Adjunction inequality and coverings of Stein surfaces

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Abstract

A stronger form of the adjunction inequality is proved for immersed real surfaces in non simply-connected Stein surfaces. The result is applied to the geometry of Stein domains and analytic continuation on complex surfaces.

1. Introduction

A complex manifold X is Stein if it admits a strictly plurisubharmonic exhaustion function $\varphi : X \to \mathbb{R}$. (A C^2-smooth function on a complex manifold is called strictly plurisubharmonic if $dd^c\varphi$ is a Kähler form.) Every connected component of a regular sublevel set $\{x \in X \mid \varphi(x) < r\}$ is a strictly pseudoconvex Stein domain, which provides an important example of an exact symplectic manifold with contact boundary.

Applying Morse theory to a perturbation of φ, one shows that X is diffeomorphic to the interior of a (infinite) handlebody without handles of index greater than the complex dimension of X. If X is a Stein complex surface, i. e. $\dim_{\mathbb{C}} X = 2$, then there are further (and subtler) restrictions on the representatives of 2-dimensional homology classes. Namely, if $\Sigma \subset X$ is a closed oriented real surface of genus g embedded in X, then the following *adjunction inequality* holds:

$$[\Sigma] \cdot [\Sigma] + \big|\langle c_1(X), [\Sigma]\rangle\big| \leq 2g - 2 \tag{1}$$

provided that Σ is *not* an embedded 2-sphere with trivial homology class $[\Sigma] \in H_2(X; \mathbb{Z})$.

Inequality (1) was independently derived by Lisca–Matić and the author from similar inequalities for real surfaces in compact Kähler surfaces and the algebraic approximation theorem for Stein manifolds (see [16], [8], and [18] for details and bibliography). Alternatively, one can argue that a strictly pseudoconvex Stein domain has a unique Spin^c-structure with non-zero Seiberg–Witten invariant (in the sense of Kronheimer–Mrowka [14]) whereas a homologically non-trivial embedded surface violating (1) would yield another such structure by the work of Ozsváth–Szabó [21].

The present note explores homotopy theoretic consequences of the adjunction inequality. One result concerns the "exceptional" case of homologically trivial two-spheres.

Theorem 1.1. *A smoothly embedded two-sphere in a Stein complex surface violates adjunction inequality* (1) *if and only if its **homotopy** class is trivial.*

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The proof of a general "homotopically enhanced" adjunction inequality for immersed real surfaces is given in §4. The argument will be rather simple (modulo (1) and a classical result of Karl Stein [22]). Two applications of Theorem 1.1 to geometry and complex analysis are discussed in §2 and §3.

Another application of our methods concerns Stein neighbourhoods of immersed real surfaces in arbitrary complex surfaces. In the early eighties, Eliashberg and Kharlamov obtained a version of Gromov's h-principle for totally real embeddings providing sufficient topological conditions for an immersed real surface to be ambiently isotopic to a surface with a base of tubular Stein neighbourhoods. (The state of the art as of 1992 and 2002 was documented by Forstnerić in [6] and [7].) There have been good reasons to believe that these topological conditions (taken from earlier works of E. Bishop and H. F. Lai) are actually necessary for the existence of "topologically small" Stein neighbourhoods of immersed surfaces. For embedded orientable surfaces, this is a straightforward consequence of the adjunction inequality (cf. [18]). With a little more effort, we show:

Theorem 1.2. *Let $\mathcal{C}$ be an isotopy class of immersed closed real surfaces* (*not necessarily orientable*) *in a complex surface X. Then the following are equivalent*:

   a) *there exists a surface $\Sigma \in \mathcal{C}$ with a Stein neighbourhood base*;
   b) *there exist a surface $\Sigma \in \mathcal{C}$ and a Stein domain $U \supset \Sigma$ such that the homomorphisms of homotopy groups $\pi_*(\Sigma) \to \pi_*(U)$ are injective*;
   c) $e(T\Sigma) + e(\nu\Sigma) + |\langle c_1(X), [\Sigma]\rangle| \leq 0$ *for every surface $\Sigma \in \mathcal{C}$.*

Here $e(\cdot) \in \mathbb{Z}$ denotes the Euler number of a vector bundle, and $\langle c_1(X), [\Sigma]\rangle := 0$ for non-orientable Σ.

The implication (a)⇒(b) is obvious. The implication (c)⇒(a) is the existence result mentioned above. We shall prove that (b)⇒(c). Orientable and non-orientable surfaces will be treated separately in §4 and §5. The method of proof provides more detailed information about the homomorphisms $\pi_*(\Sigma) \to \pi_*(U)$. Examples of interest include surfaces in $\mathbb{C}^2$ and symplectic immersions into $\mathbb{C}P^2$.

Throughout the paper, our principal tools are the adjunction inequalities and the following result from complex analysis:

Theorem 1.3 (K. Stein [22]). *Any covering of a Stein complex manifold is Stein.*

The complex structure on the covering is induced by the projection. Much more generally, any locally Stein unramified domain over a Stein manifold is a Stein manifold (see [13] or the original paper [4]). For the benefit of a topologically educated reader, a proof of Theorem 1.3 for coverings of strictly pseudoconvex domains is presented in §6.

2. Thickenings of three-manifolds

Let Y be an open orientable 3-manifold. Then the product four-manifold $Y \times \mathbb{R}$ admits a handle decomposition without handles of index > 2. Hence, it is *homeomorphic* to a Stein complex surface by the results of Gompf and Eliashberg (see [8, Ch. 11]).

Corollary 2.1. *Suppose that Y is an open 3-manifold such that $Y \times \mathbb{R}$ is diffeomorphic to a Stein complex surface. Then every embedded 2-sphere in Y bounds a homotopy 3-ball.*

Proof. If $S \subset Y$ is an embedded 2-sphere in Y, then $\Sigma = S \times \{0\} \subset Y \times \mathbb{R}$ is an embedded 2-sphere whose self-intersection index is zero. In particular, Σ cannot satisfy the adjunction inequality and hence its homotopy class is trivial by Theorem 1.1. Since the inclusion $Y \times \{0\} \subset Y \times \mathbb{R}$ is a homotopy equivalence, it follows that S is null-homotopic in Y. It is a standard result that a null-homotopic embedded 2-sphere in a 3-manifold bounds a homotopy ball (see, for instance, [11, Prop. 3.10]). □

Example 2.1. Let M be a closed orientable three-manifold and $M^{(n)}$ the open three-manifold obtained by removing $n \geq 1$ points from M. *If the smooth manifold $M^{(n)} \times \mathbb{R}$ admits a Stein complex structure, then $n = 1$ and M is a homotopy 3-sphere.* Indeed, let us apply the previous corollary to the boundary of a small ball about one of the punctures in M. It follows that this 2-sphere bounds a homotopy ball in $M^{(n)}$, which is only possible if $n = 1$ and M is a homotopy sphere. In other words, if the three-dimensional Poincaré conjecture holds true, then the standard $\mathbb{R}^4$ is the only example of a diffeomorphism class of Stein surfaces obtained by thickening punctured 3-manifolds.

3. Analytic continuation from two-spheres in $\mathbb{C}^2$

In a different vein, let us consider an application to complex analysis which improves on a result in [18, §5.1]. Let U be any domain (= open connected subset) in $\mathbb{C}^2$. Suppose that U contains an embedded 2-sphere $S \subset U$.

Corollary 3.1. *All holomorphic functions in U can be holomorphically extended to a Riemann domain $\widetilde{U} \supset U$ in which S becomes homotopically trivial.*

Recall that a Riemann domain over $\mathbb{C}^2$ is a connected complex surface V together with a locally biholomorphic projection $p_V : V \to \mathbb{C}^2$. Given two Riemann domains V and W, we write $W \supset V$ if there is a holomorphic map $j : V \to W$ such that $p_W \circ j = p_V$. (*Warning*: The map j is not necessarily injective. However, if p_V is injective so that $V \subset \mathbb{C}^2$ is a usual domain, then j is automatically a genuine inclusion.) Holomorphic extension to Riemann domains is the geometric equivalent of analytic continuation in the sense of Weierstraß.

Proof. Let us define $\widetilde{U}$ to be the maximal Riemann domain containing U such that all holomorphic functions in U can be extended to $\widetilde{U}$. This Riemann domain is called the envelope of holomorphy of U. The fundamental result of Cartan–Thullen–Oka asserts that $\widetilde{U}$ is a Stein manifold (see e. g. [10] or [13]).

The two-sphere $S \subset \widetilde{U}$ has self-intersection index zero because S is embedded in $U \subset \mathbb{C}^2$. (In general, $\widetilde{U}$ is not a domain inside $\mathbb{C}^2$ and can contain real surfaces of arbitrary self-intersection!) It follows that S violates the adjunction inequality in the Stein complex surface $\widetilde{U}$, so it must be homotopically trivial there by Theorem 1.1. □

Corollary 3.1 extends a long line of results on envelopes of holomorphy of embedded 2-spheres in $\mathbb{C}^2$. The initial approach of Bedford–Gaveau [2] was by "attaching analytic

discs" and led to very satisfactory results for 2-spheres contained in strictly pseudoconvex boundaries [3], [15]. However, the case of general embeddings has been only treated with the help of Seiberg–Witten theory (so far?).

The assertion of the corollary is false for embedded n-spheres in $\mathbb{C}^n$ for all $n \neq 2$. The case $n = 1$ is elementary and the case $n \geq 3$ follows from [19]. It is shown in [19] that an embedded n-sphere can represent a non-zero homology class in the complement to a generic complex algebraic hypersurface of degree ≥ 3 in $\mathbb{C}^n$ for $n \geq 3$. If the hypersurface is given by the equation $F = 0$, then the function $1/F$ cannot be extended to any Riemann domain in which the n-sphere is homologically trivial.

A more precise non-extendability result is available for $n = 3$. Namely, there exists an embedded 3-sphere in $\mathbb{C}^3$ with a Stein neighbourhood base. There is no "forced analytic continuation" from this 3-sphere because a Stein domain coincides with its envelope of holomorphy. The existence of such embeddings was proved by Gromov as a corollary to his h-principle for totally real embeddings (see [9, §2.4.5(C)]) and explicit examples were given by Ahern and Rudin [1]. Similar ideas are used to prove the implication (c)⇒(a) in Theorem 1.2. Note, for instance, that every real surface of genus $g > 0$ can be embedded in $\mathbb{C}^2$ in such a way that it admits a Stein neighbourhood base (see [6] for the non-trivial case $g \geq 2$).

4. Immersions of orientable surfaces

Once we are on the subject of homotopy, it is perhaps more natural to consider immersed real surfaces in a Stein surface X. Let $\iota : \mathbb{S}_g \looparrowright X$ be an immersion of the real surface $\mathbb{S}_g$ of genus g. We shall always assume that the immersion is generic, that is to say, has transverse double points only. Let $\Sigma = \iota(\mathbb{S}_g) \subset X$ be the image of ι.

Each double point $x = \iota(s_1) = \iota(s_2) \in \Sigma$ has a sign defined as the local intersection index of the two branches of Σ at x for any choice of orientation on $\mathbb{S}_g$. Let us denote by $\varkappa_\pm = \varkappa_\pm(\Sigma)$ the number of positive and negative double points of Σ.

Double points of different sign behave differently in adjunction inequalities. (I have learnt that from [5].) For an immersed surface in a Stein complex surface we have the following:

$$[\Sigma] \cdot [\Sigma] + \big|\langle c_1(X), [\Sigma]\rangle\big| \leq 2g + 2\varkappa_+ - 2 \qquad (2)$$

provided again that Σ is not a homologically trivial 2-sphere. The absence of negative double points in this formula can be explained as follows. If we perform a blow-up at each such point, then the proper pre-image of Σ has the same homological self-intersection index and the same pairing with the first Chern class but only positive double points. Replacing each positive double point by an embedded handle gives us an embedded surface of genus $g + \varkappa_+$. Then (2) follows in the same way as (1) with the help of blow-up formulas for Seiberg–Witten invariants.

Note that it is certainly impossible to replace $\varkappa_+$ in (2) by the difference $\varkappa_+ - \varkappa_-$. Indeed, an arbitrary number of double points of either sign can be added by taking interior connected sums with the standard figure-eight immersions of the two-sphere into $\mathbb{R}^4$ so

that the homotopy class of the immersion remains unchanged. The situation appears to be different, however, if the fundamental group is taken into account.

For a double point $x = \iota(s_1) = \iota(s_2) \in \Sigma$, consider a path in $\mathbb{S}_g$ connecting s_1 and s_2. A double point loop of x is the image of this path in X. The free homotopy class of a double point loop (modulo sign) is defined up to addition of homotopy classes from the subgroup $\iota_*\pi_1(\mathbb{S}_g) \subset \pi_1(X)$.

Definition 4.1. A double point $x \in \Sigma = \iota(\mathbb{S}_g)$ is called *essential* if all its double point loops are homotopically non-trivial in X. Let $\varkappa_\pm^{\mathrm{ess}} = \varkappa_\pm^{\mathrm{ess}}(\Sigma, X)$ be the number of positive and negative essential double points of Σ.

Note that double points created by taking interior connected sums with immersed spheres in $\mathbb{R}^4$ are never essential.

Theorem 4.1. *Let $\Sigma = \iota(\mathbb{S}_g) \subset X$ be an immersed real surface in a Stein complex surface. Then either Σ is a homotopically trivial two-sphere or*

$$[\Sigma] \cdot [\Sigma] + \big|\langle c_1(X), [\Sigma]\rangle\big| \leq 2g + 2\big(\varkappa_+ - \varkappa_-^{\mathrm{ess}}\big) - 2.$$

Proof. Let us assume that $\Sigma = \iota(\mathbb{S}_g) \subset X$ is an immersed surface other than a homotopically trivial two-sphere. Let $p : \widehat{X} \to X$ be the covering corresponding to the subgroup $\iota_*\pi_1(\mathbb{S}_g) \subset \pi_1(X)$. Then there exists a lift $\widehat{\iota} : \mathbb{S}_g \to \widehat{X}$ of the immersion $\iota : \mathbb{S}_g \to X$. Clearly, $\widehat{\Sigma} = \widehat{\iota}(\mathbb{S}_g)$ is a generically immersed surface of the same genus.

Notice that if Σ is an immersed sphere, then $p : \widehat{X} \to X$ is the universal covering and hence $H_2(\widehat{X}) \cong \pi_2(\widehat{X})$ by the Hurewicz theorem. In particular, if Σ is homotopically non-trivial, then its lift $\widehat{\Sigma}$ is homologically non-trivial.

Since $\widehat{X}$ is Stein, it follows that $\widehat{\Sigma}$ satisfies adjunction inequality (2), that is to say,

$$[\widehat{\Sigma}] \cdot [\widehat{\Sigma}] + \big|\langle c_1(\widehat{X}), [\widehat{\Sigma}]\rangle\big| \leq 2g + 2\varkappa_+(\widehat{\Sigma}) - 2.$$

Hence, Σ satisfies the inequality of the theorem by the following lemma (which is completely trivial in the case of an embedded surface Σ).

Lemma 4.2. *Each lift of Σ is an immersed surface $\widehat{\Sigma} = \widehat{\iota}(\mathbb{S}_g) \subset \widehat{X}$ such that*

    a) $\varkappa_\pm(\widehat{\Sigma}) = \varkappa_\pm(\Sigma) - \varkappa_\pm^{\mathrm{ess}}(\Sigma, X)$;
    b) $[\widehat{\Sigma}] \cdot [\widehat{\Sigma}] = [\Sigma] \cdot [\Sigma] - 2\varkappa_+^{\mathrm{ess}}(\Sigma, X) + 2\varkappa_-^{\mathrm{ess}}(\Sigma, X)$;
    c) $\langle c_1(\widehat{X}), [\widehat{\Sigma}]\rangle = \langle c_1(X), [\Sigma]\rangle$.

Proof. Let $\widehat{\Sigma}$ be any lift of Σ to $\widehat{X}$. If $\widehat{x} \in \widehat{\Sigma}$ is a double point of $\widehat{\Sigma}$, then $p(\widehat{x}) \in \Sigma$ is a double point of the same sign because p is locally an orientation preserving diffeomorphism. Furthermore, by the definition of $\widehat{X}$, a double point of Σ lifts to two ordinary points in $\widehat{\Sigma}$ if and only if it is essential. This proves (a).

Let $\nu\Sigma = \iota^*TX/T\mathbb{S}_g$ be the normal bundle of Σ. Note that $\nu\widehat{\Sigma}$ is isomorphic to $\nu\Sigma$. Recall the formula for the homological self-intersection index of an immersed surface:

$$[\Sigma] \cdot [\Sigma] = e(\nu\Sigma) + 2\varkappa_+(\Sigma) - 2\varkappa_-(\Sigma),$$

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where $e(\nu\Sigma)$ is the Euler number of the normal bundle. Applying this formula to Σ and $\widehat{\Sigma}$ and using relation (a) gives equality (b).

Finally, observe that $c_1(\widehat{X}) = p^* c_1(X)$ because $p : \widehat{X} \to X$ is an unramified holomorphic map. Therefore,

$$\langle c_1(\widehat{X}), [\widehat{\Sigma}] \rangle = \langle p^* c_1(X), [\widehat{\Sigma}] \rangle = \langle c_1(X), p_*[\widehat{\Sigma}] \rangle = \langle c_1(X), [\Sigma] \rangle,$$

which proves (c). $\qquad\square$

To put Theorem 4.1 in proper perspective and establish Theorem 1.2 for orientable surfaces, let us state the relevant existence result for Stein neighbourhoods (see [6] and [18, §2]). Suppose that $\Sigma \subset Y$ is an immersed orientable real surface of genus g in an arbitrary complex surface Y. If Σ satisfies the inequality

$$[\Sigma] \cdot [\Sigma] + \big| \langle c_1(Y), [\Sigma] \rangle \big| \leq 2g + 2(\varkappa_+ - \varkappa_-) - 2, \qquad (3)$$

then it is isotopic (by a C^0-small isotopy) to an immersed real surface $\Sigma' \subset Y$ with a Stein neighbourhood base. The surface Σ' can be chosen so that the Stein neighbourhoods are thin tubes around it and, in particular, have the homotopy type of Σ (see [7]). Inequality (3) is equivalent to

$$e(T\Sigma) + e(\nu\Sigma) + \big| \langle c_1(Y), [\Sigma] \rangle \big| \leq 0$$

because $e(T\Sigma) = \chi(\mathbb{S}_g) = 2 - 2g$ and $e(\nu\Sigma) = [\Sigma] \cdot [\Sigma] - 2(\varkappa_+ - \varkappa_-)$.

Note that for surfaces without negative double points, condition (3) coincides with adjunction inequality (2) and therefore holds whenever the surface admits a Stein neighbourhood in which it is not homotopically trivial. It follows from Theorem 4.1 that an immersed surface $\Sigma \subset Y$ with $\varkappa_- \neq 0$ satisfies (3) if there exists a Stein neighbourhood $U \supset \Sigma$ in which the negative double points of Σ are essential.

Corollary 4.3. *Let $\mathcal{C}$ be an isotopy class of immersions of $\mathbb{S}_g$ into a complex surface. Suppose that there exist a surface $\Sigma \in \mathcal{C}$ and a Stein domain $U \supset \Sigma$ such that the homomorphisms of homotopy groups $\pi_*(\Sigma) \to \pi_*(U)$ are injective. Then every surface in $\mathcal{C}$ satisfies inequality (3) and there exists a surface $\Sigma' \in \mathcal{C}$ with a Stein neighbourhood base.*

Proof. $\Sigma = \iota(\mathbb{S}_g)$ is homotopy equivalent to the wedge of $\mathbb{S}_g$ and $(\varkappa_+ + \varkappa_-)$ circles (the circles being double point loops of the double points). Hence, if the map $\pi_1(\Sigma) \to \pi_1(U)$ is injective, then all double points are essential and the result follows from the preceding discussion. $\qquad\square$

Example 4.1 (Immersed surfaces in $\mathbb{C}^2$). Let us use Theorem 4.1 and the notion of the envelope of holomorphy to obtain a generalization of Corollary 3.1 for an immersed real surface $\Sigma = \iota(\mathbb{S}_g) \subset \mathbb{C}^2$. There are the following options:

1. If $g + \varkappa_+ - \varkappa_- \geq 1$ (that is, (3) holds), then Σ can have a Stein neighbourhood base. In that case, there is no forced analytic continuation for holomorphic functions in a neighbourhood of Σ.

2. If $g + \varkappa_+ - \varkappa_- < 1$, then all holomorphic functions from any neighbourhood of Σ extend holomorphically to a Riemann domain in which at least $\varkappa_- - \varkappa_+ - g + 1$ negative double points of Σ become non-essential.

3. If, moreover, $g = \varkappa_+ = 0$, then all holomorphic functions extend to a Riemann domain in which Σ is null-homotopic.

For instance, if $S \subset \mathbb{C}^2$ is an immersed 2-sphere with a single positive and a single negative double point, then *the double point loop of the negative double point of S bounds an immersed disc in the envelope of holomorphy of any domain $U \supset S$.* Such "extension along 2-cells" seems to be a new phenomenon (compared to the results mentioned in §3).

Example 4.2 (Symplectic surfaces in $\mathbb{C}P^2$)**.** An oriented immersed surface $\Sigma \subset Y$ in a Kähler complex surface is called symplectic if the restriction of the Kähler form to Σ is a positive two-form. Symplectic surfaces satisfy the *adjunction formula*:

$$[\Sigma] \cdot [\Sigma] - \langle c_1(Y), [\Sigma] \rangle = 2g + 2(\varkappa_+ - \varkappa_-) - 2.$$

Notice that this equality is compatible with inequality (3) if and only if $\langle c_1(Y), [\Sigma] \rangle \leq 0$.

For instance, symplectic surfaces in $\mathbb{C}P^2$ cannot satisfy (3) because the first Chern class of $\mathbb{C}P^2$ is proportional to the class of the symplectic form and the latter is positive on every symplectic surface. In particular, a symplectic surface in $\mathbb{C}P^2$ all of whose double points are positive is not contained in any Stein domain over $\mathbb{C}P^2$ by the adjunction inequality. (For symplectic spheres, this fact was first proved in a completely different way by Ivashkovich and Shevchishin [12].)

However, a symplectic surface with negative double points can lie in a Stein domain in $\mathbb{C}P^2$ by the following argument suggested by Ivashkovich. Take any homologically non-trivial immersed surface $S \subset \mathbb{C}P^2$ satisfying (3) (e. g., a two-sphere in the primitive homology class with three positive double points). S is isotopic to an immersed surface $S' \subset \mathbb{C}P^2$ which has a Stein neighbourhood. A theorem of Gromov [9, §3.4.2(A)] shows that S' can be C^0-approximated by a symplectic immersion whose image is inside the Stein neighbourhood.

On the other hand, it follows from Theorem 4.1 that a Stein domain containing a symplectically immersed surface $\Sigma \subset \mathbb{C}P^2$ is never a "small neighbourhood" of Σ in the topological sense. Let $d := [\Sigma] \cdot [\mathbb{C}P^1] > 0$ be the degree of Σ. If $U \supset \Sigma$ is a Stein Riemann domain over $\mathbb{C}P^2$, then at least $3d$ negative double points of Σ are non-essential in U.

The argument (and the result) remains valid for all surfaces of positive degree in $\mathbb{C}P^2$ that satisfy the adjunction formula. However, it would be interesting to find a proof for genuine symplectic immersions by the Ivashkovich–Shevchishin method, which is considerably more geometric in spirit.

5. Non-orientable surfaces

Every non-orientable closed real surface is diffeomorphic to a connected sum

$$\mathfrak{S}_h = \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{h+1 \text{ times}}, \qquad h \geq 0.$$

Note that the Euler characteristic $\chi(\mathfrak{S}_h) = 1 - h$.

The fundamental group $\pi_1(\mathfrak{S}_h)$ contains the index two subgroup $\pi_1^+(\mathfrak{S}_h)$ consisting of homotopy classes of orientation preserving loops. The covering $\mathbb{S}_h \to \mathfrak{S}_h$ corresponding to that subgroup is the orientable double covering by the orientable surface of genus h.

If $\iota : \mathfrak{S}_h \to Y$ is an immersion into an orientable four-manifold (e. g., into a complex surface), then the normal bundle $\iota^* TY / T\mathfrak{S}_h$ has a well-defined Euler number in $\mathbb{Z}$.

Theorem 5.1. *Let* $\Sigma = \iota(\mathfrak{S}_h) \subset X$ *be an immersed non-orientable real surface in a Stein complex surface. Suppose that the homomorphisms* $\pi_*(\Sigma) \to \pi_*(U)$ *are injective. Then the following inequality holds*:

$$e(\nu\Sigma) \le h - 1,$$

where $e(\nu\Sigma) \in \mathbb{Z}$ *is the Euler number of the normal bundle of* Σ.

It should be pointed out that the statement for embedded surfaces is already different from the orientable case. Firstly, the assumption on the homomorphism of fundamental groups cannot be dropped. Secondly, $e(\nu\Sigma)$ is not a homological invariant. For instance, the Klein bottle $K = \mathfrak{S}_1$ can be embedded into $\mathbb{C}^2$ with normal Euler number ± 4.

Proof of the theorem. Let us consider the covering $p : \widehat{X} \to X$ corresponding to the subgroup $\iota_* \pi_1^+(\mathfrak{S}_h)$. The manifold $\widehat{X}$ with the complex structure induced by p is a Stein complex surface by Theorem 1.3. The immersion ι lifts to an embedding $\widehat{\iota} : \mathbb{S}_h \to \widehat{X}$ of the orientable covering of $\mathfrak{S}_h$. (That $\widehat{\iota}$ is an embedding follows because $\pi_1(\Sigma)$ injects into $\pi_1(X)$ and hence all double point loops are non-trivial.) Set $\widehat{\Sigma} = \widehat{\iota}(\mathbb{S}_h)$.

An easy argument shows that

$$[\widehat{\Sigma}] \cdot [\widehat{\Sigma}] = e(\nu\widehat{\Sigma}) = 2e(\nu\Sigma)$$

and

$$\langle c_1(\widehat{X}), [\widehat{\Sigma}] \rangle = 0.$$

Furthermore, $\widehat{\Sigma}$ is not a homotopically trivial two-sphere because in that case it would follow that $\iota_* : \pi_2(\Sigma) \to \pi_2(X)$ has non-trivial kernel.

The desired inequality now follows from adjunction inequality (1) and Theorem 1.1 applied to the embedded orientable surface $\widehat{\Sigma}$ in the Stein complex surface $\widehat{X}$. $\qquad\square$

Let us now state the existence result for Stein neighbourhoods of non-orientable surfaces (see [6], [7]). An immersed non-orientable real surface $\Sigma = \iota(\mathfrak{S}_h) \subset Y$ in a complex surface is isotopic to a surface with a Stein neighbourhood base provided that

$$e(\nu\Sigma) \le h - 1. \tag{4}$$

This is precisely the inequality from Theorem 1.2(c) because $h - 1 = -\chi(\mathfrak{S}_h) = -e(T\Sigma)$. Hence, Theorem 1.2 for non-orientable surfaces follows from Theorem 5.1.

Example 5.1 (Embedded non-orientable surfaces in $\mathbb{C}^2$)**.** By the results of Whitney and Massey [17], the normal Euler number of an embedded non-orientable surface $\Sigma \subset \mathbb{C}^2$

can take any of the following values:

$$-4 + 2\chi(\Sigma), 2\chi(\Sigma), 2\chi(\Sigma) + 4, \dots, 4 - 2\chi(\Sigma).$$

For instance, $\mathbb{R}P^2$ admits embeddings with normal Euler numbers ± 2.

Let $X \supset \Sigma$ be a Stein domain over $\mathbb{C}^2$ containing Σ. Suppose that the groups $\pi_1^+(\Sigma)$ and $\pi_1(\Sigma)$ have different images in the fundamental group of X. Then arguing as in the proof of Theorem 5.1, we conclude that either $e(\nu\Sigma) \leq -\chi(\Sigma)$ or $\Sigma \cong \mathbb{R}P^2$ and $e(\nu\Sigma) = 0$. However, the second possibility is ruled out by the Whitney theorem.

Consequently, analytic continuation from a non-orientable embedded surface $\Sigma \subset \mathbb{C}^2$ can be described as follows:

1. If $e(\nu\Sigma) \leq -\chi(\Sigma)$, then the surface is isotopic to an embedded surface with a Stein neighbourhood base ("no forced continuation").
2. If $e(\nu\Sigma) > -\chi(\Sigma)$, then all holomorphic functions from a neighbourhood $U \supset \Sigma$ can be extended to a Riemann domain $\widetilde{U} \supset \Sigma$ in which all orientation reversing loops in Σ become homotopic to orientation preserving loops in Σ.

For instance, consider an embedded $\mathbb{R}P^2 \cong \Sigma \subset \mathbb{C}^2$. If the normal Euler number equals -2, then Σ can have a Stein neighbourhood base. On the other hand, if the normal Euler number is $+2$, then the only non-trivial loop in Σ bounds an immersed disc in every Stein domain containing Σ.

6. Stein coverings

In this section, we outline a direct geometric proof for a special case of Theorem 1.3 sufficient for our applications. The argument can be traced back to the fundamental paper of Oka [20]. In one form or another it appears in many books on holomorphic functions of several complex variables (cf., for instance, [10, §IX.D]).

Let X be a Stein manifold of complex dimension n with a smooth strictly plurisubharmonic exhaustion function $\varphi : X \to \mathbb{R}$. Denote by $X_r = \{x \in X \mid \varphi(x) < r\}$ the sublevel set of φ for a regular value $r \in \mathbb{R}$.

Proposition 6.1. *For any covering $p : V \to X_r$, the manifold V with the induced complex structure is Stein, that is, admits a strictly plurisubharmonic exhaustion function.*

Let us first recall that X carries a Kähler metric defined by $g(\cdot, \cdot) = dd^c\varphi(\cdot, J\cdot)$, where J is the complex structure on X. To measure the "defect of plurisubharmonicity" of a C^2-function $\beta : X \to \mathbb{R}$, it is convenient to introduce the following function:

$$\lambda(\beta, x) = \min_{\xi \in T_x X, \, \|\xi\|_g = 1} dd^c\beta(\xi, J\xi) \quad \text{for } x \in X.$$

In other words, $\lambda(\beta, x)$ is the minimal eigenvalue of the hermitian form $dd^c\beta(\cdot, J\cdot)$ with respect to the Kähler metric g at the point $x \in X$. For instance, $\lambda(\varphi, x) \equiv 1$ by the definition of g. Notice that β is strictly plurisubharmonic in an open set if and only if the function $x \mapsto \lambda(\beta, x)$ is positive there.

Lemma 6.2. *There exists an exhaustion function* $\Phi : X_r \to \mathbb{R}$ *such that*

$$\lambda(\Phi, x) \geq \lambda(\varphi, x) = 1 \qquad \text{for all } x \in X_r.$$

Proof. Let $f : (-\infty, r) \to \mathbb{R}$ be a smooth function. By the chain rule, we get

$$dd^c f(\varphi) = f'(\varphi)dd^c\varphi + f''(\varphi)d\varphi \wedge d^c\varphi.$$

Recall that $d^c\varphi(\xi) = d\varphi(-J\xi)$ by definition and hence

$$d\varphi \wedge d^c\varphi(\xi, J\xi) = \big(d\varphi(\xi)\big)^2 + \big(d\varphi(J\xi)\big)^2 \geq 0 \quad \text{ for all } \xi \in TX.$$

Consequently, if f is a *convex* function such that $f'(t) \geq 1$ for all $t \in (-\infty, r)$ and $\lim\limits_{t \to r-} f(t) = +\infty$, then $\Phi = f(\varphi)$ is the desired exhaustion function on X_r. $\qquad \square$

This lemma settles the easy case of a finite covering $p : V \to X_r$, because then $v \mapsto \Phi(p(v))$ is a strictly plurisubharmonic exhaustion function on V.[1] In the general case, an additional argument is required.

Pick a regular value $r' > r$ of φ so that there are no critical values in the interval (r, r'). Then the covering $p : V \to X_r$ extends to a covering $p' : V' \to X_{r'}$. Let $G = (p')^*g$ be the pull-back metric on $V' \supset V$.

Assume henceforth that V is connected. Fix a point $v_0 \in V$. For any point $v \in V'$, let $\rho(v)$ be the distance from v to v_0 in V' with respect to the pull-back metric G. A topological argument shows that $v \mapsto \rho(v) + \Phi(p(v))$ is an exhaustion function on V. (Φ can be replaced by any exhaustion function on X_r.) Unfortunately, ρ is neither smooth nor plurisubharmonic. The following "double averaging trick" of Oka serves to resolve both problems.

Lemma 6.3. *There exists a C^2-smooth function* $\bar{\rho} : V \to \mathbb{R}$ *with the following properties*:
  (i) $\bar{\rho}(v) > \rho(v) - 1$ *for all* $v \in V$;
  (ii) $\lambda(\bar{\rho}, v) > -K$ *for all* $v \in V$ *and a constant* $K > 0$ *independent of* $v \in V$.

Strictly speaking, $\lambda(\bar{\rho}, v)$ is computed with respect to the metric G. However, this is equivalent to pushing things down to X and taking the usual λ because its definition has been completely local.

Sketch proof of Lemma 6.3. For $\varepsilon > 0$, the averaging operator A_ε on continuous functions in V' is defined by the formula

$$A_\varepsilon f(v) := \frac{1}{\text{Vol}(B(v, \varepsilon))} \int_{B(v,\varepsilon)} f \, d\text{vol},$$

where $B(v, \varepsilon)$ is the geodesic ball of radius ε about v.

The function $A_\varepsilon f$ can be defined for all points $v \in V'$ such that $B(v, \varepsilon)$ is relatively compact in V'. Clearly, $|A_\varepsilon f - f|$ is bounded by the modulus of continuity of the function f. Furthermore, $A_\varepsilon f$ is of class C^1 and there is a bound for its derivatives in terms of the metric, the modulus of continuity of f, and ε.

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[1]In particular, this proves that X_r itself is Stein for our definition.

We claim that, for $\varepsilon > 0$ small enough, the second average $\bar{\rho} := (A_\epsilon)^2 \rho$ is the desired function.

Let us choose a positive $\varepsilon < 1/2$ so that, for every point $x \in X_r$, the geodesic ball $B(x, 2\varepsilon)$ of radius 2ε about x is diffeomorphic to the euclidean ball and relatively compact in $X_{r'}$. Then the pre-image of each such ball is a disjoint union of geodesic balls of the same radius with respect to the pull-back metric on V'. It follows that $\bar{\rho} = (A_\varepsilon)^2 \rho$ is well-defined in V.

By the aforementioned properties of the averaging operator, $\bar{\rho}$ is of class C^2, the difference $|\bar{\rho}(v) - \rho(v)|$ is bounded above by $2\varepsilon < 1$, and $\lambda(\bar{\rho}, v)$ is bounded by a function depending on ε and the metric G. This implies a uniform (lower) bound for $\lambda(\bar{\rho}, v)$ on $V = p^{-1}(X_r)$ because G is the pull-back of a metric on $X_{r'}$ and X_r is relatively compact in $X_{r'}$. $\qquad\square$

To complete the proof of Proposition 6.1, let us consider the C^2-function

$$\Psi(v) := \bar{\rho}(v) + K\Phi(p(v)) + 1 \quad \text{for } v \in V.$$

Ψ is an exhaustion function because it majorates the exhaustion function $\rho(v) + K\Phi(p(v))$ by Lemma 6.3(i). Furthermore, Ψ is strictly plurisubharmonic because

$$\lambda(\Psi, v) \geq \lambda(\bar{\rho}, v) + K\lambda(\Phi, p(v)) > 0$$

by Lemma 6.3(ii) and Lemma 6.2.

Acknowledgements

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