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A monopole homology for integral homology 3-spheres

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Abstract

To an integral homology 3-sphere Y, we assign a well-defined $\mathbf{Z}$-graded (monopole) homology $MH_*(Y, I_\eta(\Theta;\eta_0))$ whose construction in principle follows from the instanton Floer theory with the dependence of the spectral flow $I_\eta(\Theta;\eta_0)$, where Θ is the unique $U(1)$-reducible monopole of the Seiberg-Witten equation on Y and η_0 is a reference perturbation datum. The definition uses the moduli space of monopoles on $Y \times \mathbf{R}$ introduced by Seiberg-Witten in studying smooth 4-manifolds. We show that the monopole homology $MH_*(Y, I_\eta(\Theta;\eta_0))$ is invariant among Riemannian metrics with same $I_\eta(\Theta;\eta_0)$. This provides a chamber-like structure for the monopole homology of integral homology 3-spheres. The assigned function $MH_{\mathrm{SWF}} : \{I_\eta(\Theta;\eta_0)\} \to \{MH_*(Y, I_\eta(\Theta;\eta_0))\}$ is a topological invariant (as Seiberg-Witten-Floer Theory).

1. Introduction

Since Donaldson [9] initiated the study of smooth 4-manifolds via the Yang-Mills theory, the gauge theory (Donaldson invariants, relative Donaldson-Floer invariants and Taubes' Casson-invariant interpretation, etc) has proved remarkably fruitful and rich to unfold some of the mysteries in studying smooth 4-manifolds. The topological quantum field theory proposed by Witten [37] stimulates the most exciting developments in low-dimensional topology. In 1994, Seiberg and Witten (see [38]) introduces a new (simpler) kind of differential-geometric equation. In a very short time after the equation was introduced, some long-standing problems were solved, new and unexpected results were discovered. For instance, Kronheimer and Mrowka [15] proved the Thom conjecture affirmatively, several authors proved variants (generalizations) of the Thom conjecture independently in [11, 24, 29], as well as the three-dimensional version of the Thom conjecture [4]. Taubes showed that there are more constraints on symplectic structures in [32, 33] and the beautiful equality $SW = Gr$ in [34, 35]. See [7] for a survey in the Seiberg-Witten theory.

Using the dimension-reduction principle, one expects the Floer-type homology of 3-manifolds via the Seiberg-Witten equation. Indeed Kronheimer and Mrowka [15] analyzed the Seiberg-Witten-Floer theory for $\Sigma \times S^1$, where Σ is a closed oriented surface. Later on Marcolli studied the Seiberg-Witten-Floer homology for 3-manifolds with first Betti number positive in [21]. For a connected compact oriented 3-manifold with positive first Betti number and zero Euler characteristic, Meng and Taubes [23] showed that a (average) version of Seiberg-Witten invariant is the same as the Milnor torsion. The interesting class of 3-manifolds as integral (rational) homology 3-spheres is lack of well-posed theory. Although various authors attempted to resolve the problem on defining a "Seiberg-Witten-Floer" theory, the new phenomenon of harmonic-spinor jumps and the dependence of Riemannian metrics is not addressed clearly. The metric-dependence (also related to the harmonic-spinors) issue is quickly realized by many experts in this field (see [7, 26]). In [26], the irreducible Seiberg-Witten-Floer homology of the Seifert spaces is shown to be dependent on the metric and the choice of a connection on the tangent bundle (as our reference η_0 in this paper).

In this paper, we construct a monopole homology from the Seiberg-Witten equation in the same way as an instanton Floer homology from the Self-Duality equation in Donaldson-Floer theory [12]. Our key point is that by using the unique $U(1)$-reducible solution Θ of the Seiberg-Witten equation on an integral homology 3-sphere Y we make use of the spectral flow of Θ to capture the dependence in certain perturbation classes of Riemannian metrics and 1-forms. The same idea was used before by the present author to establish a symplectic Floer homology of knots in [17], and the original one was in the study of the instanton Floer homology of rational homology 3-spheres by Lee and the present author in [16]. We emphasize the Riemann-metric dependence and understand the role of such a fixing spectral flow of $(\Theta; \eta_0)$ in this paper.

Our approach is similar to approaches in [5, 16, 17] to understand the perturbation data (including Riemannian metrics). The unique $U(1)$-reducible Θ gives a spectral flow $I_\eta(\Theta; \eta_0)$ as a Maslov index in [5] Part III. The spectral flow $I_\eta(\Theta; \eta_0) = \mu_\eta(\Theta) - \mu_{\eta_0}(\Theta)$ with respect to a reference η_0 fixes a class of admissible perturbations consisting of Riemannian metrics and 1-forms. As long as Riemannian metrics and 1-forms give the same spectral flow $I_\eta(\Theta; \eta_0)$, we prove that the constructed monopole homology is invariant inside the fixed class of Riemann-metrics and 1-forms ($\eta = (g_Y, \alpha)$) with same $I_\eta(\Theta; \eta_0)$. The spectral flow $I_\eta(\Theta; \eta_0)$ is not a topological invariant, and is dependent upon the Riemannian metrics. Without fixing a class of Riemannian metrics with same $I_\eta(\Theta; \eta_0)$, one cannot obtain well-defined notions such as spectral flow of irreducible Seiberg-Witten solutions on Y, and the gluing formula as well as the relative Seiberg-Witten invariant. Hence our results follow from fixing the data $I_\eta(\Theta; \eta_0)$ as a parameter for our monopole homology.

Theorem A. *(1) For an integral homology 3-sphere Y and any admissible perturbation η, there is a well-defined $\mathbf{Z}$-graded monopole homology $MH_*(Y, I_\eta(\Theta; \eta_0))$ constructed by the Seiberg-Witten equation over $Y \times \mathbf{R}$.*

(2) For any two admissible perturbations η_1 and η_2, there is a group homomorphism Ψ_* between two monopole homologies $MH_*(Y, I_{\eta_1}(\Theta; \eta_0))$ and $MH_*(Y, I_{\eta_2}(\Theta; \eta_0))$.

(3) If $I_{\eta_1}(\Theta; \eta_0) = I_{\eta_2}(\Theta; \eta_0)$, then the homomorphism Ψ_* is an isomorphism.

Our fixed-class $I_\eta(\Theta; \eta_0)$ of Riemannian metrics gains control of the birth and death of irreducible solutions of the Seiberg-Witten equation on the integral homology 3-sphere Y. Changing the reference η_0 into η_0' corresponds to an overall degree-shifting by $\mu_{\eta_0'}(\Theta) - \mu_{\eta_0}(\Theta)$ for the monopole homologies. The control in the instanton homology of rational homology 3-spheres is gained by fixing the spectral flows of all $U(1)$-reducibles from the Wilson-loop perturbations (not metrics). The control in the monopole homology of integral homology 3-spheres is gained by fixing the spectral flow of the unique $U(1)$-reducible Θ from the Riemannian metrics (not only 1-forms). Fixing $I_\eta(\Theta;\eta_0)$ enters crucially in proving Theorem A and Theorem B.

Theorem B. *For a smooth 4-manifold $X = X_0\#_Y X_1$ with $b_2^+(X_i) > 0 (i = 0,1)$ and Y an integral homology 3-sphere, the Seiberg-Witten invariant of X is given by the Kronecker pairing of $MH_*(Y; I_\eta(\Theta;\eta_0))$ with $MH_{-1-*}(-Y; I_\eta(\Theta;\eta_0))$ for the relative Seiberg-Witten invariants $q_{X_0,Y,\eta}$ and $q_{X_1,-Y,\eta}$ (see Definition 8); assume that the moduli space $\mathcal{M}_X$ does not split to $\mathcal{M}_{X_i}(\Theta)$ through the stretching-neck process,*

$$\langle , \rangle : MH_*(Y; I_\eta(\Theta;\eta_0)) \times MH_{-1-*}(-Y; I_\eta(\Theta;\eta_0)) \to \mathbf{Z}; \quad q_{SW}(X) = \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta}\rangle.$$

The paper is organized as follows. §2 provides an introduction of the Seiberg-Witten equation on 3-manifolds. §3 studies the configuration space over Y through Seiberg-Witten equation and a natural monopole complex. We show that there are admissible perturbations from Riemannian metrics and 1-forms in §4 via the method similar to [28]. The Seiberg-Witten solution on $Y\times\mathbf{R}$ with finite energy has exponentially decay property at ends, and the regularity of Seiberg-Witten solutions on $Y\times\mathbf{R}$ is also proved in §4. The spectral-flow properties and dependence on Riemannian metrics are discussed in §5. The gluing and splitting result (Theorem 6.6) is proved in §6, and the proof of Theorem A (Proposition 6.10 for (1), Proposition 7.1 for (2) and Proposition 7.2 for (3)) is occupied in §6 and §7. In §8, we study the relative Seiberg-Witten invariant and complete the proof of Theorem B as Theorem 8.4. The length of the paper is due to the desire to provide complete and self-contained proofs of transversality, decay estimates and gluing process.

2. Seiberg-Witten equation on 3-manifolds

It is well-known that every closed oriented 3-manifold is spin. The group $Spin(3) \cong SU(2) \cong Sp_1$ is the universal covering of $SO(3) = Spin(3)/\{\pm I\}$. Pick a Riemannian metric g on Y. The metric g defines the principal $SO(3)$-bundle $P_{SO}(Y)$ of oriented orthonormal frames on Y. A spin structure is a lift of $P_{SO}(Y)$ to a principal $Spin(3)$-bundle $P_{Spin}(Y)$ over Y. The set of equivalence classes of such lifts has, in a natural way, the structure of a principal $H^1(Y,Z_2)$-bundle over a point. So there is a unique spin-structure on the integral homology 3-sphere Y.

There is a natural adjoint representation

$$Ad : Spin(3)\times Sp_1 \to Sp_1; \quad (q,\alpha)\mapsto q\alpha q^{-1},$$

and associated rank-2 complex vector bundle (spinor bundle) $W = P_{Spin(3)}(Y)\times_{Ad} C^2$. Let $L = detW$ be the determinant line bundle. For the ordinary Spin-structure, one has a Clifford multiplication

$$c : T^*Y\otimes W \to W; \quad c([p,\alpha])\otimes[p,v] \to [p,\overline{\alpha}v].$$

So c induces a map $T^*Y \to Hom(W,W)$. The spinor pairing $\tau : W \otimes \overline{W} \to T^*Y$ is given by

$$[p, v_1 \otimes v_2] \to \tau(\frac{1}{4} Im(v_1 i v_2)),$$

where τ is an orientation preserving isomorphism $P_{Spin(3)}(Y) \times Sp_1 \to T^*Y$. A connection a on L together with the Levi-Civita connection of a Riemannian metric g_Y on Y form a covariant derivative on W. This maps sections of W into sections of $W \otimes T^*Y$. Followed by the Clifford multiplication, one has a Dirac operator

$$\partial_a^{g_Y} : \Gamma(W) \stackrel{\nabla_a^{g_Y}}{\to} \Gamma(W \otimes T^*Y) \stackrel{c}{\to} \Gamma(W).$$

The determinant line bundle L is trivial for the spin structure, so we may choose θ to be the trivial connection and $\partial_\theta^g : \Gamma(W) \to \Gamma(W)$ is the usual Dirac operator. Note that all bundles over the integral homology 3-sphere Y are **trivial**.

There is a unique spin-structure on $Y \times \mathbf{R}$ associated to the unique spin-structure on Y with the product metric $g_Y + dt^2$ on $Y \times \mathbf{R}$. The two spinor bundles $W_{(4)}^{\pm}$ on $Y \times \mathbf{R}$ can be identified by using a Clifford multiplication by dt, where t is denoted for the variable on $\mathbf{R}$. Both $W_{(4)}^{+}$ and $W_{(4)}^{-}$ are obtained by the pull-back of the $U(2)$-bundle $W \to Y$ from the projection map $Y \times \mathbf{R} \to Y$. Thus we have the identification of the map $\sigma : \Lambda^2 T^*(Y \times \mathbf{R}) \to Hom(W_{(4)}^{+}, W_{(4)}^{-})$ and the map $\tau^{-1} : T^*Y \to Hom(W,W)$ through the above identifications: $\sigma(\eta) = \tau^{-1}(*_g \eta)$. In other words from the identification $\Lambda^2 T^*(Y \times \mathbf{R}) = \Lambda^2 T^*Y \oplus \Lambda^1 T^*Y$ and using the Hermitian pairing on $W_{(4)}^{\pm}$, there is an induced pairing

$$\tau : \overline{W} \times W \to \Lambda^1 T^*Y.$$

In fact for every $\gamma : T^*Y \to Hom(W,W)$ (a spin structure), that is a way to determine a spin structure on $Y \times \mathbf{R}$ by

$$\sigma : T^*(Y \times \mathbf{R}) \to Hom(W \oplus W, W \oplus W); \quad \sigma(v,r) = \begin{pmatrix} 0 & \gamma(v) + r1 \\ \gamma(v) - r1 & 0 \end{pmatrix}.$$

The determinant line bundle $L_{(4)} = det W_{(4)}^{\pm}|_{Y \times \mathbf{R}}$ (a trivial line bundle) carries $U(1)$-connections $A = a + \phi dt$. So the Dirac operator D_A^g for the product metric $g_Y + dt^2$ over $Y \times \mathbf{R}$ is given by

$$D_A^{g_Y} = \begin{pmatrix} 0 & -\nabla_t + \partial_a^{g_Y} \\ \nabla_t + \partial_a^{g_Y} & 0 \end{pmatrix},$$

where $\partial_a^{g_Y}$ is a twisted self-adjoint Dirac operator on $\Gamma(W) \to \Gamma(W)$, and $\nabla_t = \frac{\partial}{\partial t} + \phi$ is a twisted skew adjoint Dirac operator over $\mathbf{R}$.

The curvature 2-form of $A = a + \phi dt$ can be calculated as $F_A = F_a + (\frac{\partial a}{\partial t} - d_a \phi) dt$. Using the identification of $\Omega^2(Y \times \mathbf{R}) \cong \Omega^2(Y) \oplus \Omega^1(Y)$, we can write F_A^+ as $*_{g_Y} F_a + (\frac{\partial a}{\partial t} - d_a \phi) \in \Omega^1(Y)$ as the self-dual component of the curvature F_A. Now the Seiberg-Witten monopole equation on 4-manifolds reduces to a Seiberg-Witten monopole equation on 3-manifolds as

$$\begin{cases} (\nabla_t + \partial_a^{g_Y})\psi & = 0 \\ *_{g_Y} F_a + (\frac{\partial a}{\partial t} - d_a \phi) & = i\tau(\psi,\psi) \end{cases} \tag{2.1}$$

for $\psi \in \Gamma(W)$. It is equivalent to the flow equation of $(a + \phi dt, \psi)$:

$$\begin{cases} \frac{\partial \psi}{\partial t} & = -\partial_a^{g_Y}\psi - \phi.\psi \\ \frac{\partial (a+\phi dt)}{\partial t} & = - *_{g_Y} F_a + d_a\phi + i\tau(\psi,\psi). \end{cases} \tag{2.2}$$

The equation (2.1) is invariant under the gauge transformation $u \in Map(Y, U(1))$, where the gauge group action on $(a + \phi dt, \psi)$ is given by

$$u \cdot (a + \phi dt, \psi) = (u^* a + (\phi - u^{-1}\frac{du}{dt})dt, \psi u^{-1}). \tag{2.3}$$

There is a temporal gauge to obtain a simpler equation. The temporal gauge u is the element which $u \cdot (a + \phi dt) = u^* a$, i.e., $\phi - u^{-1}\frac{du}{dt} = 0$. Then the equation (2.2) can be reduced to the following form.

$$\begin{cases} \frac{\partial \psi}{\partial t} & = -\partial_a^{g_Y}\psi \\ \frac{\partial a}{\partial t} & = - *_{g_Y} F_a + i\tau(\psi,\psi). \end{cases} \tag{2.4}$$

3. Configuration spaces on Y

Fix a trivialization $L = Y \times U(1)$, one can identify the space of $U(1)$-connections of Sobolev L^p_k-norm with the space $\mathcal{A}^p_k = L^p_k(\Omega^1(Y, i\mathbf{R}))$ of 1-forms on Y such that the zero element in $\Omega^1(Y, i\mathbf{R})$ corresponds to the trivial connection θ on L. The gauge group of L can be identified with $\mathcal{G}^p_k(Y) = L^p_{k+1}(Map(Y, U(1)))$ acting on $\mathcal{A}^p_k \times L^p_k(\Gamma(W))$ by (2.3). We need to assume that $k + 1 > 3/p$ so that $\mathcal{G}_Y = \mathcal{G}^p_k(Y)$ is a Lie group. We may take $k = 1$ and $p = 2$.

Let $\mathcal{C}_Y$ be the configuration space

$$\mathcal{C}_Y = L^2_k(\Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W)).$$

The quotient space is $\mathcal{B}_Y = \mathcal{C}_Y/\mathcal{G}_Y$. Denote $\mathcal{C}^*_Y = \{(a,\phi,\psi) \in \mathcal{C}_Y | \psi \neq 0\}$. For $(a,\phi,\psi) \in \mathcal{C}^*_Y$, the isotropy group $\Gamma_{(a,\phi,\psi)} = \{id\}$. For $(a,\phi,\psi) \in \mathcal{C}_Y \setminus \mathcal{C}^*_Y$, the isotropy group $\Gamma_{(a,\phi,0)} = U(1)$, these elements are called reducibles. For example, $\Theta = (\theta, 0, 0)$ is reducible by all constant maps from Y to $U(1)$. Note that $\mathcal{G}_Y$ acts freely on $\mathcal{C}^*_Y$, so $\mathcal{B}^*_Y = \mathcal{C}^*_Y/\mathcal{G}_Y$ forms an open and dense set in $\mathcal{C}_Y/\mathcal{G}_Y$.

Proposition 3.1. *$\mathcal{B}^*_Y$ is a Hilbert manifold. For $(a_0,\phi_0,\psi_0) \in \mathcal{C}^*_Y$, the tangent space of $\mathcal{B}^*_Y$ can be identified with*

$$T_{[(a_0,\phi_0,\psi_0)]}\mathcal{B}^*_Y = \{(a,\phi,\psi) \in L^2_k(\Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W))|$$

$$\|(a,\phi,\psi)\|_{L^2_{k-1}(Y)} = \|a\|_{L^2_{k-1}(Y)} + \|\phi\|_{L^2_{k-1}(Y)} + \|\psi\|_{L^2_{k-1}(Y)} < \varepsilon, \quad d^{*g_Y}_{a_0}\psi + Im(\psi_0,\psi) = 0\}.$$

Proof: This follows from the construction of slice in [9, 13]. It will be clear from context to identify (a_0, ϕ_0, ψ_0) with its gauge equivalence class in our notation. The gauge orbit of $(a_0, \phi_0, \psi_0) \in \mathcal{C}_Y^*$ is given by $\mathcal{G}_Y \to \mathcal{C}_Y^*$:

$$g = e^{iu} \to (a_0 - g^{-1}dg, \phi_0, \psi_0 g^{-1}).$$

The linearization of this map at $Id = e^0$ is

$$\delta_0 : T_{id}\mathcal{G}_Y = \Omega^0(Y, i\mathbf{R}) \to \Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W)$$

$$u \mapsto (-du, 0, -\psi_0 u).$$

So the adjoint operator δ_0^* of δ_0 is given by

$$\delta_0^* \psi = d_{a_0}^{*_{g_Y}} \psi + Im(\psi_0.\psi).$$

A neighborhood of $[(a_0, \phi_0, \psi_0)] \in \mathcal{B}_Y^*$ can be described as a quotient of $T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*/\Gamma_{(a_0,\phi_0,\psi_0)}$ for sufficiently small ε. Every nearby orbit meets the slice $(a_0, \phi_0, \psi_0) + T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*$. This is amount to solving the gauge fixing condition relative to (a_0, ϕ_0, ψ_0), i.e., there exists a unique $u \in \Omega^0(Y, i\mathbf{R})$ such that $e^{iu} \cdot (a_0 + a, \phi_0 + \phi, \psi_0 + \psi) \in T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*$ for $\psi_0 \neq 0$. Hence it follows from applying the implicit function theorem. □

There is an associated bundle $\mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W))$ over $\mathcal{C}_Y^*$ because of the free action of $\mathcal{G}_Y$ on $\mathcal{C}_Y^*$. We define a section $f : \mathcal{C}_Y^* \to \mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W))$ by

$$f(a, \phi, \psi) = [(a, \phi, \psi), *_{g_Y} F_a - d_a\phi - i\tau(\psi, \psi), \partial_a^{g_Y} \psi + \phi.\psi].$$

Note that f is $\mathcal{G}_Y$-equivariant, $f(g \cdot (a, \phi, \psi)) = g \cdot f(a, \phi, \psi)$. Hence it descends to $\mathcal{B}_Y^*$,

$$f : \mathcal{B}_Y^* \to \mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)).$$

Now $f(a, \phi, \psi) \in T_{[(a,\phi,\psi)],\varepsilon} L_{k-1}^2 \mathcal{B}_Y^* = \mathcal{L}_{[(a,\phi,\psi)]}$. So f can be thought of as a vector field on the Hilbert manifold $\mathcal{B}_Y^*$. Over $\mathcal{B}_Y^*$, f is a section of the bundle $\mathcal{L}$ with fiber $\mathcal{L}_{[(a,\phi,\psi)]}$.

Definition 3.2. The zero set of f in $\mathcal{B}_Y^*$ is the moduli space of solutions of the 3-dimensional Seiberg-Witten equation

$$f^{-1}(0) = \mathcal{R}_{SW}^*(Y, g_Y) = \{[(a, \phi, \psi)] \in \mathcal{C}_Y^* \text{ satisfies (3.1)}\}/\mathcal{G}_Y.$$

$$\begin{cases} \partial_a^{g_Y} \psi + \phi.\psi = 0 \\ *_{g_Y} F_a - d_a\phi - i\tau(\psi, \psi) = 0 \end{cases}. \tag{3.1}$$

We will show that $\mathcal{R}_{SW}^*(Y, g_Y)$ is a zero-dimensional smooth manifold and its algebraic number is the Euler characteristic of a monopole homology defined in §6 (see also [4] for instance).

The linearization of f can be computed as the following.

$$\begin{aligned} f(a_0 + sa, \phi_0 + s\phi, \psi_0 + s\psi) &= (*_{g_Y} F_{a_0+sa} - d_{a_0+sa}(\phi_0 + s\phi) - i\tau(\psi_0 + s\psi, \\ &\quad \psi_0 + s\psi), \partial_{a_0+sa}^{g_Y}(\psi_0 + s\psi) + (\phi_0 + s\phi).(\psi_0 + s\psi) \\ &= f(a_0, \phi_0, \psi_0) + sDf(a_0, \phi_0, \psi_0)((a, \phi, \psi)) + o(s^2). \end{aligned}$$

So the linearized operator $Df(a_0, \phi_0, \psi_0) : T_{[(a_0,\phi_0,\psi_0)]}\mathcal{B}^*_Y \to \mathcal{L}_{[(a_0,\phi_0,\psi_0)]}$ is given by

$$Df(a_0, \phi_0, \psi_0) : \Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W) \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W),$$

$$((a, \phi, \psi) \longmapsto \begin{pmatrix} *_{g_Y} d_{a_0} & -d_{a_0} & -iIm(\psi_0, , \cdot) \\ c(\cdot\psi_0) & c\cdot\psi_0 & \partial^{g_Y}_{a_0} + \phi_0\cdot \end{pmatrix} \begin{pmatrix} a \\ \phi \\ \psi \end{pmatrix}.$$

It forms a natural 3-dimensional monopole complex, since $\ker \delta^*_0$ is the gauge fixing slice. So

$$MC_\bullet : 0 \to \Omega^0(Y, i\mathbf{R}) \overset{\delta_0}{\to} \Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W) \overset{Df}{\to} \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W) \to 0, \tag{3.2}$$

is a short exact sequence. The operator

$$\delta^*_0 \oplus Df(a_0, \phi_0, \psi_0) : \Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W) \to \Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W)$$

$$(a, \phi, \psi) \longmapsto \begin{pmatrix} *_{g_Y} d_{a_0} & -d_{a_0} & -iIm(\psi_0, \cdot) \\ -d^{*_{g_Y}}_{a_0} & 0 & Im(\psi_0, \cdot) \\ c(\cdot\psi_0) & c\cdot\psi_0 & \partial^{g_Y}_{a_0} + \phi_0\cdot \end{pmatrix} \begin{pmatrix} a \\ \phi \\ \psi \end{pmatrix}, \tag{3.3}$$

is a first-order operator with symbol $\sigma(\delta^*_0 \oplus Df) = \sigma(\delta(a_0, 0, \psi_0; g_Y))$, where

$$\delta(a_0, 0, \psi_0; g_Y) = \begin{pmatrix} *_{g_Y} d_{a_0} & -d_{a_0} & 0 \\ -d^{*_{g_Y}}_{a_0} & 0 & 0 \\ 0 & 0 & \partial^{g_Y}_{a_0} \end{pmatrix}$$

is a first-order self-adjoint Dirac operator. Hence

$$\begin{aligned} Ind(\delta^*_0 \oplus Df) &= Ind(\delta(a_0, 0, \psi_0; g_Y)) \\ &= Ind \begin{pmatrix} *_{g_Y} d_{a_0} & -d_{a_0} \\ -d^{*_{g_Y}}_{a_0} & 0 \end{pmatrix} + Ind\partial^{g_Y}_{a_0} \\ &= 0. \end{aligned} \tag{3.4}$$

Since the operator $\begin{pmatrix} *_{g_Y} d_{a_0} & -d_{a_0} \\ -d^{*_{g_Y}}_{a_0} & 0 \end{pmatrix}$ is self-adjoint and every Dirac operator has index zero over odd (3-)dimensional manifolds, thus we have the zero index for the operator $\delta^*_0 \oplus Df$. Generically, the moduli space $\mathcal{R}_{SW}(Y, g_Y)$ is zero-dimensional.

Define $H^0(MC_\bullet) = \ker \delta_0$, $H^1(MC_\bullet) = \ker Df/im\delta_0$, $H^2(MC_\bullet) = cokerDf$. The first cohomology $H^1(MC_\bullet)$ is isomorphic for every $(a_0, \phi_0, \psi_0) \in \mathcal{B}^*_Y$, so that $(a_0, \phi_0, \psi_0) \in \mathcal{B}^*_Y$ is a nondegenerate zero of f if and only if $\ker(\delta^*_0 \oplus Df) = H^1(MC_\bullet) = 0$. For $\Theta = (\theta, 0, 0)$ and a generic metric g_Y without harmonic spinors of $\partial^{g_Y}_\theta$, we have that Θ is always isolated and nondegenerate (in the Bott sense) zero of f on the integral homology 3-sphere Y.

4. Admissible Perturbation and Transversality

In this section, we prove that there are enough perturbations to make the zero set of f transverse. There is a 1-form perturbation reduced from 4-dimensional Seiberg-Witten equation as in [7, 15, 32]. In our 3-dimensional case, *the harmonic spinor may vary or jump as metrics on Y vary.* In order to obtain any topological information, one needs to extend the perturbation-data and understand the harmonic spinors accordingly. The method we used here is essentially the one used in [13, 16, 17, 28]. See also [4, 21, 22, 27, 30, 36] for different approaches.

Let $\mathcal{P}_Y = \Sigma_Y \times \Omega^1(Y, i\mathbf{R})$ be the space of perturbation data, where Σ_Y is the space of Riemannian metrics on Y. Consider the union $\cup_{(g_Y,\alpha)\in\mathcal{P}_Y}\mathcal{R}^*_{SW}(Y; g_Y, \alpha)$ of the moduli spaces of 3-dimensional Seiberg-Witten solutions over all metrics and 1-forms. If the union is a (Banach) Hilbert manifold, then its projection to the space $\mathcal{P}_Y$ is a Fredholm map. So there exists a Baire first category in $\mathcal{P}_Y$ such that $\mathcal{R}^*_{SW}(Y; g_Y, \alpha)$ is a manifold by the Sard-Smale theorem.

Let f_η be the parametrized smooth section of the bundle $\mathcal{L} \to \mathcal{B}^*_Y \times \mathcal{P}_Y$ with $\eta = (g_Y, \alpha) \in \mathcal{P}_Y$. The map f_η is given by

$$f_\eta : \mathcal{B}^*_Y \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)$$

$$(a, \phi, \psi) \mapsto (*_{g_Y} F_a - d_a\phi - i\tau(\psi, \psi) + \alpha, \partial_a^{\nabla_0+\alpha}\psi + \phi.\psi),$$

where ∇_0 is the Levi-Civita connection for the metric g_Y. Let $f_{1\eta}(a, \phi, \psi) = \partial_a^{\nabla_0+\alpha}\psi + \phi.\psi$ be the second component of the map f_η on $\Gamma(W)$, and $f_{0\eta}(a, \phi, \psi)$ be the first component of f_η.

Lemma 4.1. *$f_{1\eta}$ is a submersion ($Df_{1\eta}$ is surjective).*

Proof: The differential $Df_{1\eta}$ is given by the formula

$$Df_{1\eta}(a, \phi, \psi; o, \alpha)(\varepsilon a, \varepsilon\phi, \varepsilon\psi, 0, \varepsilon\alpha) = \partial_a^{\nabla_0+\alpha}(\varepsilon\psi) + (\varepsilon\alpha + \varepsilon a + \varepsilon\phi).\psi + \phi.\varepsilon\psi,$$

where we vary along $\{\Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W)\} \times \{\{0\} \times \Omega^1(Y, i\mathbf{R})\}$ of $T[a, \phi, \psi]\mathcal{B}^*_Y \times \mathcal{P}_Y$. We want to show that $Df_{1\eta}$ is surjective. Suppose the contrary. Then there exists a spinor $\chi \in \Gamma(W)$ such that it is perpendicular to $ImDf_{1\eta}$.

$$\langle \partial_a^{\nabla_0+\alpha}(\varepsilon\psi), \chi\rangle = 0, \tag{4.1}$$

for all $\varepsilon\psi$. I.e., $\chi \in \ker(\partial_a^{\nabla_0+\alpha})^*$. By the elliptic regularity of (4.1), a solution χ is smooth. Choose a point $y \in Y$ such that $\chi(y) \neq 0$. By the uniqueness of continuation of the solution of the elliptic equation [2], $\partial_a^{\nabla_0+\alpha} \cdot (\partial_a^{\nabla_0+\alpha})^*\chi = 0$, there is a neighborhood U_y of y such that $\chi(y) \neq 0$ for $y \in U_y$. Thus we can find a 1-form $\varepsilon\alpha + \varepsilon a \in \Omega^1(Y, i\mathbf{R})$ such that $(\varepsilon\alpha + \varepsilon a).\psi = \lambda\chi$ with $\lambda \neq 0$ in U_y, and $\varepsilon\alpha + \varepsilon a$ has compact support. So we obtain

$$\begin{aligned} 0 &= \langle \partial_{a+\varepsilon a}^{\nabla_0+\alpha+\varepsilon\alpha}(\varepsilon\psi), \chi\rangle \\ &= \langle \partial_a^{\nabla_0+\alpha}(\varepsilon\psi), \chi\rangle + \langle(\varepsilon\alpha + \varepsilon a).\varepsilon\psi, \chi\rangle \\ &= \langle\lambda\chi, \chi\rangle = \lambda\langle\chi, \chi\rangle. \end{aligned}$$

Therefore $\chi = 0$ in U_y, so $\chi \equiv 0$ by a result in [2]. The contradiction implies that $f_{1\eta}$ is a submersion. □

By the Hodge decomposition of $\Omega^1(Y, i\mathbf{R}) = Imd \oplus Imd^{*_{g_Y}}$ for Y, we have that $f_{0\eta}(\alpha, \phi, \psi) = *_{g_Y} F_a - d_a\phi - i\tau(\psi, \psi) + \alpha$ is also a submersion onto $\Omega^1(Y, i\mathbf{R})$.

Corollary 4.2. *The spaces $f_{0\eta}^{-1}(0)$ and $f_{1\eta}^{-1}(0)$ are Banach manifolds.*

□

Now at point $(a_0, \phi_0, \psi_0; g_0, \alpha) \in \mathcal{C}_Y \times \mathcal{P}_Y$, the parametrized smooth section (still denoted by f)

$$f(a_0, \phi_0, \psi_0; g_0, \alpha) = f_{(g_0,\alpha)}(a_0, \phi_0, \psi_0) = f_\eta(a_0, \phi_0, \psi_0)$$

is a submersion.

Proposition 4.3. *The differential Df is onto at all points of the moduli space $f^{-1}(0) \subset \mathcal{B}_Y^* \times \mathcal{P}_Y$.*

Proof: The differential Df at $(a_0, \phi_0, \psi_0; g_0, \alpha) \in \mathcal{C}_Y \times \mathcal{P}_Y$ is of the form (Df_0, Df_1)

$$Df_0 = *_{g_0} d_{a_0} a + (g)_* F_{a_0} - d_{a_0}\phi - iIm(\psi_0, \psi) - a.\phi_0 + \alpha \tag{4.2}$$

$$Df_1 = \partial_{a_0}^{\nabla_0 + \alpha_0}\psi + (\alpha + a).\psi_0 + (\phi.\psi_0 + \phi_0.\psi) + r(g)) \tag{4.3}$$

where $(g)_*$ is the variation of the Hodge star operator $(g)_* = \frac{d}{ds}|_{s=0} *_{g_0+sg}$, $r(g)$ is a zero order operator applied to the variation $g_0 + sg + o(s^2)$ of metric, $a.\phi_0$ is the Clifford multiplication of 1-form a on the section $\phi_0 \in \Gamma(W)$. The surjective of Df_0 follows from Theorem 3.1 of [13], and the surjective of Df_1 follows similarly from Proposition I.3.5 of [28]. It is sufficient to prove that

$$(0, 0, \psi, 0, \alpha) \mapsto \partial_{a_0}^{\nabla_0 + \alpha_0}\psi + (\alpha).\psi_0 \tag{4.4}$$

is surjective. Let $\chi \in \Gamma(W)$ be an element perpendicular to the image of (4.4) ($\chi \in \ker(\partial_{a_0}^{\nabla_0 + \alpha_0})^*$). So we obtain

$$0 = \langle \partial_{a_0}^{\nabla_0 + \alpha_0}\psi + (\alpha).\psi_0, \chi \rangle = \langle \alpha.\psi_0, \chi \rangle,$$

for all $\alpha \in \Omega^1(Y, i\mathbf{R})$. Hence the pointwise Hermitian product $(\,,\,)$ on W for $\alpha.\psi_0$ and χ gives the corresponding function $(\alpha.\psi_0, \chi) = 0$ on Y. The sections ψ_0 and χ are solutions of the regular elliptic equations

$$\partial_{a_0}^{\nabla_0 + \alpha_0}\psi_0 + \phi_0.\psi_0 = 0; \qquad (\partial_{a_0}^{\nabla_0 + \alpha_0})^*\chi = 0.$$

So both ψ_0 and χ cannot vanish on an open subsets by [2]. Thus there exists an open dense domain $U \subset Y$ on which ψ_0 and χ are not zero. In the local coordinate $\{x_1, x_2, x_3\}$ of y, $\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ as quanterion imaginary part multiplication on the sections $\Gamma(W)$. If $\{s_i\}_{i=1,2}$ is a local basis of W at $y \in Y$, then

$$\psi_0 = s_1.e_1 + s_2.e_2; \qquad \chi = s_1^{'}.c_1 + s_2^{'}.c_2,$$

where $s_i^{'} = \gamma^1(s_i)$ with γ^1 given by spinor multiplication of $\frac{\partial}{\partial x_1}$. So we obtain the Clifford multiplication

$$\begin{aligned}\alpha.\psi_0 &= \gamma(a_1dx_1).\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \gamma(a_2dx_2).\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \gamma(a_3dx_3).\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &= \{\begin{pmatrix} -2a_1i & 0 \\ 0 & 2a_1i \end{pmatrix} + \begin{pmatrix} 0 & -2a_2 \\ 2a_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2a_3i \\ 2a_3i & \end{pmatrix}\}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &= \begin{pmatrix} -2a_1i & -2a_2+2a_3i \\ 2a_2+2a_3i & 2a_1i \end{pmatrix}\cdot\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.\end{aligned}$$

Then the Hermitian pairing

$$0 = (\alpha.\psi_0, \chi) = \mathrm{Tr}(\begin{pmatrix} -2a_1i & -2a_2+2a_3i \\ 2a_2+2a_3i & 2a_1i \end{pmatrix}\cdot\begin{pmatrix} e_1.c_1 & e_1.c_2 \\ e_2.c_1 & e_2.c_2 \end{pmatrix}),$$

for all a_1, a_2, a_3. It follows that

$$\begin{pmatrix} e_1.c_1 & e_1.c_2 \\ e_2.c_1 & e_2.c_2 \end{pmatrix} = 0,$$

since the linear span of the matrices of the form $\begin{pmatrix} -a_1i & -a_2+a_3i \\ a_2+a_3i & a_1i \end{pmatrix}$ is the whole of $End(C^2)$ except for scaler $a.Id$. Up to permutation we have $e_2 = c_2 = 0$ and e_1, c_1 are orthogonal $(e_1, c_1) = 0$. Now $\psi_0 = s_1.e_1$ and $\chi = s_1^{'}.e_1$ and $(e_1, c_1) = 0$ in the domain U. We normalize e_1 and c_1 with the property $(e_1, e_1) = (c_1, c_1) = 1$.

$$\partial_{a_0}^{\nabla_0+\alpha_0}\psi_0 + \phi_0.\psi_0 = \partial_{a_0}^{\nabla_0+\alpha_0}(s_1)\otimes e_1 + s_1.d_{a_0}e_1 + \phi_0.s_1.e_1 = 0;$$

$$(\partial_{a_0}^{\nabla_0+\alpha_0}(s_1)\otimes e_1 + s_1.d_{a_0}e_1 + \phi_0.s_1.e_1, c_1) = (\partial_{a_0}^{\nabla_0+\alpha_0}(s_1) + \phi_0.s_1)(e_1, c_1) + s_1(d_{a_0}e_1, c_1) = 0.$$

So we have $s_1(d_{a_0}e_1, c_1) = 0$. If $s_1 \neq 0$, then $(d_{a_0}e_1, c_1) = 0$. From $(e_1, c_1) = 0$, one obtains

$$(d_{a_0}e_1, c_1) + (e_1, d_{a_0}c_1) = 0.$$

This implies $(e_1, d_{a_0}c_1) = 0$, the connection a_0 is reducible. But a_0 is not the trivial connection, so $s_1 = 0$. Hence $\psi_0 = s_1.e_1 = 0$ and $(a_0, \phi_0, 0) \in \mathcal{C}_Y \setminus \mathcal{C}_Y^*$. For any $(a_0, \phi_0, \psi_0) \in \mathcal{C}_Y^*$, $\chi = 0$ by the same method. Thus the differential Df_1 is surjective. □

We consider the map $f_* : \mathcal{B}_Y^* \times \mathcal{P}_Y \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)$.

Corollary 4.4. *There is a dense subset $\mathcal{P}_Y^{'} \subset \mathcal{P}_Y$ such that for $\eta \in \mathcal{P}_Y^{'}$ the space $f_*^{-1}(0)$ is regular (i.e., a smooth Banach manifold).*

Proof: Take f_* as a section of $\mathcal{B}_Y^* \times \mathcal{P}_Y$ to $(\mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)) \times \mathcal{P}_Y$. So $f_*^{-1}(0)|_{\mathcal{B}_Y^*} = f_*^{-1}(0)/\mathcal{G}_Y$ is a Banach manifold.

$$\begin{array}{ccc} \mathcal{B}_Y^* \times \mathcal{P}_Y & \xrightarrow{f} & \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W) \\ \downarrow \pi_2 & & \\ \mathcal{P}_Y & & \end{array}$$

The projection map π_2 is a smooth Fredholm map of index zero. It follows exactly from the same argument in [9, 13] by the transversality. $\square$

Corollary 4.5. *The inverse image $\pi_2^{-1}((g_Y, \alpha))$ of a generic parameter $\eta = (g_Y, \alpha) \in \mathcal{P}_Y^{'}$, the moduli space $\mathcal{R}_{SW}(Y, \eta)$ of the 3-dimensional monopole solutions is a zero-dimensional manifold.*

In general, the class of reducible elements in $\mathcal{C}_Y \setminus \mathcal{C}_Y^*$ forms a singular strata in the quotient space $\mathcal{B}_Y$. If it is a solution of 3-dimensional Seiberg-Witten equation, it is also singular to the space of $\mathcal{R}_{SW}(Y, g_Y)$. The reducible solutions of the 3-dimensional Seiberg-Witten equation satisfy

$$\begin{aligned} \partial_a^{\nabla_0+\alpha}\psi + \phi_0.\psi &= 0 \\ - *_{g_Y} F_a + d_a\phi &= 0, \end{aligned} \tag{4.5}$$

for $\psi = 0$. Applying the temporal gauge $g \cdot (a, \phi) = (g^*a, 0)$, we get that g^*a is a flat connection on $Y \times U(1)$ over Y. For the integral homology 3-sphere Y, there is a unique $U(1)$ reducible connection, namely the trivial one. So the reducible solution is $(\theta, 0)$. There is a unique $U(1)$-reducible solution of (4.5), denoted by $\Theta = (\theta, 0)$.

Note that $\ker(\delta_0^* \oplus Df(a_0, \phi_0, \psi_0)) = \ker \partial_{a_0}^{g_Y}$ for an integral homology 3-sphere. For a generic metric g_Y, $\ker \partial_{a_0}^{g_Y} = 0$. But $\ker \partial_{a_0}^{g_t}$ may have a nontrivial kernel as the Riemannian metrics vary in an one-parameter family (see [14]). Hence the harmonic-spinor jump creates and/or destroys irreducible solutions of the 3-dimensional Seiberg-Witten equation. This is the main problem to understand the new phenomenon that the "Seiberg-Witten-Floer theory" is not entirely metric-independent (see [7, 26]). In the next section, we study such a dependence of Riemannian metrics.

Proposition 4.6. *$\mathcal{R}_{SW}^*(Y, \eta) = \mathcal{R}_{SW}(Y, \eta) \setminus \{\Theta\}$ is a zero-dimensional smooth compact manifold for a first category near $\eta = (g_Y, \alpha)$ in $\mathcal{P}_Y^{'}$.*

Proof: The results follows from the construction above, Proposition 2c.1 of [12] and the Sard-Smale theorem. The compactness follows from the standard arguments of elliptic regularity and maximal principle in [4, 7, 15, 38]. $\square$

For any solution $(a, \phi, \psi) \in \mathcal{R}_{SW}^*(Y, \eta)$, we have $\ker(\delta_0^* \oplus Df_\eta(a, \phi, \psi)) = 0$. Thus $\delta_0^* \oplus Df_\eta(a, \phi, \psi)$ defines a closed essentially self-adjoint Fredholm operator on $\mathcal{L}_{[a,\phi,\psi]}$, and its domain is the L_k^2-completion of $\mathcal{L}_{[a,\phi,\psi]}$. The eigenvalues set is discrete, unbounded in $\mathbf{R} \setminus \{0\}$, and each eigenvalue is of finite multiplicity. Note that there are finitely many elements in $\mathcal{R}_{SW}(Y, \eta)$ for $\eta \in \mathcal{P}_Y^{'}$. Therefore there is a number $\delta_\eta > 0$ which is smaller than the smallest nonzero absolute value of an eigenvalue of $\delta_0^* \oplus Df_\eta(a, \phi, \psi)$ with $(a, \phi, \psi) \in f_\eta^{-1}(0) = \mathcal{R}_{SW}(Y, \eta)$.

Note that the Chern-Simons type functional with respect to a reference connection a_0 is

$$cs_\eta(a, \psi) = -\frac{1}{2}\int_Y \{(a - a_0) \wedge (F_a + F_{a_0} + 2\alpha) + \langle \partial_a^{\nabla_0+\alpha}\psi, \psi\rangle\} dvol_{g_Y}, \tag{4.6}$$

which has the gradient $\nabla cs_\eta(a,\psi) = f_\eta(a,0,\psi)$ for the representative with a temporal gauge. The solution of the deformed Seiberg-Witten equation on $Y\times\mathbf{R}$ can be transformed into:

$$\frac{\partial}{\partial t}(a(t)+\phi(t)dt,\psi(t)) = -f_\eta(a(t),\phi(t),\psi(t)), \tag{4.7}$$

and by Proposition 8 of [15] there is always a translation-invariant representative satisfying (2.4) in a temporal gauge. For $(A,\Psi) = (a(t)+\phi(t)dt,\psi(t))$, the deformed Seiberg-Witten energy is given by $E_\eta(A,\Psi) =$

$$\frac{1}{2}\int_{Y\times\mathbf{R}}\{|\nabla_t\psi(t)|^2+|\partial_a^{\nabla_0+\alpha}\psi(t)|^2+|\nabla_t a(t)+d_a\phi(t)|^2+|*_{g_Y}F_a - i\tau(\psi,\psi)+\alpha|^2\}dvol_{g_Y}dt, \tag{4.8}$$

where $\nabla_t = \frac{\partial}{\partial t}+\phi$. Following the similar calculations in [7, 15, 30, 36], one has

$$E_\eta(A,\Psi) = cs_\eta(A,\Psi)|_{t=-\infty} - cs_\eta(A,\Psi)|_{t=+\infty}$$

$$+\int_{Y\times\mathbf{R}}\{|F_A^+ - \frac{1}{4}\tau(\Psi,\Psi)+\alpha\wedge dt + *(\alpha\wedge dt)|^2 + |\partial_A^{\nabla_0+\alpha}\Psi|^2\}dvol_{g_Y}dt,$$

where $*$ is the Hodge star operator on $Y\times\mathbf{R}$ with respect to the metric g_Y+dt^2. Let (A,Ψ) be a trajectory flow line in (4.7) with the terms $\nabla_t\psi(t)$, $\partial_a^{\nabla_0+\alpha}\psi(t)$, $\nabla_t a(t)+d_a\phi(t)$, $*_{g_Y}F_a - i\tau(\psi,\psi)+\alpha$ are in L^p for some $p\geq 2$.

Lemma 4.7. *There is a solution $(a_\infty,\phi_\infty,\psi_\infty)\in f_\eta^{-1}(0)$ over Y such that $(A,\Psi) = (a(t)+\phi(t)dt,\psi(t))$ converges to $(a_\infty+\phi_\infty dt,\psi_\infty)$ uniquely up to the gauge equivalence in the sense that $(A,\Psi)|_{Y\times\{t\}}$ converges in C^∞ over Y.*

Proof: Let the temporal-gauge representative $(a(t),0,\psi(t))$ be the element over $Y\times(0,1)$ obtained from the translation of $(A,\Psi)|_{Y\times(t,t+1)}$. The finiteness of the integrals $\nabla_t\psi(t), \partial_a^{\nabla_0+\alpha}\psi(t), \nabla_t a(t)+d_a\phi(t), *_{g_Y}F_a - i\tau(\psi,\psi)+\alpha$ over the end implies that for $p\geq 2$, as $t\to+\infty$,

$$\|\nabla_t\psi(t)\|_{L^p(Y\times(0,1))}\to 0, \qquad \|\partial_a^{\nabla_0+\alpha}\psi(t)\|_{L^p(Y\times(0,1))}\to 0,$$

$$\|\nabla_t a(t)\|_{L^p(Y\times(0,1))}\to 0, \qquad \|*_{g_Y}F_a - i\tau(\psi,\psi)+\alpha\|_{L^p(Y\times(0,1))}\to 0.$$

For any sequence $t_n\to\infty$, there is a subsequence $t_n^{'}\to\infty$ and $(A(\infty),\Psi(\infty))$ over $Y\times(0,1)$ such that after suitable gauge transformations $(A,\Psi)|_{Y\times\{t_n'\}}\to(A(\infty),\Psi(\infty))$ in C^∞ sense over any compact subset of $Y\times(0,1)$. Hence the limit element $(A(\infty),\Psi(\infty)) = (a_\infty,0,\psi_\infty)$ has zero L^p-norm over $Y\times(0,1)$. Therefore we get

$$\nabla_t\psi_\infty = \partial_{a_\infty}^{\nabla_0+\alpha}\psi_\infty = \nabla_t a_\infty = *_{g_Y}F_{a_\infty} - i\tau(\psi_\infty,\psi_\infty)+\alpha = 0.$$

I.e., $f_\eta(a_\infty,0,\psi_\infty)=0$. Up to the gauge equivalence, $(a_\infty,0,\psi_\infty)\in\mathcal{R}_{SW}(Y,\eta) = f_\eta^{-1}(0)$. □

Let $J(T) = E_\eta(A,\Psi)|_{Y\times[T,\infty)}$ be the energy functional over $Y\times[T,\infty)$. For the Seiberg-Witten solution (A,Ψ) of (2.1), $J(T) = cs_\eta(A|_T,\Psi|_T) - cs_\eta(A|_\infty,\Psi|_\infty)$ from the calculation above. Thus we have, by (4.7),

$$\frac{dJ(T)}{dT} = \nabla cs_\eta(A|_T,\Psi|_T)\cdot\frac{\partial(A|_T,\Psi|_T)}{\partial T} = -\|f_\eta(a(T),\phi(T),\psi(T))\|^2_{L^2(Y\times\{T\})}, \qquad (4.9)$$

where $(A|_T,\Psi|_T) = (a(T)+\phi(T)dt,\psi(T))$. For T sufficiently large, $(a(T)+\phi(T)dt,\psi(T)) = (a_\infty+\phi_\infty dt,\psi_\infty)+(a(t)+\phi(t)dt,\psi(t))$, the Taylor expansion of the functional f_η is

$$f_\eta(a(T),\phi(T),\psi(T)) = Df_\eta(a_\infty,\phi_\infty,\psi_\infty)(a(t),\phi(t),\psi(t)) + N(a(t),\phi(t),\psi(t)), \quad (4.10)$$

where $f_\eta(a_\infty,\phi_\infty,\psi_\infty)=0$ and $N(a(t),\phi(t),\psi(t))$ is the quadratic term of $a(t),\phi(t),\psi(t)$.

Lemma 4.8. *For* $(a,\phi,\psi)\in\{\ker(\delta_0^*\oplus Df_\eta(a_\infty,\phi_\infty,\psi_\infty))\}^\perp$ *(the subspace which is perpendicular to* $\ker(\delta_0^*\oplus Df_\eta(a_\infty,\phi_\infty,\psi_\infty))$*), there exist a constant* C_2 *and* T_0 *such that for* $t\geq T\geq T_0$*,*

$$\|(a,\phi,\psi)\|_{L^2_k(Y\times\{t\})} \leq C_2\|f_\eta(a(T),\phi(T),\psi(T))\|_{L^2_{k-1}(Y)},$$

where $\|(a,\phi,\psi)\|_{L^2_k(Y\times\{t\})} = \|(a+\phi dt,\psi)\|_{L^2_k(Y\times\{t\})} = \|a\|_{L^2_k(Y\times\{t\})}+\|\phi\|_{L^2_k(Y\times\{t\})}+\|\psi\|_{L^2_k(Y\times\{t\})}$.

Proof: Note that $\delta_\eta\|(a,\phi,\psi)\|_{L^2_k(Y)} \leq \|\delta_0^*\oplus Df_\eta(a_\infty,\phi_\infty,\psi_\infty)(a,\phi,\psi)\|_{L^2_{k-1}(Y)}$ from the smallest number of absolute eigenvalues for $(a,\phi,\psi)\in\{\ker(\delta_0^*\oplus Df_\eta(a_\infty,\phi_\infty,\psi_\infty))\}^\perp$ with $(a_\infty,\phi_\infty,\psi_\infty)\in f_\eta^{-1}(0)$. For T sufficiently large, $(a(t)+\phi(t)dt,\psi(t)) = (a(T)+\phi(T)dt,\psi(T)) - (a_\infty+\phi_\infty dt,\psi_\infty)$ is approaching to zero in the C^∞ sense, and $\|(a(t),\phi(t),\psi(t)\|_{L^2_k(Y)}$ is sufficiently small. Note that by the Hölder inequality and the Sobolev embedding theorem,

$$\|N(a(t),\phi(t),\psi(t))\|_{L^2_{k-1}(Y)} \leq C_1\|(a,\phi,\psi)\|^2_{L^4_{k-1}(Y\times\{t\})} \leq C_1'\|(a,\phi,\psi)\|^2_{L^2_k(Y\times\{t\})}.$$

Let T_0 be the number such that $C_1'\|(a,\phi,\psi)\|^2_{L^2_k(Y\times\{T_0\}}\leq \delta_\eta/2$. Thus the result follows from (4.10) for any $T\geq T_0$. □

We have the first order Taylor expansion for $J(T) = cs_\eta(A|_T,\Psi|_T) - cs_\eta(A|_\infty,\Psi|_\infty)$:

$$\begin{aligned} J(T) &= dcs_\eta(A|_\infty,\Psi|_\infty)(A|_T - A|_\infty,\Psi|_T-\Psi|_\infty) + N(a(t),\phi(t),\psi(t)) \\ &= dcs_\eta(A|_\infty,\Psi|_\infty)(a(t)+\phi(t)dt,\psi(t)) + N(a(t),\phi(t),\psi(t)). \end{aligned}$$

By the same method used in Lemma 4.8, one obtains, for $T\geq T_0$,

$$\begin{aligned} J(T) &\leq C_3\|(a,\phi,\psi)\|^2_{L^2_1(Y\times\{T\})} \\ &\leq C_3C_2\|f_\eta(A|_T,\Psi|_T)\|^2_{L^2(Y\times\{T\})} \\ &= -C_3C_2\frac{\partial J}{\partial T}, \end{aligned}$$

by (4.9). Therefore $J(T)\leq J(T_0)e^{-\gamma(T-T_0)}$ for some $\gamma>0$ and $T\geq T_0$.

Proposition 4.9. *Let (A,Ψ) be the trajectory flow line of (4.7) over $Y\times\mathbf{R}$. For the end $Y\times[T,\infty)$ or $Y\times(-\infty,-T]$, there exist a gauge transformation $g_\pm$, a constant C_4 and $\gamma_1>0$ such that $(A,\Psi)=(a(t)+\phi(t)dt,\psi(t))+g^*_\pm(a_{\pm\infty}+\phi_{\pm\infty}dt,\psi_{\pm\infty})$ for $\pm t\geq T$, and for $(a(t),\phi(t),\psi(t))$ satisfying the hypothesis of Lemma 4.8,*

$$\sup_Y\{|a(t)|,|\phi(t)|,|\psi(t)|,|f_\eta(a(t),\phi(t),\psi(t))|\}\leq C_4\cdot e^{-\gamma_1(|t|-T)},$$

for $|t|>T$. Moreover one can choose $(A|_t,\Psi|_t)$ such that all derivatives decay exponentially:

$$\sup_Y\{|\nabla^l a(t)|,|\nabla^l\phi(t)|,|\nabla^l\psi(t)|\}\leq C_5\cdot e^{-\gamma_1(|t|-T)},\qquad |t|>T,$$

where the constants C_4 and C_5 depends continuously on (A,Ψ).

Proof: Let σ be a positive number with $\sigma<\gamma$ in the decay $J(T)\leq J(T_0)e^{-\gamma(T-T_0)}$. For $T\geq T_0$, we define $J_\sigma(T)=\int_T^\infty e^{\sigma t}\|f_\eta(a(t),\phi(t),\psi(t))\|^2_{L^2(Y\times\{t\})}dt$. By (4.9) for every $t\geq T$, we have

$$J_\sigma(T)=\int_T^\infty e^{\sigma t}(-\frac{dJ(t)}{dt})dt\leq J(T_0)e^{\gamma T_0}(1+\frac{\sigma}{\sigma-\gamma})e^{-(\gamma-\sigma)T},$$

from the integration by part and the decay of $J(T)$. Thus $\|f_\eta(a(t),\phi(t),\psi(t))\|^2_{L^2(Y\times\{t\})}$ decays exponentially, and we can estimate all the covariant derivatives of $f_\eta(a(t),\phi(t),\psi(t))$ in the same way. The term $J(T)$ controls the L^2-norm of $f_\eta(a(t),\phi(t),\psi(t))$ over the compact subset $Y\times[T+1,T+2]$, and this gives a bound on all higher derivatives over $Y\times(T+1,T+2)$. By the translation and gauge-fixing condition, we have, for some $0<\gamma_1<\gamma$ and $t>T$,

$$|f_\eta(a(t),\phi(t),\psi(t))|_{L^\infty(Y)}\leq C_6e^{-\gamma_1(t-T)}.$$

For the temporal gauge representative $(A,\Psi)=(a(t)+a_\infty,\psi(t)+\psi_\infty)$, the trajectory flow satisfies $\frac{\partial}{\partial t}(a(t),\psi(t))=-f_\eta(a(t),0,\psi(t))$. So

$$\sup_Y\{|\frac{\partial a(t)}{\partial t}|,|\frac{\partial\psi(t)}{\partial t}|\}\leq C_6e^{-\gamma_1(t-T)}.$$

Hence (A,Ψ) converges to $(a_\infty,0,\psi_\infty)$ exponentially. The exponential decay of the covariant derivatives of $f_\eta(a(t),0,\psi(t))$ on Y for $t\in[T,\infty)$ implies that $(a_\infty,0,\psi_\infty)\in C^\infty(Y)$, and $\nabla^l a(t),\nabla^l\psi(t)$ converge exponentially. The gauge equivalence classes $[(A|_t,\Psi|_t)]$ converge to $[(a_\infty,0,\psi_\infty)]$ with a suitable choice of subsequences, there is a converging sequence $\{g_t\}$ to g_∞ in the C^∞ sense: $g_t^*(A|_t,\Psi|_t)\to g^*_\infty(a_\infty,0,\psi_\infty)$. Thus we can modify the representative to the $(A|_t,\Psi|_t)$ over $Y\times[T,\infty)$ such that the exponentially decay estimate holds for $t>T$. The case for $Y\times(-\infty,-T]$ is same. □

Note that if $(a_\infty,\phi_\infty,\psi_\infty)\in\mathcal{R}^*_{SW}(Y,\eta)$, then $\ker(\delta_0^*\oplus Df_\eta(a_\infty,\phi_\infty,\psi_\infty))=0$. There is a 1-dimensional subspace $\ker(\delta_0^*\oplus Df_\eta(\theta,0,0))$ for the reducible solution $\Theta=(\theta,0)$.

Define the weighted Sobolev space $L^p_{k,\delta}$ on sections ξ of a bundle over $Y\times\mathbf{R}$ to be the space of ξ for which $e_\delta\cdot\xi$ is in L^p_k, where $e_\delta(y,t)=e^{\delta|t|}$ for $|t|\geq 1$. For any

$0 \le \delta = \min\{\delta_\eta/2, \gamma_1/2\}$ and any Seiberg-Witten monopole solution (A, Ψ) on $Y \times \mathbf{R}$, the linearized operator

$$D_{A,\Psi} : L^p_{k+1,\delta}(\Omega^1(Y \times \mathbf{R}) \oplus \Gamma(W^+_{(4)})) \to L^p_{k,\delta}((\Omega^0 \oplus \Omega^2_+)(Y \times \mathbf{R}) \oplus \Gamma(W^+_{(4)}))$$

is Fredholm (the proof is a direct application of Theorem 1.3 of [20] with the choice of δ, see also [7, 12, 15, 32, 38]). We call (A, Ψ) *regular* if $\mathrm{Coker} D_{A,\Psi} = 0$ and we call $\mathcal{M}_{Y\times\mathbf{R}}$ (the moduli space of perturbed Seiberg-Witten solutions with finite energy) *regular* if it contains orbits of regular (A, Ψ)'s. Note that the weighted Sobolev spaces are mainly needed for dealing with reducible Seiberg-Witten solutions. If $\lim_{t\to\pm\infty}(A, \Psi) \in \mathcal{R}^*(Y, \eta)$, then $D_{A,\Psi} : L^p_{k+1}(\Omega^1(Y \times \mathbf{R}) \oplus \Gamma(W^+_{(4)})) \to L^p_k((\Omega^0 \oplus \Omega^2_+)(Y \times \mathbf{R}) \oplus \Gamma(W^+_{(4)}))$ is Fredholm.

Let τ_Y be a smooth cutoff function such that $\tau_Y(y, t) = |t|$ and $e_\delta = e^{\delta\tau_Y(y,t)}$ for $|t| > T_0 > 0$. We define an element (∇_0, Ψ_0) on $Y \times \mathbf{R}$ such that

$$\nabla_0|_{Y\times[T_0,\infty)} = \frac{d}{dt} + (a_+ + \phi_+ dt), \quad \Psi_0|_{Y\times[T_0,\infty)} = \psi_+,$$

$$\nabla_0|_{Y\times(-\infty,-T_0]} = \frac{d}{dt} + (a_- + \phi_- dt), \quad \Psi_0|_{Y\times(-\infty,-T_0]} = \psi_-,$$

where $(a_\pm + \phi_\pm dt, \psi_\pm) \in f_\eta^{-1}(0)$. The Fréchet space $\Omega^1_c(Y \times \mathbf{R}) \oplus \Gamma(W)$ of compact supported C^∞-sections on $T^*(Y \times \mathbf{R}) \oplus W$ can be completed to a Banach space

$$\mathcal{A}^p_{k,\delta}(Y \times \mathbf{R}) = (\nabla_0, \Psi_0) + L^p_{k,\delta}(\Omega^1_c(Y \times \mathbf{R}) \oplus \Gamma(W)),$$

where $\|(a, \phi, \psi)\|_{L^p_{k,\delta}(Y\times\mathbf{R})} = \|(a+\phi dt, \psi)\|_{L^p_{k,\delta}(Y\times\mathbf{R})} = \|e_\delta \cdot a\|_{L^p_k(Y\times\mathbf{R})} + \|e_\delta \cdot \phi\|_{L^p_k(Y\times\mathbf{R})} + \|e_\delta \cdot \psi\|_{L^p_k(Y\times\mathbf{R})}$. The gauge group is given by

$$\mathcal{G}^p_{k+1,\delta} = \{u \in L^p_{k+1,\mathrm{loc}}(Y \times \mathbf{R}, i\mathbf{R}) | u = \exp\xi \quad \text{for } |t| \ge T_0 \text{ and } \xi \in L^p_{k+1,\delta}\}.$$

The quotient space $\mathcal{B}^p_{k,\delta}(Y \times \mathbf{R}) = \mathcal{A}^p_{k,\delta}(Y \times \mathbf{R})/\mathcal{G}^p_{k+1,\delta}$ is the path space from $c_- = (a_-, \phi_-, \psi_-)$ to $c_+ = (a_+, \phi_+, \psi_+)$ with appropriate Sobolev norm.

Let $\mathcal{M}_{Y\times\mathbf{R}}$ be the moduli space of finite-energy Seiberg-Witten solutions of (4.7) on $Y \times \mathbf{R}$. By Lemma 4.7, the moduli space is a disjoint union given by $\mathcal{M}_{Y\times\mathbf{R}} = \coprod_{c,c'\in f_\eta^{-1}(0)} \mathcal{M}_{Y\times\mathbf{R}}(c,\ c')$, where $\mathcal{M}_{Y\times\mathbf{R}}(c,\ c')$ is the solutions $(A,\ \Psi)$ of (4.7) with $\lim_{t\to\infty}(A, \Psi) = c'$ and $\lim_{t\to-\infty}(A, \Psi) = c$. There is a free $\mathbf{R}$-action on $\mathcal{M}_{Y\times\mathbf{R}}$. We set by

$$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c, c') = \{(A, \Psi) \in \mathcal{M}_{Y\times\mathbf{R}}(c, c') | J(0) = E_\eta(A, \Psi)|_{Y\times[0,\infty)} = \frac{1}{2} E_\eta(A, \Psi)|_{Y\times\mathbf{R}}\}.$$

It represents the trajectory flow line with $t = 0$ splitting the energy in half (see [18] §3.1), and can be identified with $\hat{\mathcal{M}}_{Y\times\mathbf{R}}(c, c') = \mathcal{M}_{Y\times\mathbf{R}}(c, c')/\mathbf{R}$ in [12]. The moduli space $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c, c')$ is compact (possibly with boundary) with dimension determined by the spectral flow $\mathrm{Ind}\, D_{A,\Psi}$.

Proposition 4.10. *The set of all perturbations $\eta \in \mathcal{P}'_Y$ of which $\mathcal{M}_{Y\times\mathbf{R}}$ is regular is of Baire's first category.*

Proof: There is an well-defined map $l : \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \to \mathcal{B}_Y$ by restricting the balanced Seiberg-Witten solutions (A, Ψ) of (4.7) to $t = 0$ slice as $l(A, \Psi) = (a(0) + \phi(0)dt, \psi(0))$. By Aronszajin's theorem in [2], l is injective. The Seiberg-Witten solution is the zero set of the section

$$F : \mathcal{B}^p_{k,\delta}(Y \times \mathbf{R}) \times \mathcal{P}_Y \to \mathcal{B}^p_{k,\delta}(Y \times \mathbf{R}) \times_{\mathcal{G}^p_{k+1,\delta}} (\Omega^1(Y, i\mathbf{R}) \times \Gamma(W)),$$

through the identifications in §2, where $F(a+\phi dt, \psi, \eta) = (\frac{\partial a}{\partial t} - d_a\phi + *_{g_Y} F_a - i\tau(\psi, \psi) + \alpha, \frac{\partial \psi}{\partial t} + \partial_a^{\nabla_0+\alpha}\psi + \phi.\psi) = \frac{\partial}{\partial t}(a + \phi dt, \psi) + f_\eta(a, \phi, \psi)$. We can identify $\mathcal{B}^p_{k,\delta}(Y \times \mathbf{R})$ as a path space of $\mathcal{B}^p_k(Y)$ with e_δ decay in t-direction. Thus we have the following diagram

$$\begin{array}{ccc} \Omega^1(Y, i\mathbf{R}) \times \Gamma(W) & = & \Omega^1(Y, i\mathbf{R}) \times \Gamma(W) \\ \downarrow\uparrow F & & \downarrow\uparrow f_\eta \\ \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \times \mathcal{P}_Y & \stackrel{l}{\longrightarrow} & \mathcal{B}_Y \times \mathcal{P}_Y, \end{array}$$

where $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \cong F^{-1}(0)/\mathbf{R} \subset \mathcal{B}^p_{k,\delta}(Y \times \mathbf{R})/\mathbf{R}$. Furthermore we have $F = \frac{\partial}{\partial t} + f_\eta \circ l^*$ for the injective map $l^* : \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \to \mathcal{B}^*_Y$. Thus $DF = \frac{\partial}{\partial t} + Df_\eta(l^*(\cdot)) \cdot Dl^*$ is surjective for $Im(l^*) \neq \emptyset$, i.e., $(A(0), \Psi(0)) \neq (a(0), 0, 0)$. If $l : \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \to \mathcal{B}_Y \setminus \mathcal{B}^*_Y$, then we can shift $l[t](A, \Psi) = (A|_t, \Psi|_t) \in \mathcal{B}^*$ unless (A, Ψ) is a constant $c \in f_\eta^{-1}(0)$ which is regular solution by Corollary 4.4. Note that $l[t]$ is also injective. Since $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}$ is a disjoint union of compact subspaces $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c, c^{'})$. Hence there is a t_0 such that $l[t_0] : \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c, c^{'}) \to \mathcal{B}^*_Y$ provided that $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c, c^{'})$ is not a constant solution. Therefore the Baire's first category $\mathcal{P}^{'}_Y(c, c^{'})$ for which $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c, c^{'})$ is regular follows from the Sard-Smale theorem and Corollary 4.4. Then $\mathcal{P}^{'}_Y = \bigcap_{c,c' \in f_\eta^{-1}(0)} \mathcal{P}^{'}_Y(c, c^{'})$ is again Baire's first category for our result. The proof follows exactly from the same method in [12] Proposition 2c.2 with Chern-Simons Seiberg-Witten functional as defined in [15] §4 and [4, 21, 23]. □

A perturbation $\eta = (g_Y, \alpha)$ satisfying Corollary 4.5 and Proposition 4.10 is called **admissible.** We still use $\mathcal{P}^{'}_Y$ to denote it for the rest of the paper. Note that the Seiberg-Witten equation on $Y \times \mathbf{R}$ is written as

$$F(A, \Psi) = (\frac{\partial}{\partial t} + Df_\eta(c))(a(t), \phi(t), \psi(t)) + N(a(t), \phi(t), \psi(t)), \tag{4.11}$$

where the expansion is near the end with the limit $c \in f_\eta^{-1}(0)$, and $(A, \Psi) = (a(t) + \phi(t)dt, \psi(t))$ and $N(a(t), \phi(t), \psi(t))$ is the quadratic term of $(a(t), \phi(t), \psi(t))$. The index of $D_{A,\Psi} = \frac{\partial}{\partial t} + \delta_0^* + Df_\eta(c)$ does not change if δ is varied in such a way that δ avoids the spectrum of $\delta_0^* + Df_\eta(c)$ over Y. The index of $D_{A,\Psi}$ will change if δ is changed across an eigenvalue of $\delta_0^* + Df_\eta(c)$ (see the next section).

5. Spectral flow and dependence on Riemannian metrics

In this section, we use the unique $U(1)$-reducible solution Θ to capture the metric-dependent relation via the spectral flow. In [16] joined with Lee, the author used the Walker correction-term around $U(1)$-reducibles to obtain homotopy classes of admissible perturbations (realized by a family of Lagrangians), and to show the invariance among the same homotopy class of the Lagrangian perturbations. Those Walker correction-term can be interpreted as the spectral flow in [5, 16].

Proposition 5.1. *For an admissible perturbation* $\eta = (g_Y, \alpha) \in \mathcal{P}_Y^{'}$ *and a nondegenerate zero* $(a, \phi, \psi) \in \mathcal{R}_{SW}(Y, \eta) = f_\eta^{-1}(0)$, *we can associate an integer* $\mu_\eta(a, \phi, \psi) \in \mathbf{Z}$ *such that for* $(A, \Phi) \in \mathcal{B}_{Y\times\mathbf{R}}((a, \phi, \psi), (a^{'}, \phi^{'}, \psi^{'}))$

$$\begin{aligned} \mu_\eta(e^{iu} \cdot (a, \phi, \psi)) &= \mu_\eta(a, \phi, \psi), \\ IndexD_{A,\Phi} &= \mu_\eta(a, \phi, \psi) - \mu_\eta(a^{'}, \phi^{'}, \psi^{'}) - dim\,\Gamma_{(a^{'},\phi^{'},\psi^{'})}, \end{aligned}$$

where $\Gamma_{(a^{'},\phi^{'},\psi^{'})}$ *is the isotropy subgroup of* $(a^{'}, \phi^{'}, \psi^{'})$.

Proof: Let $\pi_1 : Y \times [0, 1] \to Y$ be the projection on the first factor. Let $L_{(4)} \times W_{(4)}$ be the pullback $\pi_1^*(detW^{\pm}) \times \pi_1^* W^{\pm}$ such that $(A, \Phi) \in \mathcal{A}_{L_{(4)}} \times W_{(4)}$ satisfies $(A, \Phi)|_{t\leq 0} = (a + \phi dt, \psi)$ and $(A, \Phi)|_{t\geq 1} = (a^{'} + \phi^{'} dt, \psi^{'})$. We have $D_{A,\Phi} = \frac{\partial}{\partial t} + \delta_t$ with $\delta_t = \delta_0^* \oplus Df_\eta(A(t), \Phi(t))$ in (4.2). Then the Fredholm index of $D_{A,\Phi}$ is given by the spectral flow of δ_t (see [3, 5, 12]). The second equality follows from the same proof of Proposition 2b. 2 in [12]. The first equality follows from

$$\begin{aligned} SF(e^{iu} \cdot (a, \phi, \psi), (a, \phi, \psi)) &= \text{Ind}\, D_{A,\Phi}((A, \Phi)|_{t=0}, (A, \Phi)|_{t=0})_{Y\times S^1} \\ &= \frac{1}{4}(c_1(L_{(4)})^2 - (2\chi + 3\sigma))(Y \times S^1) = 0, \end{aligned}$$

where χ and σ are the Euler number and signature of $Y \times S^1$, and $c_1(L_{(4)})^2(Y \times S^1) = 0$ for the integral homology 3-sphere Y. □

Note that the relative index is gauge-invariant, but depending on the perturbation $\eta \in \mathcal{P}_Y^{'}$ by Proposition 5.1. The absolute index may not be well-defined since $\mu_\eta(\Theta)$ depends upon $\eta \in \mathcal{P}_Y^{'}$. In the instanton case, we fix the trivialization of a principal bundle and a fixed tangent vector to the trivial connection to determine $\mu(\theta) = 0$ for the trivial connection θ. It turns out that such a fixation is independent of metrics and other perturbation data in the instanton Floer theory. But this is no longer true for the monopole case.

Proposition 5.2. (Definition) *Two admissible perturbations* η_0 *and* η_1 *in* $\mathcal{P}_Y^{'}$ *are (called) homotopic to each other through a 1-parameter family* $\eta_t (0 \leq t \leq 1)$ *in* $\mathcal{P}_Y$ *if and only if* $\mu_{\eta_0}(\Theta) = \mu_{\eta_1}(\Theta)$.

Proof: For two admissible perturbations η_0 and η_1 in §4, we can connect them into a 1-parameter family η_t such that there are at most finitely many $t \in (0, 1)$ with η_t corresponding harmonic-spinor jumps. Denote those $0 < t_0 < t_1 \cdots < t_n < 1$ and $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1} = 0$ so that λ_i is not the eigenvalues of $\delta_t = \delta_t(\theta, 0)$ for $t_{i-1} \leq t \leq t_i$, where $t_{-1} = 0$ and $t_{n+1} = 1$. Define $n_i = \dim(\delta_{t_i} - \lambda Id)$ with $\lambda \in [\lambda_{i+1}, \lambda_i]$ and $n_i = -\dim(\delta_{t_i} - \lambda Id)$ with $\lambda \in [\lambda_i, \lambda_{i+1}]$. From the operator $\frac{\partial}{\partial t} + \delta_t(\Theta)$ (denoted by

$DF_{\eta_t}(\Theta)$) and the well-known facts in [3, 5, 12], we have

$$\operatorname{Ind} DF_{\eta_t}(\Theta) = \sum_{i=0}^{n} n_i.$$

This shows that $\operatorname{Ind} DF_{\eta_t}(\Theta)$ is independent of the construction η_t and that is continuous in η_t. On the other hand,

$$\operatorname{Ind} DF_{\eta_t}(\Theta) = \mu_{\eta_0}(\Theta) - \mu_{\eta_1}(\Theta).$$

Thus the obstruction to connect two generic perturbations is the spectral flow along the metric path in Σ_Y. The Riemannian-metric space Σ_Y is path-connected. So $\operatorname{Ind} DF_{\eta_t}(\Theta) = 0$ provides that η_0 and η_1 are in the same (homotopy) class of with respect to the spectral flow. □

Thus the dependence of metrics also enters into the definition of relative indices for $(a, \phi, \psi) \in \mathcal{R}^*_{SW}(Y, \eta)$. Now we follow the instanton case to fix the relative index

$$\mu_\eta(a, \phi, \psi) = \operatorname{Ind} DF_\eta(\Theta, (a, \phi, \psi)) \in \mathbf{Z},$$

which depends on the value $\mu_\eta(\Theta)$. Any changes of $\mu_\eta(\Theta)$ shift $\mu_\eta(a, \phi, \psi)$ by an integer, and $\mu_\eta(\Theta)$ is understood with respect to some reference perturbation $\eta_0 \in \mathcal{P}_Y^{'}$. Thus we call that $\eta \in \mathcal{P}_Y^{'}$ lies in the same homotopy class of $\eta^{'}$ provided $\mu_\eta(\Theta) = \mu_{\eta'}(\Theta)$.

Lemma 5.3. *For an admissible perturbation $\eta \in \mathcal{P}_Y^{'}$, the Seiberg-Witten moduli space $\mathcal{R}_{SW}(Y, \eta) = f_\eta^{-1}(0)$ is a compact 0-dimensional oriented manifold. The orientation is well-defined for a fixed homotopy class $\eta \in \mathcal{P}_Y^{'}$.*

Proof: By Proposition 4.6, $\mathcal{R}_{SW}(Y, \eta)$ is a 0-dimensional C^∞-compact manifold. The orientation at each point of $\mathcal{R}_{SW}(Y, \eta)$ is defined by its spectral flow which depends on the perturbation homotopy class of η. (This is different phenomenon from the (instanton) Casson invariant of integral homology 3-spheres.) Hence the orientation for the monopole case is not globally defined for all $\eta \in \mathcal{P}_Y^{'}$. It is only locally defined for a homotopy class with same spectral flow at Θ. □

Note that the monopole number $\#\mathcal{R}^*_{SW}(Y, \eta)$ (counted with sign) is not a topological invariant. The number $\#\mathcal{R}^*_{SW}(Y, \eta)$ depends on the metric with harmonic-spinor jumps and the homotopy class $\eta \in \mathcal{P}_Y^{'}$.

6. Monopole homology of integral homology 3-spheres

For an admissible perturbation $\eta \in \mathcal{P}_Y^{'}$, we have a new gradient vector field f_η for which the irreducibles are all nondegenerate in §4. Since zeros of f_η are now isolated finite-many points, we use them to generate the monopole chain groups. The transitivity is proved in terms of gluing and splitting Seiberg-Witten solutions over $Y \times_T \mathbf{R}$. This gives the general structure of the Seiberg-Witten trajectory flow lines of (4.7) to obtain a homology result.

Lemma 6.1. *For any $c, c^{'} \in \mathcal{R}^*(Y,\eta)$ and $p \geq 2$, there exists a positive constant C_p such that for all $(A,\Psi) \in \mathcal{M}^{bal}_{Y\times\mathbf{R}}(c,c^{'})$ in one component and $(a,\phi,\psi) \in L^p(\Omega^1(Y,i\mathbf{R}) \oplus \Omega^0(Y,i\mathbf{R}) \oplus \Gamma(W))$, we have*

$$C_p\|(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})} \leq \|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(A,\Psi))^*(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})},$$

where $(\ )^*$ denotes the adjoint operator with respect to the L^2-norm.

Proof: Proposition 4.10 implies that $(D_{A,\Psi})^*$ has trivial kernel for $(A,\Psi) \in \mathcal{M}^{\rm bal}_{Y\times\mathbf{R}}(c,c^{'})$, where the operator $D_{A,\Psi} = \frac{\partial}{\partial t} + \delta_0^* \oplus Df_\eta(A,\Psi)$ from the gauge-fixing condition and $c, c^{'} \in \mathcal{R}^*(Y,\eta)$. Thus we obtain

$$C_{(A,\Psi),p}\|(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})} \leq \|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(A,\Psi))^*(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})},$$

for $(a,\phi,\psi) \in L^p(\Omega^1(Y,i\mathbf{R}) \oplus \Omega^0(Y,i\mathbf{R}) \oplus \Gamma(W))$. The constant $C_{(A,\Psi),p}$ is continuous in (A,Ψ). Any one component of $\mathcal{M}^{\rm bal}_{Y\times\mathbf{R}}(c,c^{'}) \cap \mathcal{B}^p_k(c,c^{'})$ is compact. Hence C_p is the smallest constant of $C_{(A,\Psi),p}$ for this compact set. □

Note that the components of $\mathcal{M}^{\rm bal}_{Y\times\mathbf{R}}(c,c^{'})$ are described by the spectral flow $\mu_\eta(c) - \mu_\eta(c^{'})$ depending upon $I_\eta(\Theta,\eta_0)$. For a fixed $I_\eta(\Theta,\eta_0)$, there is a unique component $\mathcal{M}^{\rm bal}_{Y\times\mathbf{R}}(c,c^{'})$. Unlike the instanton case, there are possibly infinitely many components for the instantons over $Y \times \mathbf{R}$ with fixed asymptotic.

Let $\chi_-(t)$ be a smooth cutoff function with $\chi_-(t) = 1$ for $t \leq T_1 - 1$ and $\chi_-(t) = 0$ for $t \geq T_1$ and $|d\chi_-| \leq C_0$ for some constant C_0, where T_1 is a parameter to be determined ($\geq T_0$). Let $(A_-,\Psi_-) \in \mathcal{M}^{\rm bal}_{Y\times\mathbf{R}}(c_-,c^{'})$ with our fixed $I_\eta(\Theta,\eta_0)$. Define $(\tilde{A}_-,\tilde{\Psi}_-) = (1-\chi_-)c^{'} + \chi_- \cdot (A_-,\Psi_-)$ to be the cutoff element as a path element of the configuration space $\Omega^1(Y,i\mathbf{R}) \oplus \Gamma(W) \oplus \Gamma(W)$.

Lemma 6.2. *There exist $T_4 \geq T_0 + 1$ and C_7 independent of (A_-,Ψ_-) such that for $T_1 > T_4$ and $(A_-,\Psi_-) \in \mathcal{M}^{bal}_{Y\times\mathbf{R}}(c_-,c^{'})$, $p,q \geq 2$, we have*

$$\|(\tilde{A}_-,\tilde{\Psi}_-) - (A_-,\Psi_-)\|_{L^q(Y\times\mathbf{R})} \leq C_7 e^{-\delta(T_1-T_4)}, \quad \|F(\tilde{A}_-,\tilde{\Psi}_-)\|_{L^p(Y\times\mathbf{R})} \leq C_7 e^{-\delta(T_1-T_4)},$$

where $(A^{'},\Psi^{'}) - (A,\Psi) = (A^{'} - A, \Psi^{'} - \Psi)$.

Proof: Note that $(\tilde{A}_-,\tilde{\Psi}_-) - (A_-,\Psi_-)$ is support on $Y \times [T_1 - 1, \infty)$. By Proposition 4.9,

$$\sup_Y |(\tilde{A}_-,\tilde{\Psi}_-) - (A_-,\Psi_-)| = \sup_Y |(1-\chi_-)(c^{'} - (A_-,\Psi_-))| \leq C_{4,(A_-,\Psi_-)} e^{-\gamma_1(t-T_0)},$$

for $t \geq T_0$. Hence

$$\|(\tilde{A}_-,\tilde{\Psi}_-) - (A_-,\Psi_-)\|_{L^q(Y\times\mathbf{R})} \leq C_{4,(A_-,\Psi_-)}(\frac{Vol(Y,g_Y)}{\gamma_1 q})^{1/q} e^{-\gamma_1(T_1-T_0-1)},$$

from the integration. Note that $\gamma_1 > \delta$, and $e^{-\gamma_1(T_1-T_0-1)} \leq e^{-\delta(T_1-T_0-1)}$. Similarly, $(\tilde{A}_-,\tilde{\Psi}_-)$ does not satisfy the Seiberg-Witten equation (4.7) on $Y \times [T_1 - 1, T_1]$. Thus the estimate follows from Proposition 4.9. The constant $C_{4,(A_-,\Psi_-)}$ is bounded by its maximal value for the compact component $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c_-, c^{'})$. Hence for $T_4 \geq T_0 + 1$ and $T_1 > T_4$, we obtain our estimates with a constant C_7 independent of (A_-, Ψ_-). □

Now we define a neighborhood of $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c_-, c^{'})$. Let

$$U_{\varepsilon_-} = \{(B,\Phi) \in \mathcal{B}_{Y\times\mathbf{R}}(c_-, c^{'}) | \text{there exists a } (A,\Psi) \in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c_-, c^{'}) \text{ such that}$$

$$\|(B,\Phi) - (A,\Psi)\|_{L^q(Y\times\mathbf{R})} < \varepsilon_-, \quad \|F(B,\Phi)\|_{L^p(Y\times\mathbf{R})} < \varepsilon_-\}.$$

By Lemma 6.2 with T_1^- such that $C_7 e^{-\delta(T_1 - T_4)} = \varepsilon_-/2$, we have $(\tilde{A}_-, \tilde{\Psi}_-) \in U_{\varepsilon_-}$ for $T_1 \geq T_1^-$.

Lemma 6.3. *There exists ε_0^- such that for $0 < \varepsilon_- < \varepsilon_0^-$ there is a constant C_8^- independent of ε_- with*

$$\|(a,\phi,\psi)\|_{L_1^p(Y\times\mathbf{R})} \leq C_8^- \|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(B,\Phi))^*(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})},$$

for all $(B,\Phi) \in U_{\varepsilon_-}$.

Proof: From $(B,\Phi) \in U_{\varepsilon_-}$ and the difference of first order operators, we have

$$\|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(B,\Phi))^*(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})} \geq \|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(A_-,\Psi_-))^*(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})}$$

$$-\|((B,\Phi) - (A_-,\Psi_-)) \cdot (a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})}.$$

The expression $((B,\Phi) - (A_-,\Psi_-))$, as zero-th order operator $D^*_{B,\Phi} - D^*_{A_-,\Psi_-}$ acting on (a,ϕ,ψ), can be estimated by Hölder inequality and the Sobolev embedding theorems:

$$\begin{aligned}\|((B,\Phi) - (A_-,\Psi_-)) \cdot (a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})} &\leq \|(B,\Phi) - (A_-,\Psi_-)\|_{L^q(Y\times\mathbf{R})} \|((a,\phi,\psi)\|_{L^4(Y\times\mathbf{R})} \\ &\leq C\varepsilon_- \|((a,\phi,\psi)\|_{L_1^p(Y\times\mathbf{R})}.\end{aligned}$$

By Lemma 6.1 and changing a reference element (∇_0, Ψ_0) to (A_-,Ψ_-) for L_1^p-norm over $Y \times \mathbf{R}$,

$$\begin{aligned}\|(a,\phi,\psi)\|_{L_1^p(Y\times\mathbf{R})} &= \|(a,\phi,\psi)\|_{L_1^p(\nabla_0,\Psi_0)} \\ &\leq C_9 \|(a,\phi,\psi)\|_{L_1^p(A_-,\Psi_-)} \\ &\leq C_{10} \|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(A_-,\Psi_-))^*(a,\phi,\psi)\|_{L^p} + C_9 \|(a,\phi,\psi)\|_{L^p} \\ &\leq (C_{10} + C_9/C_p) \|(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(A_-,\Psi_-))^*(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})}.\end{aligned}$$

Choosing ε_0^- such that $C(C_{10} + C_9/C_p)\varepsilon_0^- \leq 1/2$. Then there is a constant $C_8^- (\geq 2C_p^{-1})$ satisfies the desired inequality. □

By Lemma 6.3 and the Sobolev embedding theorem, we have $L_1^p \hookrightarrow L^q$ for $1/4 + 1/q \geq 1/p$ and the bounded right inverse operator $Q_{(B,\Phi)} = D^*_{B,\Phi} \circ (D_{B,\Phi} \circ D^*_{B,\Phi})^{-1}$ of

$(\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(B,\Phi)) = D_{B,\Phi}$ satisfies

$$\|Q_{(B,\Phi)}(a,\phi,\psi)\|_{L^q(Y\times\mathbf{R})} \le C_{11}\|Q_{(B,\Phi)}(a,\phi,\psi)\|_{L_1^p(Y\times\mathbf{R})} \le C_{11}C_8^-\|(a,\phi,\psi)\|_{L^p(Y\times\mathbf{R})}, \tag{6.1}$$

for all $(B,\Phi) \in U_{\varepsilon_-}$. The importance of (6.1) is that the constant $C_{11}C_8^-$ is independent of $(B,\Phi) \in U_{\varepsilon_-}$. Similarly, the results above hold exactly same for the neighborhood

$$U_{\varepsilon_+} = \{(B,\Phi) \in \mathcal{B}_{Y\times\mathbf{R}}(c', c_+) | \text{there exists a } (A,\Psi) \in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c', c_+) \text{ such that}$$

$$\|(B,\Phi) - (A,\Psi)\|_{L^q(Y\times\mathbf{R})} < \varepsilon_+, \quad \|F(B,\Phi)\|_{L^p(Y\times\mathbf{R})} < \varepsilon_+\},$$

where $c', c_+ \in \mathcal{R}^*(Y,\eta)$. We can define $(\tilde{A}_+, \tilde{\Psi}_+) = (1-\chi_+)c' + \chi_+(A_+,\Psi_+)$ for $(A_+,\Psi_+) \in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}(c', c_+)$, where $\chi_+ = 1$ for $t \ge -T_1 + 1$ and $\chi_+ = 0$ for $t \le -T_1$. There exist $T_1^+ > 0, \varepsilon_0^+ > 0$ and C_8^+ such that for $T_1 > T_1^+$ and $0 < \varepsilon_+ < \varepsilon_0^+$ we have $(\tilde{A}_+, \tilde{\Psi}_+) \in U_{\varepsilon_+}$, and the bounded right inverse operator $Q_{(B,\Phi)}$ is bounded by $C_{11}C_8^+$ as in (6.1) for all $(B,\Phi) \in U_{\varepsilon_+}$.

For the balanced monopole $(A_\pm, \Psi_\pm)$, we can choose $\varepsilon_+ = \varepsilon_-$ and $T_1 \ge \max\{T_1^+, T_1^-\}$ such that the deformed monopole $(\tilde{A}_\pm, \tilde{\Psi}_\pm)$ is an almost solution of Seiberg-Witten equation over $Y \times \mathbf{R}$. For $T_3 > T_2 \ge T_1 \ge \max\{T_1^\pm\}$, the 4-dimensional annulus $Y \times [T_2, T_3]$ will be used as the gluing region in forming the patching transitivity. For gluing $(\tilde{A}_-, \tilde{\Psi}_-)$ on $(Y \times \mathbf{R}, g_-)$ with $(\tilde{A}_+, \tilde{\Psi}_+)$ on $(Y \times \mathbf{R}, g_+)$ and any real positive numbers $T_3 > T_2$, we set $N_- = Y \times [T_2, T_3]$ and $N_+ = Y \times [-T_3, -T_2]$, where $g_\pm = g_Y + dt^2$ and $T_3 = T_2 K$ for another parameter $K > 4$.

Let θ_- be a smooth cutoff function from modifying the function $\chi_K = -\frac{1}{\ln K}\ln\frac{t}{T_2 K}$ on $Y \times [T_2, T_2 K] \subset Y \times \mathbf{R}$ with $\chi_K = 0$ at $t = T_2 K$, $\chi_K = 1$ at $t = T_2$ and $\|\nabla\chi_K\|_{L^4(Y\times\mathbf{R})} = CT_2^{-3/4}\frac{(1-K^{-3})^{1/4}}{\ln K} \to 0$ for $T = T_2 = \max\{T_1^\pm\} + 1$ and $K \to \infty$. Here exists $K_0 > 4$ such that $K \ge K_0$ with $\|\nabla\chi_K\|_{L^4(Y\times\mathbf{R})}$ sufficiently small. We fix such a parameter K.

Then define $f_T : N_- \to N_+$ by $f_T(y,t) = (y,-t)$ which sends the "inner part" $Y \times \{T\} \subset N_-$ to the "outer part" $Y \times \{-T\} \subset N_+$ and reduces an orientation-reversing diffeomorphism from N_- to N_+. Note that f_T is the identity map on the 3-manifold Y. In the usual sense, we define the gluing $Y \times_T \mathbf{R}$ to be

$$Y \times_T \mathbf{R} = Y \times (-\infty, TK] \cup_{f_T} Y \times [-TK, \infty),$$

where the "annuli" $N_\pm$ are identified by f_T with a fixed $K \ge K_0$. The Riemannian metric on $Y \times_T \mathbf{R}$ is again a product metric $g_Y + dt^2$ since the map f_T is isometric and orientation-reversing on the overlap.

Lemma 6.4. *Let $F : E_1 \to E_2$ be a C^1-map between Banach spaces with first order Taylor expansion $F(\xi) = F(0) + DF(0)\xi + N(\xi)$. Assume that $DF(0)$ has a finite dimensional kernel and a right inverse Q such that*

$$\|QN(\xi_1) - QN(\xi_2)\|_{E_1} \le C(\|\xi_1\|_{E_1} + \|\xi_2\|_{E_1})\|\xi_1 - \xi_2\|_{E_1},$$

for some constant C. Let $\varepsilon = 1/(8C)$. If $\|QF(0)\|_{E_1} \leq \varepsilon/3$, then there exists a C^1-function $u : K_\varepsilon \to ImQ$ with $F(\xi + u(\xi)) = 0$ for all $\xi \in K_\varepsilon$, and furthermore we have the estimate

$$\|u(\xi)\|_{E_1} \leq \frac{4}{3}\|QF(0)\|_{E_1} + \frac{1}{3}\|\xi\|_{E_1},$$

where $K_\varepsilon = \ker DF(0) \cap \{\xi \in E_1 : \|\xi\|_{E_1} < \varepsilon\}$.

See [12, 18] for the standard inverse function theorem. Applying Lemma 6.4 to $F(\tilde{A}, \tilde{\Psi})$ (the Seiberg-Witten functional on $Y \times \mathbf{R}$) as $F(0)$, to $\frac{\partial}{\partial t} + \delta_0^* + Df_\eta(\tilde{A}, \tilde{\Psi}) = D_{\tilde{A},\tilde{\Psi}}$ as $DF(0)$, to the quadratic term $N(a, \phi, \psi)$ as the remainder term of the first order Taylor expansion, to $L_1^p \cap L^q(T_{(\tilde{A},\tilde{\Psi})}\mathcal{B}_{Y\times\mathbf{R}})$ as E_1 and $L^p(\Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W))$ as E_2, we have the gluing and splitting result. For $(A_+, \Psi_+) \in \mathcal{M}_{Y\times\mathbf{R}}(c^{'}, c_+)$ and $(A_-, \Psi_-) \in \mathcal{M}_{Y\times\mathbf{R}}(c_-, c^{'})$ with $c_\pm, c^{'} \in \mathcal{R}^*(Y, \eta)$, we define the almost Seiberg-Witten solution $(\tilde{A}, \tilde{\Psi})$ by rescaling and identifying,

$$(\tilde{A}, \tilde{\Psi}) = \begin{cases} (\tilde{A}_-(t + 2TK - T), \tilde{\Psi}_-(t + 2TK - T)) & t \leq -(TK - T) \\ \rho \cdot c^{'} & -(TK - T) \leq t \leq (TK - T) \\ (\tilde{A}_+(t - (2TK - T)), \tilde{\Psi}_+(t - (2TK - T))) & (TK - T) \leq t, \end{cases}$$

where $\rho \in \Gamma_{c'}$ (the isotropic group of $c^{'}$).

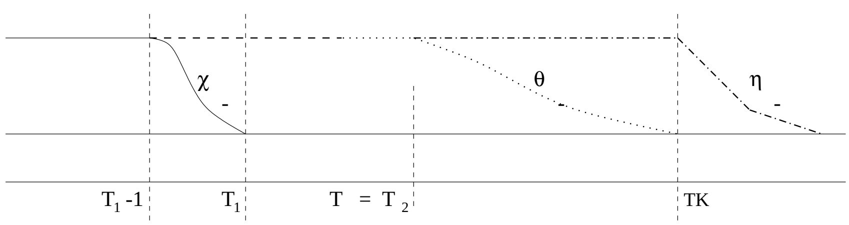

Figure 1. Smooth Cutoff Functions

Proposition 6.5. *For $0 < \varepsilon < min\{\varepsilon_0^\pm\}$ as in Lemma 6.3, T (fixed) in Lemma 6.2, there is a constant C independent of ε such that the operator $D_{\tilde{A},\tilde{\Psi}}$ has a bounded right inverse $Q_{(\tilde{A},\tilde{\Psi})}$ with, $1/4 + 1/q \geq 1/p$,*

$$\|Q_{(\tilde{A},\tilde{\Psi})}(a, \phi, \psi)\|_{L_1^p(Y\times_T\mathbf{R})} \leq C_{13}\|(a, \phi, \psi)\|_{L^p(Y\times_T\mathbf{R})},$$

$$\|Q_{(\tilde{A},\tilde{\Psi})}(a, \phi, \psi)\|_{L^q(Y\times_T\mathbf{R})} \leq C_{13}\|(a, \phi, \psi)\|_{L^p(Y\times_T\mathbf{R})}.$$

Proof: Similarly define a cutoff function θ_+ (as we did for θ_-) on $(Y \times \mathbf{R}, g_+)$, and regarding θ_+ as a function on $Y \times_T \mathbf{R}$, define $\eta_- = 1 - \theta_+ : Y \times_T \mathbf{R} \to \mathbf{R}$ (see [10] for the same construction). Now η_- is simultaneously a function on $Y \times_T \mathbf{R}$ and $(Y \times \mathbf{R}, g_-)$ for $(\tilde{A}_-, \tilde{\Psi}_-)$ such that the support of η_- is larger than θ_-, and $\eta_- = 1$ on the support of θ_-. Similarly we define η_+.

By Lemma 6.3, we have bounded right inverse $Q_{(\tilde{A}_\pm, \tilde{\Psi}_\pm)}$ for $D_{\tilde{A}_\pm, \tilde{\Psi}_\pm}$. Then using the definition of the gluing almost Seiberg-Witten solution $(\tilde{A}, \tilde{\Psi})$ and the standard parameterization method, we define

$$\tilde{Q}(a, \phi, \psi) = \eta_- Q_{(\tilde{A}_-, \tilde{\Psi}_-)}(a, \phi, \psi)_- + \eta_+ Q_{(\tilde{A}_+, \tilde{\Psi}_+)}(a, \phi, \psi)_+,$$

where $(a, \phi, \psi)_-$ is the restriction to $Y \times (-\infty, TK] \subset (Y \times \mathbf{R}, g_-)$ and $(a, \phi, \psi) = (a, \phi, \psi)_- + (a, \phi, \psi)_+$. From the definition of $\eta_\pm$,

$$\eta_-(a, \phi, \psi)_- + \eta_+(a, \phi, \psi)_+ = (a, \phi, \psi).$$

By a simple calculation, one gets

$$D_{\tilde{A}, \tilde{\Psi}} \circ \tilde{Q}(a, \phi, \psi) = (a, \phi, \psi) + d\eta_- Q_{(\tilde{A}_-, \tilde{\Psi}_-)}(a, \phi, \psi)_- + d\eta_+ Q_{(\tilde{A}_+, \tilde{\Psi}_+)}(a, \phi, \psi)_+.$$

Note that we use the right inverse $Q_{(\tilde{A}_\pm, \tilde{\Psi}_\pm)}$ from Lemma 6.3. Thus there is no more term regarding the metric-difference. Now $D_{\tilde{A}, \tilde{\Psi}} \circ \tilde{Q} - Id$ has a C^∞-kernel, and $\|d\eta_\pm\|_{L^4(Y \times \mathbf{R})}$ is sufficiently small. Thus $Q_{(\tilde{A}, \tilde{\Psi})} = \tilde{Q} \circ (D_{\tilde{A}, \tilde{\Psi}} \circ \tilde{Q})^{-1}$ has the desired properties. □

Remark: The method we used in Proposition 6.5 is similar to the one in [9, 10, 18]. For the second order elliptic differential operator, see [25] for the analysis with obstruction bundles. One may also adapt the analytic setup in [27] to work out the estimates.

Theorem 6.6. *If* $0 < \varepsilon < min\{\varepsilon_0^\pm\}$ *and* $T \geq \max\{T_1^\pm\} + 1$, *then there is a well-defined gluing map*

$$G_T : \mathcal{M}^{bal}_{Y \times \mathbf{R}}(c_-, c') \times \mathcal{M}^{bal}_{Y \times \mathbf{R}}(c', c_+) \times [TK, \infty) \to \mathcal{M}_{(Y \times_T \mathbf{R})}(c_-, c_+),$$

which is a local diffeomorphism with a fixed $K \geq K_0$.

Proof: From the above construction, we have $(\tilde{A}_\pm, \tilde{\Psi}_\pm) \in U_{\varepsilon_\pm}$ for $T > \max\{T_1^+, T_1^-\}$ with $\varepsilon_+ = \varepsilon_- < \varepsilon_0^\pm$. By Proposition 6.5, the bounded right inverse operator $Q_{(\tilde{A}, \tilde{\Psi})}$ satisfies

$$\|Q_{(\tilde{A}, \tilde{\Psi})}(a, \phi, \psi)\|_{L^q(Y \times_T \mathbf{R})} \leq C_{13} \|(a, \phi, \psi)\|_{L^p(Y \times_T \mathbf{R})}.$$

Thus we have

$$\|Q_{(\tilde{A}, \tilde{\Psi})} F(\tilde{A}, \tilde{\Psi})\|_{L^q(Y \times_T \mathbf{R})} \leq C_{13}(\varepsilon_+ + \varepsilon_-).$$

By the Hölder inequality and quadratic expression, we get

$$\|Q_{(\tilde{A}, \tilde{\Psi})} N(a, \phi, \psi) - Q_{(\tilde{A}, \tilde{\Psi})} N(a', \phi', \psi')\|_{L^q(Y \times_T \mathbf{R})}$$

$$\leq C_{14}(\|(a, \phi, \psi)\|_{L^q(Y \times_T \mathbf{R})} + \|(a', \phi', \psi')\|_{L^q(Y \times_T \mathbf{R})}) \cdot \|(a, \phi, \psi) - (a', \phi', \psi')\|_{L^q(Y \times_T \mathbf{R})}.$$

By Lemma 6.4, there exists a C^1-map $u : \ker D_{\tilde{A},\tilde{\Psi}} \to \mathcal{M}_{Y\times_T\mathbf{R}}(c_-, c_+)$ such that $(\tilde{A},\tilde{\Psi})+u((\tilde{A},\tilde{\Psi});\rho)$ is a solution of (4.7) with $\|u((\tilde{A},\tilde{\Psi});\rho)\|_{L^q(Y\times_T\mathbf{R})}$ small and $\rho = id \in U(1)$, and is smooth by standard elliptic regularity. Thus there is a well-defined C^∞-map

$$G_T((A_-,\Psi_-), id, (A_+,\Psi_+)) = (\tilde{A},\tilde{\Psi}) + u((\tilde{A},\tilde{\Psi}); id).$$

For any $(A,\Psi) \in U_\varepsilon(\mathrm{Im}\,(G_T))$, there is a representative $(A,\Psi) = (\tilde{A},\tilde{\Psi}) + u((\tilde{A},\tilde{\Psi}); id)$. Suppose the contrary. There exists a sequence $\varepsilon_n \to 0$ ($T_n \to +\infty$) with $\varepsilon_n < \varepsilon_\pm$ and $(A_n,\Psi_n) \in U^c_{\varepsilon_n} \cap \mathcal{M}_{Y\times_{T_n}\mathbf{R}}(c_-, c_+)$ cannot be written as the image of G_{T_n}, where $U^c_{\varepsilon_n}$ is the complement of the neighborhood $U_\varepsilon(\mathrm{Im}\,(G_{T_n}))$. By the compactness result from [7, 15, 30, 38], we have a subsequence converging to $(A_-,\Psi_-)\coprod(A_+,\Psi_+)$. Note that $(A_\pm,\Psi_\pm)$ are Seiberg-Witten solutions of (4.7) (since $\varepsilon_n \to 0$) which has a singular point at $(y_0, 0)$ conformally corresponding to the infinite point. By the removablity result in [31], we have $(A_\pm,\Psi_\pm)$ are solutions of (4.7) over $(Y\times\mathbf{R}, g_\pm)$. Note that $\lim_{t\to\pm\infty}(A_\mp,\Psi_\mp) = c^{'}$ since the elements are in the neighborhood of the image G_{T_n} which are close to $c^{'}$ from the construction. So we get $(A_-,\Psi_-) \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_-, c^{'})$ and $(A_+,\Psi_+) \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c^{'}, c_+)$ with possible shifting in the t-direction. Hence we can perform the gluing process to obtain a $G_{T_n}((A_-,\Psi_-),\rho,(A_+,\Psi_+))$ such that

$$\|(A_n,\Psi_n) - G_{T_n}((A_-,\Psi_-),\rho,(A_+,\Psi_+))\|_{L^q(Y\times_{T_n}\mathbf{R})} < \varepsilon/2,$$

and $(A_n,\Psi_n) \in U_{\varepsilon/2} \cap \mathcal{M}_{Y\times_{T_n}\mathbf{R}}(c_-, c_+)$. This shows that (A_n,Ψ_n) cannot lie in the complement of U_{ε_n}. The contradiction shows that the gluing map G_T is a local diffeomorphism. □

Remark: The monopole boundary operator requires the consideration of moduli spaces with $\dim \mathcal{M}_{Y\times\mathbf{R}}(c_-, c_+) \le 2$. If $c^{'} = \Theta$ and $\dim \mathcal{M}_{Y\times\mathbf{R}}(c_-, c_+) = 2$, then

$$\dim \mathcal{M}_{Y\times\mathbf{R}}(c_-, \Theta) + \dim \Gamma_\Theta + \dim \mathcal{M}_{Y\times\mathbf{R}}(\Theta, c_+) = 2.$$

Hence there will be no boundary stratum $\hat{\mathcal{M}}_{Y\times\mathbf{R}}(c_-,\Theta)\times\hat{\mathcal{M}}_{Y\times\mathbf{R}}(\Theta, c_+)$ of $\hat{\mathcal{M}}_{Y\times\mathbf{R}}(c_-, c_+)$ with $\dim \mathcal{M}_{Y\times\mathbf{R}}(c_-, c_+) = 2$ in a generic sense (see [6, 22]). For $c_\pm \in \mathcal{R}^*_{\mathrm{SW}}(Y,\eta)$ and $c^{'} = \Theta$, the general gluing result is not needed for our definition of the monopole homology, but is needed for the equivariant version of the monopole homology. See [22] §2.4 and §5.2 and [36].

Now our transitivity can be expressed as Theorem 6.6:

$$G_T : \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_-, c^{'}) \times \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c^{'}, c_+) \times [TK, \infty) \to \mathcal{M}^{\mathrm{bal}}_{Y\times_T\mathbf{R}}(c_-, c_+) \cong \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_-, c_+),$$

where the gluing parameter is T with $T \ge \max\{T_1^\pm\}$ and a fixed $K \ge K_0$.

Definition 6.7. Let (a,ϕ,ψ) and $(a^{'},\phi^{'},\psi^{'})$ be zeros of f_η. A *chain solution* from (a,ϕ,ψ) to $(a^{'},\phi^{'},\psi^{'})$ is $((A_1,\Phi_1),...,(A_n,\Phi_n))$ Seiberg-Witten solutions over $Y\times\mathbf{R}$ which converge to $c_{i-1}, c_i \in f_\eta^{-1}(0)$ as $t \to \mp\infty$ such that $(a,\phi,\psi) = c_0$, $c_n = (a^{'},\phi^{'},\psi^{'})$, and $(A_i,\Phi_i) \in \mathcal{M}_{Y\times\mathbf{R}}(c_{i-1}, c_i)$ for $0 \le i \le n$.

We say that the sequence $\{(A_\alpha,\Phi_\alpha)\} \in \mathcal{M}_{Y\times\mathbf{R}}((a,\phi,\psi),(a^{'},\phi^{'},\psi^{'}))$ is *(weakly) convergent* to the chain solution $((A_1,\Phi_1),...,(A_n,\Phi_n))$ if there is a sequence of n-tuples of real numbers $\{t_{\alpha,1} \le \cdots \le t_{\alpha,n}\}_\alpha$, such that $t_{\alpha,i} - t_{\alpha,i-1} \to \infty$ as $\alpha \to \infty$, and if, for each i, the translates $t^*_{\alpha,i}(A_\alpha, \Phi_\alpha) = (A_\alpha(\circ - t_{\alpha,i}), \Phi_\alpha(\circ - t_{\alpha,i}))$ converge weakly to (A_i, Φ_i).

Theorem 6.8. *Let $\{(A_\alpha, \Phi_\alpha)\} \in \mathcal{M}_{Y\times\mathbf{R}}((a,\phi,\psi),(a^{'},\phi^{'},\psi^{'}))$ be a sequence of Seiberg-Witten solutions with uniformly bounded action over $Y \times \mathbf{R}$. Then there exists a subsequence converging to a chain solution $((A_1,\Phi_1),...,(A_n,\Phi_n))$ such that*

$$Ind\, D_{A_\alpha,\Phi_\alpha} = \sum_{i=1}^n Ind\, D_{A_i,\Phi_i} = \sum_{i=1}^n (\mu_\eta(c_i) - \mu_\eta(c_{i-1})).$$

Proof: The gluing and splitting theorem shows that the local diffeomorphism G preserves the energy E_η and the spectral flow. So it follows from the same proof as in [12] §3 and [15], and the compactness of Seiberg-Witten moduli space on 4-dimensional manifolds. □

Proposition 6.9. *The compactification of $\mathcal{M}_{Y\times\mathbf{R}}(c_0, c_{n+1})$ can be described as*

$$\overline{\mathcal{M}_{Y\times\mathbf{R}}(c_0, c_{n+1})} = \cup(\times_{i=1}^{n+1}\mathcal{M}_{Y\times\mathbf{R}}(c_{i-1}, c_i)),$$

the union over all sequence $c_0, c_1, \cdots, c_{n+1} \in \mathcal{R}^*_{SW}(Y,\eta)$ such that $\mathcal{M}_{Y\times\mathbf{R}}(c_{i-1},c_i)$ is nonempty for all $1 \le i \le n+1$.

For any sequence $c_0, c_1, \cdots, c_{n+1} \in \mathcal{R}^*_{SW}(Y,\eta)$, there is a gluing map

$$G : \times_{i=1}^{n+1}\mathcal{M}^{bal}_{Y\times\mathbf{R}}(c_{i-1}, c_i) \times \Delta^{n+1} \to \overline{\mathcal{M}_{Y\times\mathbf{R}}(c_0, c_{n+1})},$$

where $\Delta^{n+1} = \{(\lambda_0, \cdots, \lambda_n) \in [-\infty, \infty]^{n+1} : 1 + \lambda_{i-1} < \lambda_i, 1 \le i \le n\}$.

1. *The image of G is a neighborhood of $\times_{i=1}^{n+1}\mathcal{M}^{bal}_{Y\times\mathbf{R}}(c_{i-1}, c_i)$ in the compactification with chain solutions.*
2. *The restriction of G to $\times_{i=1}^{n+1}\mathcal{M}^{bal}_{Y\times\mathbf{R}}(c_{i-1}, c_i) \times Int(\Delta^{n+1})$ is an orientation-preserving diffeomorphism onto its image.*

Proof: Since there is no bubbling in the Seiberg-Witten moduli space, the map G is the well-known transitivity in the Morse-Smale theory by repeatedly applying Theorem 6.6 (see also Proposition 3.10 of [19]). □

Let $\mathcal{R}^n_{SW}(Y,\eta)$ be the set of irreducible zeros (a,ϕ,ψ) of f_η whose relative index $\mu_\eta(a,\phi,\psi) - \mu_\eta(\Theta) = n$. The **monopole chain group** $MC_n(Y,\eta)$ is defined to be the free Abelian group generated by $\mathcal{R}^n_{SW}(Y,\eta)$, where the admissible perturbation η specifies the spectral flow $\mu_\eta(\Theta)$. We write $I_\eta(\Theta;\eta_0)$ to be the integer $\mu_\eta(\Theta) - \mu_{\eta_0}(\Theta)$ with respect to a reference $\eta_0 \in \mathcal{P}_Y$. Hence $\mu_\eta(\Theta)$ is fixed with the fixation of $I_\eta(\Theta;\eta_0)$. From results in §5, the algebraic number $\#\mathcal{R}_{SW}(Y,\eta)$ is an invariant for $\eta \in \mathcal{P}^{'}_Y$ in the fixed homotopy class. Therefore we can use $\mathcal{R}_{SW}(Y,\eta)$ to form a chain group for $\eta \in \mathcal{P}^{'}_Y$ with the fixed number $I_\eta(\Theta;\eta_0)$.

Define the boundary operator $\partial : MC_n(Y,\eta) \to MC_{n-1}(Y,\eta)$:

$$\partial(a,\phi,\psi) = \sum_{(a^{'},\phi^{'},\psi^{'})\in MC_{n-1}(Y,\eta)} \#\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}((a,\phi,\psi),(a^{'},\phi^{'},\psi^{'}))\cdot(a^{'},\phi^{'},\psi^{'}).$$

Proposition 6.10. *Let $\partial : MC_n(Y,\eta) \to MC_{n-1}(Y,\eta)$ be defined as above. Then $\partial \circ \partial = 0$.*

Proof: The proof follows the same argument as in ([12], Theorem 2) except that we have to rule out the possibility of reducible connections entering into the picture by Theorem 6.6. Note that

$$\partial^2(c_0) = \sum_{c_1 \in \mathcal{R}^{n-1}_{SW}(Y,\eta)} \sum_{c_2 \in \mathcal{R}^{n-2}_{SW}(Y,\eta)} \#\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_0,c_1)\cdot \#\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_1,c_2)c_2,$$

where $c_i = (a_i, \phi_i, \psi_i) \in \mathcal{R}^*_{SW}(Y,\eta)(i = 0,1,2)$. Consider in this sum all the terms associated to a fixed $c_2 \in \mathcal{R}^{n-2}_{SW}(Y,\eta)$. For the pair (c_0, c_2), there is the 2-dimensional moduli space $\mathcal{M}^2_{Y\times\mathbf{R}}(c_0,c_2)$. By Proposition 6.9, the ends of $\mathcal{M}^{\mathrm{bal},2}_{Y\times\mathbf{R}}(c_0,c_2)$ consists of all the components $\mathcal{M}^{\mathrm{bal},1}_{Y\times\mathbf{R}}(c_0,c_1)\times\mathcal{M}^{\mathrm{bal},1}_{Y\times\mathbf{R}}(c_1,c_2)$ with $c_1 \in \mathcal{R}^{n-1}_{SW}(Y,\eta)$. It is impossible for c_1 to be the $U(1)$-reducible zero of f_η because the isotropy subgroup Γ_{c_1} would add to the extra gluing parameter, and as a result would contradict the dimension count by Proposition 5.1 and Proposition 5.2 with fixed $I_\eta(\Theta;\eta_0)$. For the fixed datum $I_\eta(\Theta;\eta_0)$, the orientations are consistent from the spectral flow calculations by Lemma 5.3. Thus

$$\sum_{c_1 \in \mathcal{R}^{n-1}_{SW}(Y,\eta)} \#\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_0,c_1)\cdot \#\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(c_1,c_2) = \partial\mathcal{M}^{\mathrm{bal},2}_{Y\times\mathbf{R}}(c_0,c_2) = 0.$$

□

As a consequence of Proposition 6.10, for a given integral homology 3-sphere Y and an admissible data $\eta \in \mathcal{P}'_Y$ with the fixed datum $I_\eta(\Theta;\eta_0)$, we have a well-defined definition of a **Monopole Homology**

$$MH_*(Y;\eta) = \ker\partial_*/\mathrm{Im}\partial_{*+1}, \qquad * \in \mathbf{Z}.$$

Now the monopole homology $MH_*(Y;\eta)$ is sensitive to the number $I_\eta(\Theta;\eta_0)$, and $MH_*(Y;\eta)$ is not a topological invariant since its Euler characteristic $\#\mathcal{R}^*_{SW}(Y,\eta)$ is metric-dependent.

7. Homomorphisms induced by cobordisms

From the troublesome path of metrics in Σ_Y of creating/destroying harmonic spinors (see [14]), the invariance of the monopole homology of integral homology 3-spheres is in question. The cobordism argument used in [12] does not apply here. We have to construct a different cobordism between admissible perturbations with the fixed spectral flow $I_\eta(\Theta;\eta_0) = \mu_\eta(\Theta) - \mu_{\eta_0}(\Theta)$. In this section, we show that our monopole homology is independent of admissible perturbations within the homotopy class $I_\eta(\Theta;\eta_0)$.

Let X be an oriented 4-manifold with two cylindrical ends $Y_1 \times \mathbf{R}_+$ and $Y_2 \times \mathbf{R}_-$, where Y_1 and Y_2 are integral homology 3-spheres. Let $\tau_X : X \to [0,\infty)$ be a smooth cutoff function such that $\tau_X(x) = 0$ for x lying outside of $Y_1 \times \mathbf{R}_+ \cup Y_2 \times \mathbf{R}_-$ and $\tau_X(y,t) = |t|$ for $(y,t) \in Y_1 \times \mathbf{R}_+ \cup Y_2 \times \mathbf{R}_-$ and $|t| > t_0 > 0$ and $e_\delta = e^{\delta\tau_X(x)}$. Then using the cutoff function τ_X and a background connection we can extend $\frac{d}{dt}+\alpha$ and $\frac{d}{dt}+\beta$ to a connection ∇_0 on X such that

$$\nabla_0|_{Y_1\times[T_5,\infty)} = \frac{d}{dt}+\alpha, \quad \nabla_0|_{Y_2\times(-\infty,-T_5]} = \frac{d}{dt}+\beta.$$

Similarly, we can extend sections on $W_X^{\pm}$. The Fréchet space $\Omega^1_{\text{comp}}(X, AdP)\oplus\Gamma_{\text{comp}}(W_X^{\pm})$ of compact supported C^∞-sections on $(T^*X\otimes AdP)\oplus\Gamma(W_X^{\pm})$ can be completed to a Banach space

$$\mathcal{A}^p_{k,\delta}(X) = (\nabla_0, 0) + L^p_{k,\delta}(\Omega^1(X, AdP)\oplus\Gamma(W_X^{\pm})),$$

where $\|c\|_{L^p_{k,\delta}} = \|e_\delta\cdot c\|_{L^p_k}$ for $c\in\Omega^1_{\text{comp}}(X, AdP)\oplus\Gamma_{\text{comp}}(W_X^{\pm})$. The gauge group $\mathcal{G}^p_{k+1,\delta}$ is given by $L^p_{k+1,\delta}$-norm of $\text{Aut}(\det W_X^{\pm})$. So the quotient space is $\mathcal{B}^p_{k,\delta}(X) = \mathcal{A}^p_{k,\delta}(X)/\mathcal{G}^p_{k+1,\delta}$. The perturbation data $\eta_1 = (g_{Y_1}, \alpha_1)$ and $\eta_2 = (g_{Y_2}, \alpha_2)$ at the ends provide the gradient vector fields f_{η_1} and f_{η_2} so that the zeros of f_{η_1} on Y_1 and of f_{η_2} on Y_2 are generic. Clearly these perturbation data η_1 and η_2 can be pulled back to the cylindrical ends $Y_1\times\mathbf{R}_+$ and $Y_2\times\mathbf{R}_-$, and produce perturbations on the time-invariant monopole equation on $\mathcal{B}^p_{k,\delta}(Y_1\times\mathbf{R}_+)$ and $\mathcal{B}^p_{k,\delta}(Y_2\times\mathbf{R}_-)$ (same δ as before). According to ([12] (1c.2) and [15, 32, 38]), there exists a Baire's first category subset in the space $\mathcal{M}et(X)\times\Pi_X$ of Riemannian metrics g_X and perturbation data α_X such that $\mathcal{M}_{\eta_X}(c, c^{'})$ ($\eta_X = (g_X, \alpha_X)$) is a smooth manifold with

$$\dim\mathcal{M}_{\eta_X}(c, c^{'}) = \mu_{\eta_1}(c) - \mu_{\eta_2}(c^{'}) + \frac{1}{2}(2\chi+3\sigma)(X). \tag{7.1}$$

In addition, $\mathcal{M}_{\eta_X}(c, c^{'})$ is oriented with an orientation specified by the orientations on $H^1(X,\mathbf{R})$ and $H^0(X,\mathbf{R})\oplus H^2_+(X,\mathbf{R})$ (see [7, 15, 32, 38]).

Define a homomorphism $\Psi_* = \Psi_*(X;\eta_X) : MC_*(Y_1;\eta_1)\to MC_*(Y_2;\eta_2)$ of the monopole chain complexes by the formula

$$\Psi_*(c) = \sum_{c^{'}\in\mathcal{R}^*_{SW}(Y_2,\eta_2)} \#\mathcal{M}^0_{\eta_X}(c, c^{'})\cdot c^{'}, \qquad c\in\mathcal{R}^*_{SW}(Y_1,\eta_1),$$

where $\mathcal{M}^0_{\eta_X}(c, c^{'})$ is the 0-dimensional oriented moduli space connecting c to $c^{'}$ on X and $\mu_{\eta_1}(c) - \mu_{\eta_2}(c^{'}) = -\frac{1}{2}(2\chi+3\sigma)(X)$.

Proposition 7.1. *Given a cobordism X and perturbation data $\eta_X\in\mathcal{M}et(X)\times\Pi_X$ as before, the homomorphism Ψ_* is a chain map shifting the degree by $\frac{1}{2}(2\chi+3\sigma)(X)$. Furthermore the induced homomorphism*

$$\Psi_* = \Psi_*(X;\eta_X) : MH_*(Y_1;\eta_1)\to MH_*(Y_2;\eta_2)$$

on the monopole homologies depends only on the cobordism X and the data $I_{\eta_1}(\Theta_{Y_1};\eta_{0,Y_1})$ and $I_{\eta_2}(\Theta_{Y_2};\eta_{0,Y_2})$.

Proof: By fixing the spectral flows $I_{\eta_1}(\Theta_{Y_1};\eta_{0,Y_1})$ and $I_{\eta_2}(\Theta_{Y_2};\eta_{0,Y_2})$ at Y_1 and Y_2, the result follows the same argument as Theorem 3 in [12] and §5 of [16]. Note that $\partial_{Y_2}\circ\Psi_*(c)$ is given by

$$\sum_{c^{'}\in\mathcal{R}^*_{SW}(Y_2,\eta_2)}\#\mathcal{M}^0_{\eta_X}(c,c^{'})\cdot\sum_{c_1\in\mathcal{R}^*_{SW}(Y_2,\eta_2)}\#\mathcal{M}^{\text{bal}}_{Y_2\times\mathbf{R}}(c^{'},c_1)\cdot c_1,$$

which is the one end of the 1-dimensional space $\mathcal{M}^1_{\eta_X}(c,c_1)$. The other end is given by

$$\sum_{d^{'}\in\mathcal{R}^*_{SW}(Y_1,\eta_1)}\#\mathcal{M}^{\text{bal}}_{Y_1\times\mathbf{R}}(c,d)\cdot\sum_{d\in\mathcal{R}^*_{SW}(Y_1,\eta_1)}\#\mathcal{M}^0_{\eta_X}(d,c_1).$$

The $c^{'}$ and d cannot be Θ_{Y_2} and Θ_{Y_1} respectively due to the index reason with our gluing result in Theorem 6.6. Hence for the fixed data $I_{\eta_1}(\Theta_{Y_1};\eta_{0,Y_1})$ and $I_{\eta_2}(\Theta_{Y_2};\eta_{0,Y_2})$, there are compatible orientations given by the index (or spectral flow) which shows that $\partial_{Y_2}\circ\Psi_*(c)=\Psi_*\circ\partial_{Y_1}$. Therefore we obtain the induced homomorphism on the monopole homologies. □

We show below that $\Psi_*(X;\eta_X)$ is functorial with respect to the composite cobordism. Given two cobordisms $(U;\eta_U)$ connecting Y_1 to Y_2 and $(V;\eta_V)$ connecting Y_2 to Y_3 so that η_U and η_V agree on Y_2, we can form the composite cobordism $(W;\eta_W)$ connecting Y_1 to Y_3. Then

$$\Psi_*(W;\eta_W)=\Psi_*(V;\eta_V)\circ\Psi_*(U;\eta_U). \tag{7.2}$$

A different strategy from Floer's has to be taken to prove that $MH_*(Y,\eta)$ is independent of admissible perturbations $\eta=(g_Y,\alpha)$ within the class of $I_\eta(\Theta;\eta_0)$. We consider the time-dependent perturbations of the Seiberg-Witten equation and its associated moduli space. Given two admissible perturbation data of generic metrics g_Y^{-1} and g_Y^1 and 1-forms α_{-1} and α_1 with $I_{\eta_{-1}}(\Theta;\eta_0)=I_{\eta_1}(\Theta;\eta_0)$ (here $\eta_t=(g_Y^t,\alpha_t)$), there is an one-parameter family of admissible perturbations $\Lambda=\{\eta_t=(g_Y^t,\alpha_t)|-\infty\leq t\leq\infty\}$ joining them. Assume that the pair $\eta_t=(g_Y^{-1},\alpha_{-1})$ for $t\leq -1$ and $\eta_t=(g_Y^1,\alpha_1)$ for $t\geq 1$. On the cylinder $Y\times\mathbf{R}$, we consider the perturbed Seiberg-Witten equation

$$\frac{\partial\psi}{\partial t}+\partial_{a_t}^{\nabla_{g_Y^t}+\alpha_t}\psi=0,\quad \frac{\partial a_t}{\partial t}+*_{g_Y^t}F(a_t)+\alpha_t=i\tau_{g_Y^t}(\psi,\psi). \tag{7.3}$$

Given $c\in\mathcal{R}^*_{SW}(Y,\eta_{-1})$ and $c^{'}\in\mathcal{R}^*_{SW}(Y,\eta_1)$, we denote by $\mathcal{M}_\Lambda(c,c^{'})$ the subspace in $\mathcal{B}^p_{k,\delta}(c,c^{'})$ consisting of solutions of (7.3). Then there exists a homomorphism

$$\Psi_\Lambda: MC_n(Y;\eta_{-1})\to MC_n(Y;\eta_1)$$

of the monopole chain complexes defined by

$$\Psi_\Lambda(c)=\sum_{c^{'}\in\mathcal{R}^n_{SW}(Y,\eta_1)}\#\mathcal{M}^0_\Lambda(c,c^{'})\cdot c^{'},\qquad c\in\mathcal{R}^n_{SW}(Y,\eta_{-1}).$$

Proposition 7.2. *Let $\Lambda=\{\eta_t=(g_Y^t,\alpha_t)|t\in\mathbf{R}\}$ be an family of admissible perturbations as defined above such that $Ind DF_{\eta_t}(\Theta)=0$ (the same homotopy class). Then*

1. *If Λ is a constant family of admissible perturbations ($g_Y^t = g_Y, \alpha_t = \alpha$), then $\Psi_\Lambda = id$.*
2. *Ψ_Λ is a chain map: $\partial \Psi_\Lambda = \Psi_\Lambda \partial$.*
3. *Given two families Λ and $\Lambda^{'}$ of admissible perturbations joining (g_Y^{-1}, α_{-1}) to (g_Y^0, α_0) and from (g_Y^0, α_0) to (g_Y^1, α_1), we have $\Psi_{\Lambda \circ \Lambda'} = \Psi_\Lambda \circ \Psi_{\Lambda'}$.*
4. *If a family Λ_0 of admissible perturbations connecting (g_Y^{-1}, α_{-1}) and (g_Y^1, α_1) can be deformed into another Λ_1 by admissible families $\Lambda_\lambda (0 \leq \lambda \leq 1)$, then the two monopole chain maps Ψ_{Λ_0} and Ψ_{Λ_1} are chain homotopic to each other.*

Proof: (1) If the perturbation is time independent $\eta_t = (g_Y, \alpha)$, then $\mathcal{M}^0_\Lambda(c, c^{'})$ is just the space $\mathcal{M}^0_{Y \times \mathbf{R}}(c, c^{'})$. For the 0-dimensional component $\mathcal{M}^0_\Lambda(c, c^{'})$, this means time-invariant solutions c_t on $Y \times \mathbf{R}$, and we have $[c_t] = c = c^{'}$. Therefore $\# \mathcal{M}^0_\Lambda(c, c^{'}) = \delta_{cc'}$ and $\Psi_\Lambda = id$.

(2) We consider the compactification of $\mathcal{M}_\Lambda(c, c^{'})$ as developed in [13, 16]. By Theorem 6.6 and Proposition 6.9 and [15, 32, 38], $\mathcal{M}_\Lambda(\alpha, \beta)$ can be compactified such that the codimension-one boundary consists of

$$\cup_{c_{-1}} \mathcal{M}^{\mathrm{bal}}_{Y \times \mathbf{R}}(c, c_{-1}) \times_{c_{-1}} \mathcal{M}_\Lambda(c_{-1}, c^{'}) \coprod \cup_{c_1} \mathcal{M}_\Lambda(c, c_1) \times_{c_1} \mathcal{M}^{\mathrm{bal}}_{Y \times \mathbf{R}}(c_1, c^{'}). \tag{7.4}$$

Here $c_{\pm 1} \in \mathcal{R}_{SW}(Y, \eta_{\pm 1})$ and $\mathcal{M}_{Y \times \mathbf{R}}(c, c_{-1})$ is the moduli space of monopoles on $Y \times (-\infty, -1)$ with respect to the perturbation η_{-1}. Similarly $\mathcal{M}^{\mathrm{bal}}_{Y \times \mathbf{R}}(c_1, c^{'})$ is obtained from the perturbation data η_1. Consider the 1-dimensional components $\mathcal{M}^1_\Lambda(c, c^{'})$ of $\mathcal{M}_\Lambda(c, c^{'})$, whose boundary by (7.4) gives two types of oriented points counted as $\partial \Psi_\Lambda = \Psi_\Lambda \partial$. We can rule out the possibilities of the reducible Θ for $c_{\pm 1}$. If they occurred, then they would have an additional $U(1)$-symmetry on these moduli spaces. This is impossible by the dimension reasoning from Proposition 5.1, Proposition 5.2 and our hypothesis $I_{\eta_{-1}}(\Theta; \eta_0) = I_{\eta_1}(\Theta; \eta_0)$ (see below also).

(3) For a composite cobordism and its induced homomorphism, we study the moduli space $\mathcal{M}_{\Lambda * \Lambda'}(T_6; \alpha, \beta)$ of solutions of the Seiberg-Witten equation on $Y \times \mathbf{R}$ with respect to the following time-dependent admissible perturbation data $\Lambda *_{T_6} \Lambda^{'}$, where

$$\Lambda *_{T_6} \Lambda^{'} = \begin{cases} \eta_{-1} = (g_Y^{-1}, \alpha_{-1}) & -\infty < t \leq -T_6 - 1 \\ \Lambda = (g_Y^{t+T_6}, \alpha_{t+T_6}) & -T_6 - 1 \leq t \leq -T_6 \\ \eta_0 & -T_6 \leq t \leq T_6 \\ \Lambda^{'} = (g_Y^{t-T_6}, \alpha_{t-T_6}) & T_6 \leq t \leq T_6 + 1 \\ \eta_1 & T_6 + 1 \leq t < +\infty. \end{cases}$$

Let T_6 be sufficiently large. Thus $\mathcal{M}_{\Lambda * \Lambda'}(T_6; c, c^{'}) (T_6 \geq T_7)$ is approximated by the union

$$\cup_{c_0} \overline{\mathcal{M}}_\Lambda(c, c_0) \times_{c_0} \overline{\mathcal{M}}_{\Lambda'}(c_0, c^{'}). \tag{7.5}$$

where $\overline{\mathcal{M}}_\Lambda(c, c_0) = \mathcal{M}_\Lambda(c, c_0)/(\Gamma_c \times \Gamma_{c_0})$. Note that the 0-dimensional components in $\overline{\mathcal{M}}_\Lambda(c, c_0) \times_{c_0} \overline{\mathcal{M}}_{\Lambda'}(c_0, c')$ correspond to the c'-coefficients in

$$\Psi_{\Lambda'} \circ \Psi_\Lambda(c) = \sum_{c_0} \#\overline{\mathcal{M}}^0_\Lambda(c, c_0) \cdot \#\overline{\mathcal{M}}^0_{\Lambda'}(c_0, c') \cdot c'.$$

On the other hand, as $T_6 \to 0$, the 0-dimensional component of the moduli space $\mathcal{M}_{\Lambda * \Lambda'}(T_6; c, c')$ gives the c'-coefficients in $\Psi_{\Lambda*\Lambda'}(c) = \sum \mathcal{M}^0_{\Lambda*\Lambda'}(c, c') \cdot c'$. Because $\cup_{0 \le T_6 \le T_7} \mathcal{M}^0_{\Lambda*\Lambda'}(T_6; c, c')$ is the cobordism between $\mathcal{M}^0_{\Lambda*\Lambda'}(0; c, c')$ and $\mathcal{M}^0_{\Lambda*\Lambda'}(T_7; c, c')$, so the assertion (3) follows by ruling out the reducible Θ. Note that

$$\dim\overline{\mathcal{M}}_\Lambda(c, c_0) = \mu_{\eta_{-1}}(c) - \lim_{\eta_t \in \Lambda, \eta_t \to \eta_0} \mu_{\eta_t}(c_0) - \dim\Gamma_{c_0};$$

$$\dim\overline{\mathcal{M}}_{\Lambda'}(c_0, c') = \lim_{\eta_t \in \Lambda', \eta_t \to \eta_0} \mu_{\eta_t}(c_0) - \mu_{\eta_1}(c'). \tag{7.6}$$

By Proposition 5.1 and Proposition 5.2, we obtain

$$\lim_{\eta_t \in \Lambda, \eta_t \to \eta_0} \mu_{\eta_t}(c_0) = \lim_{\eta_t \in \Lambda', \eta_t \to \eta_0} \mu_{\eta_t}(c_0) = \mu(c_0).$$

So it satisfies the equations $\mu_{\eta_{-1}}(c) - \mu(c_0) = 1$ ($c_0 = \Theta$) and $\mu(c_0) - \mu_{\eta_1}(c') = 0$. This is impossible because of $\mu_{\eta_{-1}}(c) = \mu_{\eta_1}(c')$. *If these spectral flows $I_{\eta_{\pm 1}}(\Theta; \eta_0)$ are not fixed to be same, then the above argument becomes invalid.*

(4) Let $\Lambda_i (i = 0, 1)$ be a family of time-independent admissible perturbations which connect up η_{-1} and η_1. Suppose that Λ_0 and Λ_1 can be smoothly deformed from one to another by a 1-parameter family $\Lambda_s = \{\eta^s_t = (g^{s,t}_Y, \alpha^s_t), 0 \le s \le 1,\ -1 \le t \le 1\}$ of the same type of admissible perturbations. Set $\Lambda_s = \Lambda_0$ for $0 \le s \le \frac{1}{4}$ and $\Lambda_s = \Lambda_1$ for $\frac{3}{4} \le s \le 1$. Associated to this situation, there is a 1-parameter family of moduli spaces denoted by $\mathcal{H}\tilde{\mathcal{M}}(c, c') = \cup_{0 \le s \le 1} \tilde{\mathcal{M}}_{\Lambda_s}(c, c')$,

$$\mathcal{H}\tilde{\mathcal{M}}(c, c') = \{(\Phi, s) | \Phi \in \tilde{\mathcal{M}}_{\Lambda_s}(c, c'), 0 \le s \le 1\} \subset \mathcal{B}^p_{k,\delta}(c, c') \times [0, 1],$$

where $\mathcal{H}\tilde{\mathcal{M}}$ is the set of regular solutions of Seiberg-Witten equation with respect to η^s_t, and is a smooth manifold with dimension $\mu_{\eta_{-1}}(c) - \mu_{\eta_1}(c') + 1$. The codimension-one boundary consists of

$$\mathcal{M}_{\Lambda_1}(c, c') \times \{0\} \coprod \mathcal{M}_{\Lambda_0}(c, c') \times \{1\},$$

$$\cup_{(s, c_0)} \tilde{\mathcal{M}}_{\Lambda_s}(c, c_0) \times \mathcal{M}_{\eta_1}(c_0, c') \coprod \cup_{(s, \gamma)} \mathcal{M}_{\eta_{-1}}(c, c_0) \times \tilde{\mathcal{M}}_{\Lambda_s}(c_0, c').$$

Since $\tilde{\mathcal{M}}_{\Lambda_s}(c, c_0)$ and $\tilde{\mathcal{M}}_{\Lambda_s}(c_0, c')$ are solutions of the Seiberg-Witten equation with virtual dimension -1, they can only occur for $0 < s < 1$. The homomorphism $H : MC_*(Y; \eta_{-1}) \to MC_*(Y; \eta_1)$ of degree $+1$ is defined by

$$H(c) = \sum_{c_0} \sum_s \#\tilde{\mathcal{M}}^0_{\Lambda_s}(c, c_0) \cdot c_0, \quad \text{for } c \in \mathcal{R}^n_{SW}(Y, \eta_{-1}), c_0 \in \mathcal{R}^{n+1}_{SW}(Y, \eta_1).$$

That c_0 is reducible is eliminated by the extra $U(1)$-symmetries in $\mathcal{M}_{\eta_1}(c_0, c^{'})$ and $\mathcal{M}_{\eta_{-1}}(c, c_0)$ and $I_{\eta_1}(\Theta; \eta_0) = I_{\eta_{-1}}(\Theta; \eta_0)$. Summing up $c^{'} \in \mathcal{R}^n_{SW}(Y, \eta_1)$, we have

$$\Psi_{\Lambda_0}(c) - \Psi_{\Lambda_1}(c) = H \circ \partial_{\eta_{-1}}(c) + \partial_{\eta_1} \circ H(c).$$

Therefore Ψ_{Λ_0} and Ψ_{Λ_1} are monopole chain homotopic to each other. □

So the monopole homology groups $MH_*(Y; \eta_{\pm 1})$ associated to two admissible perturbation data are canonically isomorphic to each other whenever $I_{\eta_1}(\Theta; \eta_0) = I_{\eta_{-1}}(\Theta; \eta_0)$ for the unique $U(1)$-reducible Θ on Y. Thus it is more appropriate to denote $MH_*(Y; \eta)$ by $MH_*(Y; I_\eta(\Theta; \eta_0))$. For an integral homology 3-sphere Y, the monopole homology can be extended to a function

$$MH_{SWF} : \{I_\eta(\Theta; \eta_0) : \eta \in \mathcal{P}^{'}_Y\} \to \{MH_*(Y, I_\eta(\Theta; \eta_0)) : \eta \in \mathcal{P}^{'}_Y\}.$$

(Changing a reference η_0 corresponds to the same homology groups with grading $I_{\eta^{'}_0}(\Theta; \eta_0)$-shift) This function MH_{SWF} is a topological invariant of the integral homology 3-sphere Y, up to the degree-shifting of monopole homologies. Hence such a function MH_{SWF} may be called a Seiberg-Witten-Floer theory, which is completely different from the instanton Floer homology, but more related to the treatment in [16]. The set $\{I_\eta(\Theta; \eta_0) : \eta \in \mathcal{P}^{'}_Y\}$ is the chamber-like structure for the monopole homology of integral homology 3-spheres Y.

8. Relative Seiberg-Witten invariants

The Seiberg-Witten invariant (see [7, 32, 38]) has proved so useful and at least as powerful as the Donaldson invariant in many cases, and is much easier to compute. In this section we are going to extend the Seiberg-Witten invariant to the relative one on smooth 4-manifolds with boundary integral homology 3-spheres. The "relative Seiberg-Witten invariants" is no longer a topological invariant since it lies in a monopole homology depending upon Riemannian metrics of integral homology 3-spheres. But the natural pairing between "relative Seiberg-Witten invariants" does recover the Seiberg-Witten invariant of closed smooth 4-manifolds.

Let X be a smooth 4-manifold with $b_1(X) > 0$ and boundary Y (an integral homology 3-sphere). The collar of X can be identified with $Y \times [-1, 1]$, and the admissible perturbation data on Y can be extended inside X as we did in §7. *Fixing $I_\eta(\Theta; \eta_0)$ should be understood through this section.*

Definition 8.1. For a smooth 4-manifold X with boundary Y (an integral homology 3-sphere), the 0-degree relative Seiberg-Witten invariant is defined by

$$q_{X,Y,\eta} = \sum_{c \in \mathcal{R}^*_{SW}(Y,\eta)} \#\mathcal{M}^0_X(c) \cdot c,$$

where $\mathcal{R}^*_{SW}(Y, \eta)$ is the set of all nondegenerate zeros of f_η with prescribed $I_\eta(\Theta; \eta_0)$.

By the index calculation and our convention $\mu_\eta(c) = SF(c, \Theta)$, we have

$$\dim\mathcal{M}^0_X(c) + \mu_\eta(c) = \dim\mathcal{M}_X(\Theta) = \frac{1}{4}(c_1(\pi^*(L))^2 - (2\chi + 3\sigma))(X) = -\frac{1}{4}(2\chi + 3\sigma)(X),$$

since $c_1(L) = 0$ for the integral homology 3-sphere Y. Thus $q_{X,Y,\eta}$ is in the monopole chain group with grading $-\frac{1}{4}(2\chi + 3\sigma)(X)$.

Proposition 8.2. *For $q_{X,Y,\eta} \in MC_{\mu_X}(Y,\eta)$ with $\mu_X = -\frac{1}{4}(2\chi + 3\sigma)(X)$ and a fixed class $I_\eta(\Theta;\eta_0)$, we have $\partial_Y \circ q_{X,Y,\eta} = 0$, i.e., $q_{X,Y,\eta}$ is a monopole cycle in the sense of §6.*

Proof:

$$\partial_Y \circ q_{X,Y,\eta}(c) = \sum_{c \in \mathcal{R}^{\mu}_{SW}(Y,\eta)} \sum_{c' \in \mathcal{R}^{\mu-1}_{SW}(Y,\eta)} \#\mathcal{M}^0_X(c) \cdot \#\hat{\mathcal{M}}^1_{Y\times\mathbf{R}}(c, c^{'}) \cdot c^{'}.$$

For both c and $c^{'}$ irreducible (nondegenerate) zeros of f_η, we take one-dimensional moduli space $\mathcal{M}^1_X(c^{'})$ for fixed $c^{'}$. Then we count the ends of the moduli space to conclude the result. Again it is a technical point to avoid the reducible Θ entering the boundary $\mathcal{M}_X(\Theta) \times \mathcal{M}_{Y\times\mathbf{R}}(\Theta, c^{'})$. For the reducible Θ, we have the dimension counting

$$\dim\{\mathcal{M}_X(\Theta) \times \mathcal{M}_{Y\times\mathbf{R}}(\Theta, c^{'})\} = \dim\mathcal{M}_X(\Theta) + \dim\Gamma_\Theta + \dim\mathcal{M}_{Y\times\mathbf{R}}(\Theta, c^{'}) \geq 0 + 1 + 1 = 2.$$

So c cannot be the reducible Θ, and $\partial_Y \circ q_{X,Y,\eta} = 0$. Hence $q_{X,Y,\eta}$ is indeed a monopole cycle. □

Let $q_{X,Y,\eta}(g_X)$ be the relative Seiberg-Witten invariant with respect to the metric g_X and the admissible perturbation $\eta \in \mathcal{P}^{'}_Y$. Now we show that the monopole homology class $[q_{X,Y,\eta}(g_X)]$ defined by Proposition 8.2 is independent of metrics g_X with $g_X|_Y$ in the fixed class of $I_\eta(\Theta;\eta_0)$.

Proposition 8.3. *Let $g^i_X (i = 1, 2)$ be two generic metrics on X with induced metric g^i_Y generic such that $I_{\eta_1}(\Theta;\eta_0) = I_{\eta_2}(\Theta;\eta_0)$ and $\eta_i = (g^i_Y, \alpha_i)$. Then there exist $c^{'} \in MC_{\mu_X+1}$ with $\mu_X = -\frac{1}{4}(2\chi + 3\sigma)(X)$ such that we have*

$$q_{X,Y,\eta_2}(g^2_X) - q_{X,Y,\eta_1}(g^1_X) = \partial(c^{'}).$$

In particular, $[q_{X,Y,\eta_2}(g^2_X)] = [q_{X,Y,\eta_1}(g^1_X)]$ as the monopole homology class in $MH_{\mu_X}(Y, I_{\eta_i}(\Theta;\eta_0))$.

Proof: Let $\{g^{t+1}_X\}_{0\leq t\leq 1}$ be a family of metrics on X such that $I_{\eta_{t+1}}(\Theta;\eta_0)$ is independent of t with $\eta_{t+1} = (g^{t+1}_X|_Y, \alpha_{t+1})$ and $\mathcal{M}^0_X(g^{t+1}_X)(c)$ has virtual dimension 0 with respect to c irreducible. Therefore $\{\mathcal{M}^0_X(g^{t+1}_X)(c)\}_{0\leq t\leq 1}$ is an one-dimensional moduli space of Seiberg-Witten solutions on X. The corresponding codimension-one boundary in $[0,1] \times \mathcal{B}_X(g^{t+1}_X)(c)$ is given by

$$\partial(\{\mathcal{M}^0_X(g^{t+1}_X)(c)\}_{0\leq t\leq 1}) =$$

$$\{0\} \times \mathcal{M}^0_X(g^1_X)(c) \coprod -\{1\} \times \mathcal{M}^0_X(g^2_X)(c) \coprod \partial\Big(\sum_{\mu_{\eta_{t+1}}(c) - \mu_{\eta_{t+1}}(c') = -1} \#([0,1] \times \mathcal{M}^{-1}_X(g^{t+1}_X)(c^{'}))\Big).$$

The number $\langle \partial_Y c^{'}, c\rangle$ is the algebraic number of $([0,1]\times \mathcal{M}_X^{-1}(g_X^{t+1})(c^{'}))$. The $c^{'}$ cannot be the reducible Θ by the fixed $I_{\eta_1}(\Theta;\eta_0)$ with the same argument as before. So

$$q_{X,Y,\eta_2}(g_X^2)(c) - q_{X,Y,\eta_1}(g_X^1)(c) = \langle \partial_Y c^{'}, c\rangle.$$

Hence $q_{X,Y,\eta_i}(g_X^i)(i=1,2)$ (as a monopole cycle) gives the same monopole homology class. □

Note that orientation reversing from Y to $-Y$ changes the grading from $\mu_\eta(c)$ to $-1-\mu_\eta(c)$ (certainly does not change the solutions of the Seiberg-Witten equation on the 3-manifold), so there is a nature identification between $MC_{\mu_\eta}(Y,\eta)$ and $CF_{-1-\mu_\eta}(-Y,\eta)$.

Theorem 8.4. *For a smooth 4-manifold $X = X_0\#_Y X_1$ with $b_2^+(X_i)>0 (i=0,1)$ and Y an integral homology 3-sphere, the Seiberg-Witten invariant of the 4-manifold X is given by the Kronecker pairing of $MH_*(Y; I_\eta(\Theta;\eta_0))$ with $MH_{-1-*}(-Y; I_\eta(\Theta;\eta_0))$ for $q_{X_0,Y,\eta}$ and $q_{X_1,-Y,\eta}$; assume that the moduli space $\mathcal{M}_X$ does not split to $\mathcal{M}_{X_i}(\Theta)$ through the stretching-neck process,*

$$\langle , \rangle : MH_*(Y; I_\eta(\Theta;\eta_0)) \times MH_{-1-*}(-Y; I_\eta(\Theta;\eta_0)) \to \mathbf{Z}; \quad q_{SW}(X) = \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta}\rangle.$$

More precisely, $q_{SW}(X_0\#_Y X_1) = \sum_c \#\mathcal{M}^0_{X_0,Y,\eta}(c)\cdot \#\mathcal{M}^0_{X_1,-Y}(-c)$, where $I_\eta(\Theta;\eta_0)$ is fixed. The invariant $q_{SW}(X)$ is independent of the choice of $I_\eta(\Theta;\eta_0)$.

Proof: If Y admits a metric of positive scalar curvature, then the proof is given in [31, 38] with $I_\eta(\Theta;\eta_0)=0$ the special case. The assumption implies that $b_2^+(X)>1$. So we can rule out the existence of reducible solutions on X by the standard method (see [7, 15, 32, 38]). Note that

$$\dim \mathcal{M}_{X_0}(c) + \dim \Gamma_c + \dim \mathcal{M}_{X_1}(c) = \dim \mathcal{M}_X.$$

By the dimension equation and the assumption, we have the term $\#\mathcal{M}^0_{X_0,Y,\eta}(c)\cdot\#\mathcal{M}^0_{X_1,-Y,\eta}(-c)$ only with $c\neq\Theta$. Then the 0-dimensional moduli space on X is obtained by gluing the solutions on (X_0,Y) with ones on $(X_1,-Y)$ along irreducible solutions of $\mathcal{R}^*_{SW}(Y,\eta)$. Using the standard technique on stretching the neck (see [9, 15], similar to our gluing construction in Theorem 6.6, one gets the equality $q_{SW}(X) = \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta}\rangle$. Since $q_{SW}(X)$ is a topological invariant, so the pairing is independent of the choice of $I_\eta(\Theta;\eta_0)$. □

For higher degree relative Seiberg-Witten invariants, one can obtain the similar results as in [16].

Remark: We believe that the assumption in Theorem 8.4 on the splitting through $\mathcal{M}_{X_0,Y}(\Theta)$ and $\mathcal{M}_{X_1,-Y}(\Theta)$ can be removed. For $b_2^+(X_i)>0 (i=0,1)$ and fixed $I_\eta(\Theta,\eta_0)$ for the unique $U(1)$-reducible solution Θ at Y, the Seiberg-Witten invariant of $X = X_0\#_Y X_1$ should be zero as in the principle of the Donaldson invariant (see Theorem B of [8] and its proof in §(iv) page 268 - 287 of [8]). One should be able to identify the Seiberg-Witten invariant for this case from gluing $\mathcal{M}_{X_i}(\Theta)(i=0,1)$ to a sum of Euler number of finitely many $U(1)$-bundles. The detailed proof of Theorem B in [8] is quite involved, the similar details for the Seiberg-Witten invariant are expected (even for $Y=S^3$ the proof in [31] is quite long). We leave this for a future study.

Computing the monopole homology is extremely complicated due to the Riemannian metric, the harmonic spinor, the spectral flow and the solution of the first-order Dirac-type nonlinear differential equation. Even for the 3-sphere, a complete calculation of the function MH_{SWF} is very difficult at this moment. Understand the harmonic spinors on S^3 with a subfamily of Riemannian metrics (metrics are $SU(2)$-left invariant and $U(1)$-right invariant) is already quite involved by the work of Hitchin [14]. On the other hand, Theorem 8.4 gives us a flexibility to understand the Seiberg-Witten invariant of closed smooth 4-manifolds through the relative ones with some preferred Riemannian metric(s) on the integral homology 3-sphere.

Remark: The method we developed in this paper also can be extended to rational homology 3-spheres with fixed spectral flows along all $U(1)$-reducible solutions of Seiberg-Witten equation on the rational homology 3-sphere (see [16] for more detail).

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