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On cofinite subgroups of mapping class groups

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Abstract

For every positive integer n, we exhibit a cofinite subgroup Γ_n of the mapping class group of a surface of genus at most two such that Γ_n admits an epimorphism onto a free group of rank n. We conclude that $H^1(\Gamma_n;\mathbb{Z})$ has rank at least n and the dimension of the second bounded cohomology of each of these mapping class groups is the cardinality of the continuum. In the case of genus two, the groups Γ_n can be chosen not to contain the Torelli group. Similarly for hyperelliptic mapping class groups. We also exhibit an automorphism of a subgroup of finite index in the mapping class group of a sphere with four punctures (or a torus) such that it is not the restriction of an endomorphism of the whole group.

1. Introduction

It is well-known that the first homology group of the mapping class group of a closed orientable surface of genus g is trivial for $g \geq 3$ and isomorphic to $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$ if $g = 1$ and $g = 2$ respectively. It follows that the first cohomology of this group is trivial. N. V. Ivanov (Problem 2.11(A) in [11]) asked whether $H^1(\Gamma;\mathbb{Z})$ is trivial for any subgroup Γ of finite index in the mapping class group. In the case $g \geq 3$, this question was answered affirmatively by J. D. McCarthy [14] for subgroups Γ containing the Torelli group, the subgroup of the mapping class group consisting of those mapping classes that act trivially on the first homology of the surface. For arbitrary subgroups of finite index, the problem is still open. It was also shown by McCarthy [14] and Taherkhani [18] that the mapping class group of a closed orientable surface of genus 2 contains subgroups of finite index with nontrivial first cohomology. All of the examples of McCarthy contain the Torelli group. More precisely, he shows that if r is an integer divisible by 2 or 3, then the kernel of the action of the mapping class group on the mod r homology of the surface has nontrivial first cohomology. It is not clear whether the examples of Taherkhani contain the Torelli group, because his calculations are carried out by computer.

The purpose of this paper is to give an elementary construction of a sequence Γ_n of subgroups of finite index in the mapping class group of an orientable surface of genus at most 2 and in the hyperelliptic mapping class group such that Γ_n admits a homomorphism onto a finitely generated free group of rank n. In the case of a closed orientable surface of genus 2, we can choose these subgroups in such a way that they do not contain the Torelli group. This shows that for any positive integer n, there is a subgroup of finite index whose first cohomology has rank at least n. Another application is that the dimension of the second bounded cohomology of each of these mapping class groups is the cardinal of the continuum. The fact that they are infinite dimensional was also proved by Bestvina and Fujiwara [1] by completely different arguments.

The last section is independent of the other results in the paper. In this section we prove that there is a subgroup Γ of finite index in the mapping class group of a sphere with four punctures and in that of a torus (or a torus with one puncture), and an automorphism $\varphi : \Gamma \to \Gamma$ such that φ is not the restriction of any endomorphism of the whole group. It is known that if the surface is not a sphere with four punctures or a torus with at least two punctures, then any isomorphism between two subgroups of finite index in the mapping class group is the restriction of an automorphism of the whole group (cf. [10, 12]). In the case of a torus with two punctures, the answer to the related obvious question is unknown.

2. Definitions and preliminaries

Let S be an orientable surface of genus g with p marked points (=punctures) and with q boundary components. The mapping class group $\mathrm{Mod}_{g,p}^q$ is defined to be the group of isotopy classes of orientation preserving diffeomorphisms $S \to S$ which restrict to the identity on the boundary and preserve the set of punctures. The isotopies are assumed to fix the punctures and the points on the boundary. If p and/or q is zero, then we omit it from the notation, so that $\mathrm{Mod}_g^q, \mathrm{Mod}_{g,p}$ and Mod_g denote $\mathrm{Mod}_{g,0}^q, \mathrm{Mod}_{g,p}^0$ and $\mathrm{Mod}_{g,0}^0$ respectively.

The pure mapping class group $\mathrm{PMod}_{g,p}^q$ is the kernel of the action of $\mathrm{Mod}_{g,p}^q$ on the set of punctures. The quotient of $\mathrm{Mod}_{g,p}^q$ by $\mathrm{PMod}_{g,p}^q$ is isomorphic to the symmetric group on p letters.

The Torelli group is the subgroup of the mapping class group Mod_g consisting of those mapping classes which act trivially on the first homology of the surface S.

Suppose now that S is closed and it is embedded in the xyz-space as in Figure 1 in such a way that it is invariant under the rotation $J(x,y,z) = (-x,y,-z)$ about the y-axis. Let $\jmath$ denote the isotopy class of J. The centralizer

$$\Delta_g = \{ f \in \mathrm{Mod}_g \mid f\jmath = \jmath f \}$$

of $\jmath$ in Mod_g is called the hyperelliptic mapping class group. Note that if $g = 1$ or 2, then the hyperelliptic mapping class group is equal to the mapping class group.

The involution J has $2g+2$ fixed points. Thus the quotient of S by J gives a branched covering $\pi : S \to R$ branching over $2g+2$ points, where R is the 2-sphere. This branched covering induces a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \Delta_g \xrightarrow{\pi_*} \mathrm{Mod}_{0,2g+2} \longrightarrow 1, \tag{1}$$

where $\mathbb{Z}_2$ is the subgroup of Δ_g generated by the hyperelliptic involution $\jmath$ (cf. [3]).

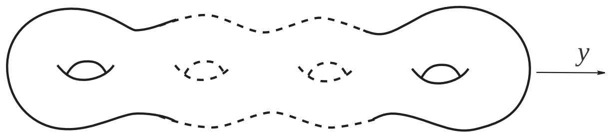

Figure 1. A closed orientable surface embedded in $\mathbb{R}^3$ so that it is invariant under J.

3. Finite index subgroups with large H^1

In this section we give the construction of subgroups admitting homomorphisms onto free groups.

Theorem 3.1. *Suppose that $g \leq 2$. If $g = 0$, suppose, in addition, that $p + q \geq 4$. For any positive integer n, there is a subgroup Γ_n of finite index in $\mathrm{Mod}_{g,p}^q$ such that there is an epimorphism from Γ_n onto a free group F_n of rank n and $\Gamma_{n+1} \subset \Gamma_n$. In the case of $g = 2$ and $p = q = 0$, the group Γ_n can be chosen not to contain the Torelli group.*

Proof. For $n \geq 4$ it is well-known that forgetting one of the punctures on a sphere with n punctures gives rise to a short exact sequence

$$1 \to F_{n-2} \to \mathrm{PMod}_{0,n} \to \mathrm{PMod}_{0,n-1} \to 1,$$

where F_{n-2} is the fundamental group of a sphere with $n-1$ punctures, which is a free group of rank $n-2$. It can easily be shown that $\mathrm{PMod}_{0,3}$ is trivial. It follows that $\mathrm{PMod}_{0,4}$ is a free group of rank 2.

We first prove the theorem for $n = 2$. That is, we prove that there is a finite index subgroup Γ_2 in $\mathrm{Mod}_{g,p}^q$ and an epimorphism $\Gamma_2 \to F_2$.

Suppose first that $g = 0$. There is an epimorphism $\mathrm{PMod}_{0,p}^q \to \mathrm{PMod}_{0,4}$ obtained by gluing a disc with one puncture to each boundary component and then forgetting some $p+q-4$ punctures. The subgroup $\mathrm{PMod}_{0,p}^q$ is of index $p\,!$ in $\mathrm{Mod}_{0,p}^q$. In this case, we can take Γ_2 to be the subgroup $\mathrm{PMod}_{0,p}^q$.

Suppose next that $g = 1$. Gluing a disc along each boundary component and forgetting all punctures yields an epimorphism $\varphi : \mathrm{Mod}_{1,p}^q \to \mathrm{Mod}_1$. The group Mod_1 is isomorphic to $SL(2, \mathbb{Z})$. The commutator subgroup $[\mathrm{Mod}_1, \mathrm{Mod}_1]$ of Mod_1 is a free group of rank 2 and its index in Mod_1 is 12. Thus we can take Γ_2 to be $\varphi^{-1}([\mathrm{Mod}_1, \mathrm{Mod}_1])$.

Suppose finally that $g = 2$. Again, gluing a disc along each boundary component and forgetting the punctures give an epimorphism $\varphi : \mathrm{Mod}_{2,p}^q \to \mathrm{Mod}_2$. Note that $\mathrm{Mod}_2 = \Delta_2$. Consider the natural map $\pi_* : \mathrm{Mod}_2 \to \mathrm{Mod}_{0,6}$ in (1). Since there is an epimorphism from $\mathrm{PMod}_{0,6}$ onto the free group $\mathrm{PMod}_{0,4}$ of rank 2, we may take $\Gamma_2 = \varphi^{-1}(\pi_*^{-1}(\mathrm{PMod}_{0,6}))$. The index of Γ_2 in $\mathrm{Mod}_{2,p}^q$ is 720.

For $n \geq 3$, consider an epimorphism $f : \Gamma_2 \to F_2$. Let F_n be a subgroup of F_2 of index $n-1$. Then F_n is a free group of rank n. The subgroup $\Gamma_n = f^{-1}(F_n)$ is of finite index in $\mathrm{Mod}_{g,p}^q$ and the restriction of f maps Γ_n onto F_n. This completes the proof of the first assertion.

In the case $g = 2$ and $p = q = 0$, we can choose Γ_n so that it does not contain the Torelli group as follows. Let S be a closed connected oriented surface of genus 2 embedded in the xyz-space as in Figure 1. We assume that S is symmetric with respect to the origin. Let a be the separating simple closed curve which is the intersection of the xz-plane with S. We note that a passes through no fixed points of J and $J(a) = a$. The quotient of S by the action of J gives rise to a branched covering $\pi : S \to R$, where R is a 2-sphere. Let us denote the image of the fixed points of J by $P_1, P_2, \ldots, P_6$, so that we see them as punctures on R. The simple closed curve $\pi(a)$ separates the punctures on R into two sets each containing three elements. We can assume that P_1, P_2, P_3 are separated from the other three punctures by $\pi(a)$.

Now assume that P_3 and P_6 are not marked points on R. Let c denote the image of $\pi(a)$ on this sphere with four punctures P_1, P_2, P_4, P_5. Choose an embedded arc δ on R connecting the punctures P_2 and P_4 so that it intersects c only once and its interior does not contain any puncture. If d denotes the boundary of a regular neighborhood of δ, it can easily be shown that the Dehn twists t_c and t_d generate $\mathrm{PMod}_{0,4} = F_2$ freely. For $n \geq 3$, the subgroup of F_2 generated by $t_d, t_c t_d t_c^{-1}, t_c^2 t_d t_c^{-2}, \ldots, t_c^{n-2} t_d t_c^{-n+2}$ and t_c^{n-1} is a free group F_n of rank n and the index of F_n in F_2 is $n-1$. Note that for $n \geq 4$, the element t_c^2 is not contained in F_n.

Since the curve a does not contain any fixed point of J, the restriction of π to a gives an honest two-sheeted covering $a \to \pi(a)$. It is easy to see that $\pi_*(t_a) = t_{\pi(a)}^2$, which is contained in $\mathrm{PMod}_{0,6}$. If $\phi : \mathrm{PMod}_{0,6} \to \mathrm{PMod}_{0,4}$ is the epimorphism obtained by forgetting the punctures P_3 and P_6, then clearly we have $\phi(t_{\pi(a)}) = t_c$, and so $\phi(\pi_*(t_a)) = \phi(t_{\pi(a)}^2) = t_c^2$.

If we define Γ_2 and Γ_3 to be the subgroup $\pi_*^{-1}(\phi^{-1}(F_4))$ and Γ_n to be $\pi_*^{-1}(\phi^{-1}(F_n))$ for $n \geq 4$, obviously there exists an epimorphism from Γ_k onto a free group of rank k for all $k \geq 2$. The element $t_a \in \mathrm{Mod}_2$ is contained in the Torelli group but not in Γ_k.

This completes the proof of the theorem. □

Remark 3.1. Suppose that $p + q \leq 3$. In this case the mapping class group $\mathrm{Mod}_{0,p}^q$ is

- trivial if $p \leq 1$ and $q \leq 1$,
- the cyclic group of order 2 if $(p, q) = (2, 0)$,
- the symmetric group on 3 letters if $(p, q) = (3, 0)$,
- $\mathbb{Z}$ if $(p, q) = (0, 2)$ or $(2, 1)$,
- $\mathbb{Z} \oplus \mathbb{Z}$ if $(p, q) = (1, 2)$, and
- $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ if $(p, q) = (0, 3)$.

Thus none of these groups have a subgroup admitting a homomorphism onto a free group of rank greater than 1.

Remark 3.2. For each n, the subgroup Γ_n of Mod_2 in the above theorem can also be chosen to contain the Torelli group.

Corollary 3.2. *Suppose that $g \geq 2$. For any positive integer n, there is a subgroup Γ_n of finite index in the hyperelliptic mapping class group Δ_g such that there is an epimorphism from Γ_n onto a free group F_n of rank n.*

Proof. The corollary follows easily from the fact that Δ_g admits an epimorphism onto $\mathrm{Mod}_{0,2g+2}$ and Theorem 3.1. □

The next two corollaries follow from Theorem 3.1 and Corollary 3.2.

Corollary 3.3. *Suppose that $g \leq 2$. If $g = 0$, suppose, in addition, that $p + q \geq 4$. For any positive integer n, there is a subgroup Γ_n of finite index in $\mathrm{Mod}_{g,p}^q$ such that the rank of $H^1(\Gamma_n; \mathbb{Z})$ is at least n.*

Corollary 3.4. *For any positive integer n, there is a subgroup Γ_n of finite index in the hyperelliptic mapping class group Δ_g such that the rank of $H^1(\Gamma_n; \mathbb{Z})$ is at least n. Moreover, in the case of $g = 2$ the subgroup Γ_n of Mod_2 can be chosen so that it does not contain the Torelli group.*

4. The second bounded cohomology

In this section, we show how to deduce from Theorem 3.1 that the dimension of the second bounded cohomology group of the mapping class group $\mathrm{Mod}_{g,p}^q$ for $g \leq 2$ and that of the hyperelliptic mapping class group Δ_g is the cardinality of the continuum.

Let G be a discrete group and let

$$C_b^k(G; \mathbb{R}) = \{ f : G^k \to \mathbb{R} \mid f(G^k) \text{ is bounded } \}.$$

There is a coboundary operator $\delta_b^k : C_b^k(G; \mathbb{R}) \to C_b^{k+1}(G; \mathbb{R})$ defined by

$$\begin{aligned} \delta_b^k(f)(x_0, \ldots, x_k) &= f(x_1, \ldots, x_k) + \sum_{i=1}^{k} (-1)^i f(x_0, \ldots, x_{i-1}x_i, \ldots, x_k) \\ &\quad + (-1)^{k+1} f(x_0, \ldots, x_{k-1}). \end{aligned}$$

The cohomology of the complex $\{C_b^k(G; \mathbb{R}), \delta_b^k\}$ is called the bounded cohomology of G and is denoted by $H_b^*(G; \mathbb{R})$. The space $C_b^k(G; \mathbb{R})$ is a Banach space with the norm

$$||f|| = \sup\{ \, |f(x_1, x_2, \ldots, x_k)| \ \mid x_i \in G \, \},$$

which induces a semi-norm on $H_b^k(G; \mathbb{R})$. The bounded cohomology $H_b^k(G; \mathbb{R})$ is always a Banach space for $k = 2$ (cf. [9]) but it need not be a Banach space for $k \geq 3$ (cf. [17]).

The first result in the theory of bounded cohomology is that the first bounded cohomology of any group is trivial. This is because a bounded 1-cochain is a bounded homomorphism $G \to \mathbb{R}$ and any such homomorphism is trivial. So the first interesting bounded cohomology is in dimension two.

In the above definition, if we replace $C_b^k(G; \mathbb{R})$ by the space $C^k(G; \mathbb{R})$ of all functions $G^k \to \mathbb{R}$ and if the coboundary operator is defined by the same formula, then we obtain the cohomology $H^*(G;\mathbb{R})$ of G. The inclusion $C_b^k(G;\mathbb{R}) \hookrightarrow C^k(G;\mathbb{R})$ induces a natural map $H_b^k(G;\mathbb{R}) \to H^k(G;\mathbb{R})$. When $k=2$, following Grigorchuk [7], let us denote the kernel of this map by $H_{b,2}^2(G;\mathbb{R})$.

For a group G, let $PX(G)$ denote the space of pseudo characters (pseudo homomorphisms) on G. That is, $PX(G)$ is the space of all functions $f: G \to \mathbb{R}$ satisfying $|f(x)+f(y)-f(xy)| \leq C$ and $f(x^n) = nf(x)$ for all $x, y \in G$ and for some C depending on f. Let $X(G)$ denote the space of all homomorphisms $G \to \mathbb{R}$. Grigorchuk proved that $H_{b,2}^2(G;\mathbb{R})$ is isomorphic to $PX(G)/X(G)$ as a vector space.

The next lemma was proved in the proof of Proposition 4.7 in [7].

Lemma 4.1. *Let G be a finitely generated group and let H be a subgroup of finite index in G. The map $\tau: PX(G) \to PX(H)$ induced by the restriction is injective and the quotient space $PX(H)/\tau(PX(G))$ is finite dimensional.*

Theorem 4.2 ([4])**.** *Let G and F be two groups and let $\sigma: G \to F$ be an epimorphism. Then σ induces an injective linear map $H_b^2(F;\mathbb{R}) \to H_b^2(G;\mathbb{R})$.*

Theorem 4.3 ([13])**.** *Suppose that $n \geq 2$. If F_n is a free group of rank n, then the dimension of the space $H_b^2(F_n;\mathbb{R})$ is equal to the cardinal of the continuum.*

Theorem 4.4. *Let G be a finitely presented group and let H be a subgroup of finite index in G. Suppose that there is a homomorphism from H onto a free group F_n of rank $n \geq 2$. Then the dimension of the space $H_b^2(G;\mathbb{R})$ is equal to the cardinal of the continuum.*

Proof. If K is a finitely generated group, then it is countable. It follows that the dimension of $C_b^k(K;\mathbb{R})$, and hence that of $H_b^k(K;\mathbb{R})$, is at most the cardinal of the continuum for any positive integer k.

Since F_n is a quotient of H, it follows from Theorems 4.2 and 4.3 that the dimension of $H_b^2(H;\mathbb{R})$ is the cardinal of the continuum. Since H is finitely presented, $H^2(H;\mathbb{R})$ and $X(H)$ are finite dimensional. It follows that the dimensions of $H_{b,2}^2(H;\mathbb{R})$ and $PX(H)$ are the cardinal of the continuum. We conclude from Lemma 4.1 that the dimension of $H_{b,2}^2(G;\mathbb{R})$ is also the cardinal of the continuum. Since $H_{b,2}^2(G;\mathbb{R})$ is a subspace of $H_b^2(G;\mathbb{R})$, the theorem follows. □

Theorem 4.5. *Suppose that $g \leq 2$. If $g = 0$, suppose, in addition, that $p + q \geq 4$. Then the dimension of $H_b^2(\mathrm{Mod}_{g,p}^q;\mathbb{R})$ is equal to the cardinal of continuum.*

Proof. The proof follows from Theorems 3.1 and 4.4 and the fact that $\mathrm{Mod}_{g,p}^q$ is finitely presented. □

Theorem 4.6. *The dimension of the second bounded cohomology group $H_b^2(\Delta_g;\mathbb{R})$ of the hyperelliptic mapping class group Δ_g is equal to the cardinal of continuum.*

Proof. The proof follows from Corollary 3.2 and Theorem 4.4 and the fact that Δ_g is finitely presented. □

Remark 4.1. If $p + q \leq 3$ then the group $\mathrm{Mod}_{0,p}^q$ is either a finite group or a free abelian group. All these groups are amenable and amenable groups have trivial bounded cohomology.

5. Automorphisms of cofinite subgroups of mapping class groups

Let S be a surface of genus g with p punctures. It was shown in [10] and [12] that any isomorphism between two subgroups of finite index in the extended mapping class group of S is the restriction of an automorphism of the extended mapping class group provided that $p \geq 5$ if $g = 0$, $p \geq 3$ if $g = 1$ or $g \geq 2$. The extended mapping class group of S is defined as the group of isotopy classes of all diffeomorphisms $S \to S$ including orientation reversing ones. Since the mapping class group $\mathrm{Mod}_{g,p}$ is characteristic in the extended mapping class group, it follows that any isomorphism between two subgroups of finite index in $\mathrm{Mod}_{g,p}$ is the restriction of an automorphism of $\mathrm{Mod}_{g,p}$ under the restrictions on g and p above. If $g = 0$ and $p \leq 1$, then the mapping class group is trivial. If $(g, p) = (0, 2)$, then the mapping class group is a cyclic group of order 2. If $(g, p) = (0, 3)$ then the mapping class group is the symmetric group on three letters. Obviously, in these cases any isomorphism between two subgroups of $\mathrm{Mod}_{g,p}$ is the restriction of an automorphism of $\mathrm{Mod}_{g,p}$. We prove in this section that this result does not hold if (g, p) is equal to $(0, 4)$, $(1, 0)$ or $(1, 1)$, leaving the case $(g, p) = (1, 2)$ open.

Lemma 5.1. *Let F_n be a nonabelian free group of rank n. If H is a proper subgroup of finite index in F_n, then there exists an automorphism $\varphi : H \to H$ such that φ is not the restriction of any endomorphism of F_n.*

Proof. Suppose that F_n is generated by $\{y, x_1, x_2, \dots, x_{n-1}\}$. Assume that the index of H is $k \geq 2$, so that H is a free group of rank $k(n-1)+1$. If K is another subgroup of F_n of index k, then there exists an automorphism of F_n restricting to an isomorphism between H and K. Therefore, we can assume without loss of generality that H is the subgroup generated by $\{y^j x_i y^{-j}, y^k \mid 1 \leq i \leq n-1,\ 0 \leq j \leq k-1\}$.

Define an automorphism $\varphi : H \to H$ by $\varphi(yx_1y^{-1}) = yx_1y^{-1}x_1$ and the identity on all other generators of H. The automorphism φ does not extend to any endomorphism $F_n \to F_n$. Because if there is such an extension $\tilde{\varphi}$, then we conclude from $\tilde{\varphi}(y^k) = y^k$ that $\tilde{\varphi}(y) = y$. Since $\tilde{\varphi}$ also fixes all generators x_i of F_n, it must be the identity. But φ is not the identity. □

Theorem 5.2. *If (g, p) is equal to $(0, 4), (1, 0)$ or $(1, 1)$, then there exists a subgroup Γ of finite index in the mapping class group $\mathrm{Mod}_{g,p}$ and an automorphism $\varphi : \Gamma \to \Gamma$ such that φ is not the restriction of any endomorphism of $\mathrm{Mod}_{g,p}$.*

Proof. Suppose first that $g = 1$. Note that in this case $\mathrm{Mod}_{1,0}$ and $\mathrm{Mod}_{1,1}$ are isomorphic to $SL(2, \mathbb{Z})$. It is well known that the commutator subgroup of $SL(2, \mathbb{Z})$, denoted by F_2, is a free group of rank 2 and its index in $SL(2, \mathbb{Z})$ is 12. Let Γ be any proper subgroup of finite index in F_2. By Lemma 5.1, there exists an automorphism $\varphi : \Gamma \to \Gamma$ which is not the restriction of any endomorphism of F_2. Since any endomorphism of $SL(2, \mathbb{Z})$ induces an endomorphism of F_2, we are done in this case.

Suppose now that $g = 0$ and $p = 4$. Let S be a sphere with four punctures, say P_1, P_2, P_3, P_4. For $i = 1, 2, 3$ let α_i be three disjoint embedded arcs from P_i to P_{i+1}. Let a and b denote the boundary component of a regular neighborhood of α_1 and α_2, respectively. The pure mapping class group $\mathrm{PMod}_{0,4}$ of S is a free group of rank two freely generated by the Dehn twists t_a and t_b. Let w_i denote the half twist about α_i, so that w_i interchanges P_i and P_{i+1}, $(w_1)^2 = t_a$, $(w_2)^2 = t_b$, and $(w_3)^2$ is the right Dehn twist about the boundary of a regular neighborhood of α_3. Theorem 4.5 in [2] gives a presentation of $\mathrm{Mod}_{0,n}$ for all n. It follows from this, in particular, that w_1, w_2 and w_3 generate $\mathrm{Mod}_{0,4}$ and $H_1(\mathrm{Mod}_{0,4})$ is a cyclic group of order 6 generated by the class of any w_i. Thus, the classes of t_a and t_b in $H_1(\mathrm{Mod}_{0,4})$ both have orders 3.

We now define an automorphism $\varphi : \mathrm{PMod}_{0,4} \to \mathrm{PMod}_{0,4}$ by $\varphi(t_a) = t_a$ and $\varphi(t_b) = t_a t_b$. Suppose that there is an endomorphism $\tilde{\varphi}$ of $\mathrm{Mod}_{0,4}$ extending φ. Since w_1 and w_2 are conjugate, so are $t_a = \tilde{\varphi}(w_1)^2$ and $t_a t_b = \tilde{\varphi}(w_2)^2$. This implies that the classes of t_a and $t_a t_b$ in $H_1(\mathrm{Mod}_{0,4})$ are equal. Therefore, $t_b = (w_2)^2$ represents 0 in $H_1(\mathrm{Mod}_{0,4})$. By this contradiction, φ cannot be extended to an automorphism of $\mathrm{Mod}_{0,4}$. □

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