defined on the x-axis. So equation (7) with $a = 0$ gives an explicit SL 3-fold N in $\mathbb{C}^3$. It is the union of two nonsingular SL 3-folds intersecting in a real curve, which are constructed in [6, Ex. 7.4] by evolving paraboloids in $\mathbb{C}^3$.

One can show that when $a = 0$, all four examples yield *singular solutions* of (8) in $\mathbb{R}^2$, in the sense of Definition 4.1.

6. Generating u, v from a potential f

If u, v satisfy (9) then as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ there exists a *potential* f for u, v with $\frac{\partial f}{\partial y} = u$, $\frac{\partial f}{\partial x} = v$. So we easily prove [8, Prop. 7.1]:

Proposition 6.1. *Let S be a domain in $\mathbb{R}^2$ and $u, v \in C^1(S)$ satisfy* (9) *for $a \neq 0$. Then there exists $f \in C^2(S)$ with $\frac{\partial f}{\partial y} = u$, $\frac{\partial f}{\partial x} = v$ and*

$$\Big(\Big(\frac{\partial f}{\partial x}\Big)^2 + y^2 + a^2\Big)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 f}{\partial y^2} = 0. \tag{11}$$

This f is unique up to addition of a constant, $f \mapsto f + c$. Conversely, all solutions of (11) *yield solutions of* (9).

Equation (11) is a second-order quasilinear elliptic equation, singular when $a = 0$, which may be written in divergence form. The following condensation of [8, Th. 7.6] and [9, Th.s 9.20 & 9.21] proves existence and uniqueness for the *Dirichlet problem* for (11).

Theorem 6.2. *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, and $k \geqslant 0$, $\alpha \in (0, 1)$. Let $a \in \mathbb{R}$ and $\phi \in C^{k+3,\alpha}(\partial S)$. Then if $a \neq 0$ there exists a unique $f \in C^{k+3,\alpha}(S)$ with $f|_{\partial S} = \phi$ satisfying* (11). *If $a = 0$ there exists a unique $f \in C^1(S)$ with $f|_{\partial S} = \phi$, which is twice weakly differentiable and satisfies* (11) *with weak derivatives.*

Define $u = \frac{\partial f}{\partial y}$ and $v = \frac{\partial f}{\partial x}$. If $a \neq 0$ then $u, v \in C^{k+2,\alpha}(S)$ satisfy (9), *and if $a = 0$ then $u, v \in C^0(S)$ are a singular solution of* (8), *in the sense of Definition* 4.1. *Furthermore, f depends continuously in $C^1(S)$, and u, v depend continuously in $C^0(S)$, on (ϕ, a) in $C^{k+3,\alpha}(\partial S) \times \mathbb{R}$.*

Here is a very brief sketch of the proof. As f, u, v satisfy certain linear elliptic equations, using the maximum principle the maxima and minima of f, u, v are achieved on ∂S. Thus $|f| \leqslant \sup_{\partial S} |\phi|$. Also, using linear functions $f' = \alpha + \beta x + \gamma y$ as comparison solutions of (11) and the strict convexity of S we can bound u, v on ∂S, and hence on S, in terms of the first two derivatives of ϕ. So we have *a priori* bounds for f in $C^1(S)$ and u, v in $C^0(S)$.

When $a \neq 0$, bounds on v and y imply that (11) is *uniformly elliptic*. We can therefore use standard results on the Dirichlet problem for second order, quasilinear, uniformly elliptic equations to prove the existence of a unique $f \in C^{k+3,\alpha}(S)$ satisfying (11) with $f|_{\partial S} = \phi$.

However, when $a = 0$ equation (11) is not uniformly elliptic, and there do not appear to be standard results available to complete the theorem. Therefore in [9, §9] we define $f_a \in C^{k+3,\alpha}(S)$ for $a \in (0, 1]$ to be the unique solution of (11) with $f|_{\partial S} = \phi$, and we show that f_a converges in $C^1(S)$ as $a \to 0_+$ to a solution f of (11) for $a = 0$, with weak second derivatives.

The key step in doing this is to prove *a priori* estimates for f_a and its first two derivatives, that hold uniformly for $a \in (0, 1]$. Proving such estimates, and making them strong enough to ensure that u, v are continuous, was responsible for the length and technical difficulty of [9].

Combining Proposition 4.1 and Theorem 6.2 gives existence and uniqueness for a large class of U(1)-invariant SL 3-folds in $\mathbb{C}^3$, with boundary conditions, including *singular* SL 3-folds. It is interesting that this existence and uniqueness is *entirely unaffected* by singularities appearing in S°.

7. Results motivated by complex analysis

In [8, §6] and [10, §7] we study the zeroes of $(u_1, v_1) - (u_2, v_2)$, where (u_j, v_j) satisfy (8) or (9). A key tool is the idea of *winding number.*

Definition 7.1. Let C be a compact oriented 1-manifold, and $\gamma : C \to \mathbb{R}^2 \setminus \{0\}$ a differentiable map. Then the *winding number of* γ *about* 0 *along* C is $\frac{1}{2\pi}\int_C \gamma^*(\mathrm{d}\theta)$, where $\mathrm{d}\theta$ is the closed 1-form $x^{-1}\mathrm{d}y - y^{-1}\mathrm{d}x$ on $\mathbb{R}^2 \setminus \{0\}$.

The motivation is the following theorem from elementary complex analysis:

Theorem 7.1. *Let S be a domain in $\mathbb{C}$, and suppose $f : S \to \mathbb{C}$ is a holomorphic function, with $f \neq 0$ on ∂S. Then the number of zeroes of f in S°, counted with multiplicity, is equal to the winding number of $f|_{\partial S}$ about 0 along ∂S.*

As (8) and (9) are nonlinear versions of the Cauchy–Riemann equations for holomorphic functions, it is natural to expect that similar results should hold for their solutions. The first step is to define the *multiplicity* of an isolated zero in S°, following [10, Def. 7.1].

Definition 7.2. Let S be a domain in $\mathbb{R}^2$, and $a \in \mathbb{R}$. Suppose $u_j, v_j : S \to \mathbb{R}$ for $j = 1, 2$ are solutions of (9) in $C^1(S)$ if $a \neq 0$, and singular solutions of (8) in $C^0(S)$ if $a = 0$, in the sense of Definition 4.1.

We call a point $(b, c) \in S$ a *zero* of $(u_1, v_1) - (u_2, v_2)$ in S if $(u_1, v_1) = (u_2, v_2)$ at (b, c). A zero (b, c) is called *singular* if $a = c = 0$ and $v_1(b, 0) = v_2(b, 0) = 0$, so that (b, c) is a *singular point* of u_1, v_1 and u_2, v_2. Otherwise we say (b, c) is a *nonsingular zero.* We call a zero (b, c) *isolated* if for some $\epsilon > 0$ there exist no other zeroes (x, y) of $(u_1, v_1) - (u_2, v_2)$ in S with $0 < (x - b)^2 + (y - c)^2 \leqslant \epsilon^2$.

Let $(b, c) \in S^\circ$ be an isolated zero of $(u_1, v_1) - (u_2, v_2)$. Define the *multiplicity* of (b, c) to be the winding number of $(u_1, v_1) - (u_2, v_2)$ about 0 along the positively oriented circle $\gamma_\epsilon(b, c)$ of radius ϵ about (b, c), where $\epsilon > 0$ is chosen small enough that $\gamma_\epsilon(b, c)$ lies in S° and (b, c) is the only zero of $(u_1, v_1) - (u_2, v_2)$ inside or on $\gamma_\epsilon(b, c)$.

From [8, §6.1] and [10, Cor. 7.6], we have:

Theorem 7.2. *In the situation above, the multiplicity of any isolated zero (b, c) of $(u_1, v_1) - (u_2, v_2)$ in S° is a positive integer.*

The proof is different depending on whether (b, c) is a singular or a nonsingular zero. If (b, c) is nonzero one can show [8, Prop. 6.5]:

Proposition 7.3. *In the situation above, suppose $(u_1,v_1)-(u_2,v_2)$ has an isolated, nonsingular zero at $(b,c)\in S^\circ$. Then there exists $k\geqslant 1$ and $C\in\mathbb{C}\setminus\{0\}$ such that*

$$\begin{aligned}\lambda u_1(x,y)+iv_1(x,y)=\lambda u_2(x,y)+iv_2(x,y)+C\bigl(\lambda(x-b)+i(y-c)\bigr)^k\\ +O\bigl(|x-b|^{k+1}+|y-c|^{k+1}\bigr),\end{aligned}\tag{12}$$

where $\lambda=\sqrt{2}\bigl(v_1(b,c)^2+c^2+a^2\bigr)^{1/4}$.

The point is that $\lambda(u_1-u_2)+i(v_1-v_2)$ is like a holomorphic function of $\lambda x+iy$ near $\lambda b+ic$, so to leading order it is a multiple of $(\lambda(x-b)+i(y-c))^k$ for some $k\geqslant 1$. Comparing (12) with Definition 7.2 we see that k is the *multiplicity* of (b,c), which proves that multiplicities of nonsingular zeroes are positive integers.

Proposition 7.3 also gives an alternative, more familiar characterization of the multiplicity of a nonsingular zero, [8, Def. 6.3]: an isolated, nonsingular zero (b,c) of $(u_1,v_1)-(u_2,v_2)$ has multiplicity $k\geqslant 1$ if $\partial^j u_1(b,c)=\partial^j u_2(b,c)$ and $\partial^j v_1(b,c)=\partial^j v_2(b,c)$ for $j=0,\dots,k-1$, but $\partial^k u_1(b,c)\neq\partial^k u_2(b,c)$ and $\partial^k v_1(b,c)\neq\partial^k v_2(b,c)$.

Proving Theorem 7.2 for singular zeroes is more tricky. Basically it follows from the nonsingular case by a limiting argument involving part (iv) of Definition 4.1, but there are subtleties in showing that a singular zero cannot have multiplicity zero.

If $(u_1,v_1)\not\equiv(u_2,v_2)$ then all zeroes of $(u_1,v_1)-(u_2,v_2)$ in S° are isolated, [8, Cor. 6.6], [10, Th. 7.8].

Theorem 7.4. *Let S be a domain in $\mathbb{R}^2$, and $a\in\mathbb{R}$. If $a\neq 0$ let $u_j,v_j\in C^1(S)$ satisfy (9) for $j=1,2$, and if $a=0$ let $u_j,v_j\in C^0(S)$ be singular solutions of (8) for $j=1,2$. Then either $(u_1,v_1)\equiv(u_2,v_2)$, or there are at most countably many zeroes of $(u_1,v_1)-(u_2,v_2)$ in S°, all isolated.*

To prove this, suppose $(u_1,v_1)\not\equiv(u_2,v_2)$ and (b,c) is a nonsingular zero of $(u_1,v_1)-(u_2,v_2)$ in S°. As u_j,v_j are real analytic near (b,c), if $\partial^j u_1(b,c)=\partial^j u_2(b,c)$ and $\partial^j v_1(b,c)=\partial^j v_2(b,c)$ for all $j\geqslant 1$ then $(u_1,v_1)\equiv(u_2,v_2)$, a contradiction. Hence there exists a smallest $k\geqslant 1$ such that $\partial^k u_1(b,c)\neq\partial^k u_2(b,c)$ or $\partial^k v_1(b,c)\neq\partial^k v_2(b,c)$. One can then show following Proposition 7.3 that (12) holds for some $C\in\mathbb{C}\setminus\{0\}$, and therefore (b,c) is an isolated zero.

Hence, if $(u_1,v_1)\not\equiv(u_2,v_2)$ then nonsingular zeroes of $(u_1,v_1)-(u_2,v_2)$ in S° are isolated. To prove that singular zeroes are isolated requires a careful study of singular solutions (u,v) of (8) with $v=0$ on an interval on the x-axis, carried out in [10, §6].

Following Theorem 7.1, we can now prove [8, Th. 6.7], [10, Th. 7.7]:

Theorem 7.5. *Let S be a domain in $\mathbb{R}^2$, and $a\in\mathbb{R}$. If $a\neq 0$ let $u_j,v_j\in C^1(S)$ satisfy (9) for $j=1,2$, and if $a=0$ let $u_j,v_j\in C^0(S)$ be singular solutions of (8) for $j=1,2$. Suppose $(u_1,v_1)\neq(u_2,v_2)$ at every point of ∂S. Then $(u_1,v_1)-(u_2,v_2)$ has finitely many zeroes in S, all isolated. Let there be n zeroes, with multiplicities $k_1,\dots,k_n$. Then the winding number of $(u_1,v_1)-(u_2,v_2)$ about 0 along ∂S is $\sum_{i=1}^n k_i$.*

Suppose u_j,v_j come from a potential f_j as in §6, with $f_j|_{\partial S}=\phi_j$. One can show directly that if the winding number of $(u_1,v_1)-(u_2,v_2)$ about 0 along ∂S is k, and

$\phi_1 - \phi_2$ has l local maxima and l local minima on ∂S, then $|k| \leqslant l-1$. So we prove [8, Th. 7.11], [10, Th. 7.10]:

Theorem 7.6. *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under $(x,y) \mapsto (x,-y)$, and $a \in \mathbb{R}$, $k \geqslant 0$, $\alpha \in (0,1)$, and $\phi_1, \phi_2 \in C^{k+3,\alpha}(\partial S)$. Let $u_j, v_j \in C^0(S)$ be the (singular) solution of* (8) *or* (9) *constructed in Theorem* 6.2 *from ϕ_j, for $j = 1, 2$.*

Suppose $\phi_1 - \phi_2$ has l local maxima and l local minima on ∂S. Then $(u_1, v_1) - (u_2, v_2)$ has finitely many zeroes in S°, all isolated. Let there be n zeroes in S° with multiplicities $k_1, \ldots, k_n$. Then $\sum_{i=1}^n k_i \leqslant l-1$.

In particular, if $l = 1$ then $(u_1, v_1) \neq (u_2, v_2)$ in S°. This will be useful in §9.

8. A rough classification of singular points

We can now use the work of §7 to study singular points of u, v, following [10, §9].

Definition 8.1. Let S be a domain in $\mathbb{R}^2$, and $u, v \in C^0(S)$ a singular solution of (8), as in Definition 4.1. Suppose for simplicity that S is invariant under $(x,y) \mapsto (x,-y)$. Define $u', v' \in C^0(S)$ by $u'(x,y) = u(x,-y)$ and $v'(x,y) = -v(x,-y)$. Then u', v' is also a singular solution of (8).

A *singular point*, or *singularity*, of (u,v) is a point $(b,0) \in S$ with $v(b,0) = 0$. Observe that a singularity of (u,v) is automatically a zero of $(u,v) - (u',v')$. Conversely, a zero of $(u,v) - (u',v')$ on the x-axis is a singularity. A singularity of (u,v) is called *isolated* if it is an isolated zero of $(u,v) - (u',v')$. Define the *multiplicity* of an isolated singularity $(b,0)$ of (u,v) in S° to be the multiplicity of $(u,v) - (u',v')$ at $(b,0)$, in the sense of Definition 7.2. By Theorem 7.2, this multiplicity is a positive integer.

From Theorem 7.4 we deduce [10, Th. 9.2]:

Theorem 8.1. *Let S be a domain in $\mathbb{R}^2$ invariant under $(x,y) \mapsto (x,-y)$, and $u, v \in C^0(S)$ a singular solution of* (8)*. If $u(x,-y) \equiv u(x,y)$ and $v(x,-y) \equiv -v(x,y)$ then (u,v) is singular along the x-axis in S, and the singularities are nonisolated. Otherwise there are at most countably many singularities of (u,v) in S°, all isolated.*

We divide isolated singularities $(b,0)$ into four types, depending on the behaviour of $v(x,0)$ near $(b,0)$.

Definition 8.2. Let S be a domain in $\mathbb{R}^2$, and $u, v \in C^0(S)$ a singular solution of (8), as in Definition 4.1. Suppose $(b,0)$ is an isolated singular point of (u,v) in S°. Then there exists $\epsilon > 0$ such that $\overline{B}_\epsilon(b,0) \subset S^\circ$ and $(b,0)$ is the only singularity of (u,v) in $\overline{B}_\epsilon(b,0)$. Thus, for $0 < |x-b| \leqslant \epsilon$ we have $(x,0) \in S^\circ$ and $v(x,0) \neq 0$. So by continuity v is either positive or negative on each of $[b-\epsilon, b) \times \{0\}$ and $(b, b+\epsilon] \times \{0\}$.

(i) if $v(x) < 0$ for $x \in [b-\epsilon, b)$ and $v(x) > 0$ for $x \in (b, b+\epsilon]$ we say the singularity $(b,0)$ is of *increasing type*.

(ii) if $v(x) > 0$ for $x \in [b-\epsilon, b)$ and $v(x) < 0$ for $x \in (b, b+\epsilon]$ we say the singularity $(b,0)$ is of *decreasing type*.

(iii) if $v(x) < 0$ for $x \in [b-\epsilon, b)$ and $v(x) < 0$ for $x \in (b, b+\epsilon]$ we say the singularity $(b, 0)$ is of *maximum type*.
(iv) if $v(x) > 0$ for $x \in [b-\epsilon, b)$ and $v(x) > 0$ for $x \in (b, b+\epsilon]$ we say the singularity $(b, 0)$ is of *minimum type*.

The type determines whether the multiplicity is even or odd, [10, Prop. 9.4].

Proposition 8.2. *Let $u, v \in C^0(S)$ be a singular solution of* (8) *on a domain S in $\mathbb{R}^2$, and $(b, 0)$ be an isolated singularity of (u, v) in S° with multiplicity k. If $(b, 0)$ is of increasing or decreasing type then k is odd, and if $(b, 0)$ is of maximum or minimum type then k is even.*

Theorem 7.6 immediately yields a criterion for finitely many singularities, [10, Th. 9.7]:

Theorem 8.3. *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, and $\phi \in C^{k+3,\alpha}(\partial S)$ for $k \geqslant 0$ and $\alpha \in (0, 1)$. Let $u, v \in C^0(S)$ be the singular solution of* (8) *constructed in Theorem* 6.2 *from ϕ with $a = 0$.*

Define $\phi' \in C^{k+3,\alpha}(\partial S)$ by $\phi'(x, y) = -\phi(x, -y)$. Suppose $\phi - \phi'$ has l local maxima and l local minima on ∂S. Then (u, v) has finitely many singularities in S°. Let there be n singularities in S° with multiplicities $k_1, \ldots, k_n$. Then $\sum_{i=1}^n k_i \leqslant l - 1$.

By applying Theorem 6.2 with S the unit disc in $\mathbb{R}^2$ and ϕ a linear combination of functions $\sin(j\theta), \cos(j\theta)$ on the unit circle ∂S, we show [10, Cor. 10.10]:

Theorem 8.4. *There exist examples of singular solutions u, v of* (8) *with isolated singularities of every possible multiplicity $n \geqslant 1$, and with both possible types allowed by Proposition* 8.2.

Combining this with Proposition 4.1 gives examples of SL 3-folds in $\mathbb{C}^3$ with singularities of an *infinite number* of different geometrical/topological types. We also show in [10, §10.4] that singular points with multiplicity $n \geqslant 1$ occur in *real codimension n* in the family of all SL 3-folds invariant under the U(1)-action (1), in a well-defined sense.

9. Special Lagrangian fibrations

We will now use our results to construct large families of *special Lagrangian fibrations* of open subsets of $\mathbb{C}^3$ invariant under the U(1)-action (1), including singular fibres. These will be important when we discuss the *SYZ Conjecture* in §10, which concerns fibrations of Calabi–Yau 3-folds by SL 3-folds.

Definition 9.1. Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, let U be an open set in $\mathbb{R}^3$, and $\alpha \in (0, 1)$. Suppose $\Phi : U \to C^{3,\alpha}(\partial S)$ is a continuous map such that if $(a, b, c) \neq (a, b', c')$ in U then $\Phi(a, b, c) - \Phi(a, b', c')$ has exactly one local maximum and one local minimum in ∂S.

Let $\boldsymbol{\alpha} = (a, b, c) \in U$, and let $f_{\boldsymbol{\alpha}} \in C^{3,\alpha}(S)$ be the unique (weak) solution of (11) with $f_{\boldsymbol{\alpha}}|_{\partial S} = \Phi(\boldsymbol{\alpha})$, which exists by Theorem 6.2. Define $u_{\boldsymbol{\alpha}} = \frac{\partial f_{\boldsymbol{\alpha}}}{\partial y}$ and $v_{\boldsymbol{\alpha}} = \frac{\partial f_{\boldsymbol{\alpha}}}{\partial x}$. Then

$(u_{\boldsymbol\alpha}, v_{\boldsymbol\alpha})$ is a solution of (9) if $a \neq 0$, and a singular solution of (8) if $a = 0$. Also $u_{\boldsymbol\alpha}, v_{\boldsymbol\alpha}$ depend continuously on $\boldsymbol\alpha \in U$ in $C^0(S)$, by Theorem 6.2.

For each $\boldsymbol\alpha = (a, b, c)$ in U, define $N_{\boldsymbol\alpha}$ in $\mathbb{C}^3$ by

$$N_{\boldsymbol\alpha} = \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v_{\boldsymbol\alpha}(x, y) + iy, \quad z_3 = x + iu_{\boldsymbol\alpha}(x, y), \\ |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S^\circ\big\}.$$

Then $N_{\boldsymbol\alpha}$ is a noncompact SL 3-fold without boundary in $\mathbb{C}^3$, which is nonsingular if $a \neq 0$, by Proposition 4.1.

By [10, Th. 8.2] the $N_{\boldsymbol\alpha}$ are the fibres of an *SL fibration*.

Theorem 9.1. *In the situation of Definition* 9.1*, if* $\boldsymbol\alpha \neq \boldsymbol\alpha'$ *in* U *then* $N_{\boldsymbol\alpha} \cap N_{\boldsymbol\alpha'} = \emptyset$. *There exists an open set* $V \subset \mathbb{C}^3$ *and a continuous, surjective map* $F : V \to U$ *such that* $F^{-1}(\boldsymbol\alpha) = N_{\boldsymbol\alpha}$ *for all* $\boldsymbol\alpha \in U$. *Thus,* F *is a special Lagrangian fibration of* $V \subset \mathbb{C}^3$*, which may include singular fibres.*

The main step in the proof is to show that distinct $N_{\boldsymbol\alpha}$ do not intersect, so that they fibre $V = \bigcup_{\boldsymbol\alpha \in U} N_{\boldsymbol\alpha}$. Suppose $\alpha = (a, b, c)$ and $\alpha' = (a', b', c')$ are distinct elements of U. If $a \neq a'$ then $N_{\boldsymbol\alpha} \cap N_{\boldsymbol\alpha'} = \emptyset$, since $|z_1|^2 - |z_2|^2$ is $2a$ on $N_{\boldsymbol\alpha}$ and $2a'$ on $N_{\boldsymbol\alpha'}$.

If $a = a'$ then $\Phi(\boldsymbol\alpha) - \Phi(\boldsymbol\alpha')$ has one local maximum and one local minimum in ∂S, by Definition 9.1. So Theorem 7.6 applies with $l = 1$ to show that $(u_{\boldsymbol\alpha}, v_{\boldsymbol\alpha}) - (u_{\boldsymbol\alpha'}, v_{\boldsymbol\alpha'})$ has no zeroes in S°, and again $N_{\boldsymbol\alpha} \cap N_{\boldsymbol\alpha'} = \emptyset$. Thus distinct $N_{\boldsymbol\alpha}$ do not intersect.

Here is a simple way [10, Ex. 8.3] to produce families Φ satisfying Definition 9.1, and thus generate many SL fibrations of open subsets of $\mathbb{C}^3$.

Example 9.1. Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, let $\alpha \in (0, 1)$ and $\phi \in C^{3,\alpha}(\partial S)$. Define $U = \mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \to C^{3,\alpha}(\partial S)$ by $\Phi(a, b, c) = \phi + bx + cy$. If $(a, b, c) \neq (a, b', c')$ then $\Phi(a, b, c) - \Phi(a, b', c') = (b - b')x + (c - c')y \in C^\infty(\partial S)$. As $b - b', c - c'$ are not both zero and S is strictly convex, it easily follows that $(b - b')x + (c - c')y$ has one local maximum and one local minimum in ∂S.

Hence the conditions of Definition 9.1 hold for S, U and Φ, and so Theorem 9.1 defines an open set $V \subset \mathbb{C}^3$ and a special Lagrangian fibration $F : V \to \mathbb{C}^3$. One can also show that changing the parameter c in $U = \mathbb{R}^3$ just translates the fibres $N_{\boldsymbol\alpha}$ in $\mathbb{C}^3$, and

$$V = \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : (\operatorname{Re} z_3, \operatorname{Im} z_1 z_2) \in S^\circ\big\}.$$

Here is very explicit example, taken from [11].

Example 9.2. Define $F : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ by

$$F(z_1, z_2, z_3) = (a, b), \quad \text{where} \quad 2a = |z_1|^2 - |z_2|^2$$

$$\text{and} \quad b = \begin{cases} z_3, & a = z_1 = z_2 = 0, \\ z_3 + \bar z_1 \bar z_2 / |z_1|, & a \geqslant 0, \;\; z_1 \neq 0, \\ z_3 + \bar z_1 \bar z_2 / |z_2|, & a < 0. \end{cases}$$

This is a piecewise-smooth SL fibration of $\mathbb{C}^3$. It is not smooth on $|z_1| = |z_2|$.

The fibres $F^{-1}(a, b)$ are T^2-cones singular at $(0, 0, b)$ when $a = 0$, and nonsingular $\mathcal{S}^1 \times \mathbb{R}^2$ when $a \neq 0$. They are isomorphic to the SL 3-folds $N_{|a|}$ of Example 5.1 under transformations of $\mathbb{C}^3$, but they are assembled to make a fibration in a novel way. As a goes from positive to negative the fibres undergo a surgery, a Dehn twist on $\mathcal{S}^1$.

10. The SYZ Conjecture

Mirror Symmetry is a mysterious relationship between pairs of Calabi–Yau 3-folds $X, \hat{X}$, arising from a branch of physics known as *String Theory*, and leading to some very strange and exciting conjectures about Calabi–Yau 3-folds, many of which have been proved in special cases.

Roughly speaking, String Theorists believe that each Calabi–Yau 3-fold X has a quantization, a *Super Conformal Field Theory* (SCFT). Invariants of X such as the Dolbeault groups $H^{p,q}(X)$ and the number of holomorphic curves in X translate to properties of the SCFT. However, different Calabi–Yau 3-folds $X, \hat{X}$ may have the same SCFT.

One way for this to happen is for the SCFT's of $X, \hat{X}$ to be related by a certain simple involution of SCFT structure, which does *not* correspond to a classical automorphism of Calabi–Yau 3-folds. We then say that X and $\hat{X}$ are *mirror* Calabi–Yau 3-folds. One can argue using String Theory that $H^{1,1}(X) \cong H^{2,1}(\hat{X})$ and $H^{2,1}(X) \cong H^{1,1}(\hat{X})$. The mirror transform also exchanges things related to the complex structure of X with things related to the symplectic structure of $\hat{X}$, and vice versa.

The *SYZ Conjecture*, due to Strominger, Yau and Zaslow [14] in 1996, gives a geometric explanation of Mirror Symmetry. Here is an attempt to state it.

The SYZ Conjecture. *Suppose X and $\hat{X}$ are mirror Calabi–Yau 3-folds. Then (under some additional conditions) there should exist a compact topological 3-manifold B and surjective, continuous maps $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *There exists a dense open set $B_0 \subset B$, such that for each $b \in B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are nonsingular special Lagrangian 3-tori T^3 in X and $\hat{X}$. Furthermore, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are in some sense dual to one another.*

(ii) *For each $b \in \Delta = B \setminus B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are expected to be singular special Lagrangian 3-folds in X and $\hat{X}$.*

We call $f, \hat{f}$ *special Lagrangian fibrations*, and the set of singular fibres Δ is called the *discriminant*. It is not yet clear what the final form of the SYZ Conjecture should be: there are problems to do with the singular fibres, and with what extra conditions on $X, \hat{X}$ are needed for $f, \hat{f}$ to exist.

Much mathematical research on the SYZ Conjecture has simplified the problem by supposing that $f, \hat{f}$ are *Lagrangian fibrations*, making only limited use of the 'special' condition, and supposing in addition that $f, \hat{f}$ are *smooth* maps. Gross [1, 2, 3, 4], Ruan [12, 13], and others have built up a beautiful, detailed picture of how dual SYZ fibrations work at the level of global symplectic topology, in particular for examples such as the quintic and its mirror, and for Calabi–Yau 3-folds constructed as hypersurfaces in toric 4-folds, using combinatorial data.

The author's approach to the SYZ Conjecture [11] has a different viewpoint, and more modest aims. We take the special Lagrangian condition seriously from the outset, and focus on the local behaviour of SL fibrations near singular points, rather than on global topological questions. Also, we are interested in *generic* SL fibrations.

The best way to introduce a genericity condition is to consider SL fibrations $f : X \to B$ in which X is a generic almost Calabi–Yau 3-fold. The point of allowing X to be an *almost* Calabi–Yau 3-fold is that the family of almost Calabi–Yau structures is infinite-dimensional, and so picking a generic one is a strong condition, and should simplify the singular behaviour of f.

Now from §9 we know a lot about U(1)-invariant SL fibrations of subsets of $\mathbb{C}^3$. By considering when these are appropriate local models for singularities of SL fibrations of almost Calabi–Yau 3-folds, the author makes the following tentative suggestions:

- In a generic SL fibration $f : X \to B$, the singularities of codimension 1 in B are locally modelled on the explicit SL fibration F given in Example 9.2.
- Similarly, in generic SL fibrations $f : X \to B$, one kind of singular behaviour of codimension 2 in B is modelled on a U(1)-invariant SL fibration of the kind considered in §9, including a 1-parameter family of singular fibres with isolated singular points of multiplicity 2, in the sense of Definition 8.1.
- However, I do not expect codimension 3 singularities in generic SL fibrations to be locally U(1)-invariant, so this approach will not help.

Here are some broader conclusions, also conjectural.

- For generic almost Calabi–Yau 3-folds X, SL fibrations $f : X \to B$ will not be smooth maps, but only *piecewise smooth*.
- In a generic SL fibration $f : X \to B$ the discriminant Δ is of codimension 1 in B, and the singular fibres are singular at finitely many points.

  In contrast, in the smooth Lagrangian fibrations $f : X \to B$ considered by Gross and Ruan, the discriminant Δ is of codimension 2 in B, and the typical singular fibre is singular along an $\mathcal{S}^1$.
- If $X, \hat{X}$ are a mirror pair of generic (almost) Calabi–Yau 3-folds and $f : X \to B$ and $\hat{f} : \hat{X} \to B$ are dual SL fibrations, then in general the discriminants Δ of f and $\hat{\Delta}$ of $\hat{f}$ will *not* coincide in B. This contradicts part (ii) of the SYZ Conjecture, as stated above.

For more details, see [11]. In the author's view, these calculations support the idea that the SYZ Conjecture in its present form should be viewed primarily as a limiting statement, about what happens at the 'large complex structure limit', rather than as simply being about pairs of Calabi–Yau 3-folds. A similar conclusion is reached by Mark Gross in [4, §5].

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