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Polyhedral approximations of Riemannian manifolds

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Abstract

I'm trying to understand which Riemannian manifolds can be Lipschitz approximated by polyhedral spaces of the same dimension with curvature bounded below. The necessary conditions I found consist of some special inequality for curvature at each point (the geometric curvature bound). This inequality is also sufficient condition for local approximation. I conjecture that it is also a sufficient condition for global approximation, and I can prove it if the curvature bound is positive. In general I can prove it only with the additional assumption that tangent bundle of the manifold is stably trivial.

0. Introduction

Let P_n be a sequence of m-dimensional polyhedral k-spaces (i.e. spaces with piecewise constant curvature $= k$) with curvature bounded below (see 1.A for precise definition), such that it converges to a Riemannian manifold (M, g) of the same dimension. One has the right to ask the following question:

(i) What one can say about (M, g)?

or even simpler one:

(ii) What one can say about the curvature tensor of (M, g)?

Obviously P_n are Alexandrov spaces with curvature $\geq k$, therefore one immediately gets that (M, g) must have sectional curvature $\geq k$. In fact, it is possible to say much more about curvature of M. The first indication of this phenomenon one can find in Cheeger's generalization of Bochner formulas to metric spaces with cone-like singularities [Ch], which suggests, in particular, that polyhedral spaces must have (in some sense) positive curvature operator.

In fact, the curvature condition on M is even stronger the positive curvature operator. I would like to call it "geometric curvature bound" ($G_p \geq 0$). I say that the curvature operator $R_p \in S^2(\Lambda^2(T_p))$ at point $p \in M$ is geometrically non-negative if it can be

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expressed as

$$R_p = \sum_i (x_i \wedge y_i)^2,$$

where $x_i, y_i \in T_p$ and $T_p = T_p(M)$ denotes tangent space at $p \in M$. For $k \in \mathbb{R}$, I will also write $G_p \geq k$ if

$$R_p - kI = \sum_i (x_i \wedge y_i)^2$$

for some collection of x_i and $y_i \in T_p$ (where I is the curvature operator of standard sphere) and we will write $G(M) \geq k$ if $G_p \geq k$ for any $p \in M$.

The curvature operator $(x \wedge y)^2$ is similar to curvature operator of $S^2 \times \mathbb{R}^{m-2}$ and therefore the above definition can be viewed as the following "The curvature operator is geometrically non-negative if it can be expressed as a convex combination of curvature operators of $S^2 \times \mathbb{R}^{m-2}$" (see section 1.E for relation of $G \geq 0$ to the other curvature bounds).

Next I am ready to formulate theorems:

Local Theorem 0.1. *Let P_n be a sequence of m-dimensional polyhedral spaces with curvature $\geq k$, which Lipschitz converge to a Riemannian manifold (M, g) of the same dimension, then $G(M) \geq k$.*

Moreover if M is a Riemannian manifold with $G(M) \geq k$ then each point has a neighborhood which is a Lipschitz limit of a sequence of polyhedral spaces with curvature $\geq k - \epsilon$ for arbitrary small $\epsilon > 0$.

This seems to be a satisfactory answer to the second question. The following is what I can do for the first one:

Global Theorem 0.2. *If (M, g) is Riemannian m-manifold with $G(M) \geq k$. Assume that M (or its finite cover) has stably trivial tangent bundle then M can be realized as a Lipschitz limit of a sequence of m-dimensional polyhedral metrics with curvature $\geq k - \epsilon$ for arbitrary $\epsilon > 0$.*

I conjecture that the condition on the tangent bundle can be removed from this formulation, but so far I can not even construct an approximation of (CP^2) with canonical metric (which has $G \geq 0$) by polyhedral metric with curvature $\geq -\epsilon$.

The Global Theorem above can be reduced to the cases $k = \{-1, 0, 1\}$, in the first and last case using rescaling one can get an approximation of (M, g) with polyhedral metrics with curvature $\geq k$. In case $k = 1$, the condition $G \geq 1$ implies, in particular, that curvature operator of M is strictly positive (see 1.E). In particular, from Micallef-Moore Theorem [MM] it follows that universal cover $\widetilde{M}$ must be homeomorphic to a sphere. In particular, $\widetilde{M}$ has a stably trivial tangent bundle and therefore we get the following:

Corollary 0.3. *An m-manifold (M, g) can be Lipschitz approximated by m-dimensional polyhedral metrics with curvature ≥ 1 if and only if $G(M) \geq 1$.*

The rest of this section is devoted to the ideas of the proofs.

The necessity of geometric curvature bound roughly follows from the fact that all curvature of a polyhedral metric lives on hyper-edges (a simplexes of codimension 2), and around every hyper-edge the metric looks exactly as $C \times \mathbb{R}^{m-2}$, where C is a two-dimensional cone. Thus the "curvature" at the vertex of C looks pretty much like curvature of S^2 with zero radius, and this allows me to view the "curvature" at the edge as the curvature of $S^2 \times \mathbb{R}^{m-2}$ (which is $(x \wedge y)^2$) multiplied by a Hausdorff measure of edge. When a space is approximated by polyhedral metrics the curvature tensors of different edges could mix with each other i.e the limit manifold must have curvature tensor which is convex combination of the curvatures of above form, in other words, it will satisfy condition $G \geq 0$. This is only rough idea, the real proof contains much of technical work. This part of the proof is not included in this paper but in the Appendix B I present an unpublished result of Perelman on which this part of the proof is based.

Much more interesting things happen in the proof of the sufficient condition. First I am constructing an isometric embedding (M^m, g) into the Euclidean space $\mathbb{R}^q$ in such a way that locally the corresponding submanifold (which I also call M) is an intersection of $q - m$ open convex hyper-surfaces such that angle between each pair of them is less than $\pi/2$. This condition on angles between hyper-surfaces in fact implies that its intersection M has $G(M) > 0$. Then I consider an approximation of convex hyper-surfaces by convex polyhedral hyper-surfaces with the same condition on angles and the needed polyhedral approximation is simply intersection of these polyhedral hyper-surfaces. That proves the local theorem.

The idea of the proof of the Global Theorem is as follows:

Assume that M is simply connected. Using that $T(M)$ is stably trivial I represent whole M as an intersection of open convex hyper-surfaces with the same conditions on angles and then do the same approximation as above. The proof of this last representation is technical, but it is obvious here that once such representation exists we have that $N(M)$, the normal bundle of M, is trivial; in particular the tangent bundle $T(M)$ should be stably trivial and it explains reasons for the strange condition on the tangent bundle in the Global Theorem.

The fun of this paper lies in the fact that although this problem was always on the surface the answer does not coincide with any curvature bound which were studied in geometry so far. On the other hand this paper shows that method of polyhedral approximation (which is known to be very powerful in dimension $= 2$) is not that good for higher dimensions, at least if you want to do something for positive scalar curvature, Ricci curvature or (in dimension ≥ 4) for sectional curvature.

I want to thank Gregory Perelman for sharing ideas and making me interested in this problem, Vladimir Voevodsky for bringing paper of Hilbert to my attention. I want to express my very special thanks to Jost Eschenburg and Sergey Kozlov who constructed for me a weird examples of curvature tensors which pulled me out from a dead end in this research and to Rostislav Matveyev and Dmitry Panov for their helpful interest in this topic.

1. Notation, Definitions and Preliminaries

1.A. Polyhedral spaces with curvature bounded below or above.

Pseudomanifold is a simplicial complex with the following property: link of any simplex is connected or $S^0 = \{-1, 1\}$.

Polyhedral k-space is a pseudo-manifold with metric such that each m-simplex is isometric to a simplex in the simply connected space of constant curvature k.

A polyhedral k-space has curvature bounded below or above if and only if space of direction at each point is an Alexandrov space with curvature ≥ 1 or $CAT(1)$ space respectively. In this case the polyhedral has curvature $\geq k$ or $\leq k$ respectively.

For case of lower curvature bound there is an equivalent description: A polyhedral k-space has curvature $\leq k$ if and only if the sum of angles around any hyper-edge (i.e. simplex of codimension $= 2$) is $\leq 2\pi$.

For shortness, I will call polyhedral k-space with curvature bounded below (above) by polyhedral space with curvature $\geq k$ ($\leq k$).

In all that follows we will assume $k = 0$, but if it is not specially mentioned, everything below is true for any k, once we replace $\mathbb{R}^q$ by the simply connected q-manifold with constant curvature k. See [Mil] for a general discussion of polyhedral spaces of nonnegative curvature.

1.B. Convex submanifolds of higher codimension.

Definition 1.1. A submanifold $M \subset \mathbb{R}^q$ is called *locally convex* if each point of M has a neighborhood U, for which there is a collection of (strictly) convex, possibly open, hypersurfaces F_i, such that $U = \cap_i F_i$ and moreover at each point of U the angle between outward normals to any pair of F_i is $> \pi/2$.

If M is C^2-smooth then the above property is equivalent to the following condition: at each point $x \in M$ there is an orthonormal basis $\{e_i\} \subset N_x(M)$, where $N(M)$ is normal bundle of $M \subset \mathbb{R}^q$, such that the representation of the second fundamental form $s : S^2(T_x(M)) \to N_x(M)$ through this basis $s = \sum_i e_i s_i$ has all quadratic forms $s_i \in S^2(T)$ positively defined.

This property can be also interpreted as the following: the submanifold is locally convex if it can be viewed locally as a convex hyper-surface in a convex hyper-surface in ... in $\mathbb{R}^q$.

It is easy to see that locally convex submanifold has positive curvature (for the induced intrinsic metric in the sense of Alexandrov). If the submanifold is smooth, one can say more about its curvature tensor, but to make it precise we must discuss a little the curvature tensor of Riemannian manifold and curvature of submanifolds in $\mathbb{R}^q$.

1.C. Nice curvature tensor for submanifolds.

Here we introduce an extrinsic curvature for submanifolds. The source for this subsection is $[\mathrm{Grom}]_{\mathrm{PDR}}$ 3.1.5.

Let T be a vector space with a scalar product, T^n will denote its tensor power of degree n, $S^n(T)$ and $\Lambda^n(T)$ will denote respectively subspaces of symmetric and antisymmetric elements of T^n. The scalar product on T canonically induces a scalar product on T^n and all its subspaces.

I will need one special subspace, the space of algebraic curvature tensors $A^4(T) = \Lambda^4(T)^\perp \cap S^2(\Lambda^2(T))$ that is exactly space of all possible curvature tensors of Riemannian manifolds. In an equivalent way this subspace $A^4 \subset S^2(\Lambda^2(T))$ can be described as the space of all tensors in $S^2(\Lambda^2(T))$ satisfying the first Bianchi identity

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$$

therefore it does not depend on the choice of the scalar product on T.

Let $M \subset \mathbb{R}^q$ be a smooth submanifold and $s_x : S^2(T_x(M)) \to N_x(M)$ be its second fundamental form at $x \in M$, here $T(M)$ and $N(M)$ are respectively the tangent and normal bundle over M. Consider the Φ-*curvature tensor*

$$\Phi(X,Y,Z,W) = \langle s(X,Y), s(Z,W)\rangle,$$

here Φ is a section of $S^2(S^2(T(M))$.

The tensor Φ admits the canonical representation

$$\Phi(X,Y,Z,W) = E(X,Y,Z,W) + \frac{1}{3}\Big(R(X,Z,Y,W) + R(X,W,Y,Z)\Big)$$

where E is total symmetrization of Φ, i.e.

$$E(X,Y,Z,W) = \frac{1}{3}\Big(\Phi(X,Y,Z,W) + \Phi(Y,Z,X,W) + \Phi(Z,X,Y,W)\Big) \in S^4(T)$$

and

$$R(X,Y,Z,W) = \Phi(X,Z,Y,W) - \Phi(X,W,Y,Z) \in A^4(T)$$

is the Riemannian curvature tensor of M.

Tensor E represent *extrinsic curvature*, $E \in S^4(T) \subset S^2(S^2(T))$ and it behaves as an entropy of the embedding, the more the embedding is wrinkled the bigger E gets. Let $f(X) = E(X,X,X,X) = |s(X,X)|^2$, f is homogeneous polynomial of degree 4 and it describes E completely.

There are two good things about tensors Φ and E. The first is that it depends only on elements of T, in particular it does not depend even on the dimension of ambient space which makes it specially useful for studying embeddings of manifolds. Second – direct construction shows that Φ describes the second fundamental form up to an isometric rotation of N, i.e. two second fundamental forms $s_1, s_2 : S^2(T) \to N$ give the same $\Phi \in S^2(S^2(T))$ if and only if there is an isometric rotation $j : N \to N$, such that $j \circ s_1 = s_2$. In particular, since Φ is a sum of Riemannian curvature tensors and E, we have that if (M,g) is a Riemannian manifold and $(M,g) \to \mathbb{R}^q$ is an isometric embedding then E-tensor together with g describes the second fundamental form at each point up to an isometric rotation.

1.D. Positiveness of elements of $S^2(S^2(T))$ and convexity of submanifold.

Most of this subsection is extracted from [Grom]$_{\mathrm{PDR}}$ 2.4.9B(4).

I will need to consider some cones in some tensor spaces, in general if C is an open convex cone in an Euclidean space $\mathbb{R}^n$ set

$$C^* = \{r \in \mathbb{R}^n; \langle r, r' \rangle > 0 \text{ for all } r' \in C\}.$$

Let me start with a definition

Definition 1.2. A tensor $\Phi \in S^2(S^2(T))$ is positive ($\Phi > 0$), if there is a representation $\Phi = \sum_i s_i^2$, where s_i are positively defined quadratic forms on T. If $i : M \to \mathbb{R}^q$ is a smooth embedding we will write $\Phi(i) > 0$ if Φ-tensor of $i(M) \subset \mathbb{R}^q$ is positive at $i(x)$ for all $x \in M$.

The cone of positive tensors in $S^2(S^2(T))$ form a convex $GL(T)$-invariant cone of tensors. If $\dim T \geq 2$ then there are other $GL(T)$ invariant cones in $S^2(S^2(T))$, one of these cones will be of particular interest for me: namely the cone of all elements $\Phi = \sum_i s_i^2$ for arbitrary elements $s_i \in S^2(T)$. This cone describes all elements of $S^2(S^2(T))$ which can appear as Φ-curvature of a submanifold.

Note that existence of the representation of the second fundamental form $s = \sum_i s_i e_i$ with positively defined $s_i \in S^2(T)$ implies, in particular, that $\Phi = \sum_i s_i^2$, i.e. $\Phi > 0$. Since Φ-tensor describes the second fundamental form completely the last property is equivalent to the fact that $M \subset \mathbb{R}^q$ is "stably" locally convex. Namely, if $\Phi > 0$ on M then for some k we have that corresponding submanifold $M \subset \mathbb{R}^q = \mathbb{R}^q \times 0 \subset \mathbb{R}^q \times \mathbb{R}^k$ is locally convex, as a submanifold in $\mathbb{R}^q \times \mathbb{R}^k$ *

1.E. Positiveness of curvature tensor and symmetric 4-tensors.

The space $S^2(S^2(T))$ has two subspaces, the first is $S^4(T) \subset S^2(S^2(T))$ and the second is $A^4_+(T) = S^4(T)^\perp \cap S^2(S^2(T))$ which is canonically isomorphic to space of algebraic curvature tensors $A^4(T) = \Lambda^4(T)^\perp \cap S^2(\Lambda^2(T)$.

$A^4(T)$. Consider cone C_+ which consists of all tensors

$$R(X, Y, Z, T) = \sum_i s_i(X, Z)s_i(Y, T) - s_i(X, T)s_i(Y, Z),$$

where s_i are positive elements of $S^2(T)$. We will call such a the curvature tensor geometrically positive and write $G > 0$ or $G(R) > 0$. (For a Riemannian manifold M we will write $G_p > 0$ if the curvature tensor at $p \in M$ is geometrically positive and $G(M) > 0$ if curvature tensor of M is geometrically positive at all $p \in M$.)

*Given $k \in \mathbb{N}$ it is not hard to construct an example of submanifold $M \subset \mathbb{R}^q$ which is not convex as a submanifold $M \subset \mathbb{R}^q \times 0 \subset \mathbb{R}^q \times \mathbb{R}^k$ but is convex as a submanifold $M \subset \mathbb{R}^q \times 0 \subset \mathbb{R}^q \times \mathbb{R}^{k+1}$

Obviously geometrically positive curvature tensors are exactly those which can be curvature tensors of submanifolds with positive Φ-curvature (or equivalently convex submanifolds, see 1.D), i.e.

$$G(R) > 0 \Leftrightarrow R(X,Y,Z,W) = \Phi(X,Z,Y,W) - \Phi(X,W,Y,Z) \text{ for some } \Phi > 0,$$

and, as you will see, any closed Riemannian manifold M with $G(M) > 0$ admits a smooth isometric embedding $i : M \to \mathbb{R}^q$ with $\Phi(i) > 0$ (in fact, q can be taken equal to $(n+2)(n+5)/2$, see [Grom]$_{\mathrm{PDR}}$ 3.1.5(A) and 3.1.2(C)]). Cone $\bar{G}_+$, the closure of G_+ can also be described as minimal convex $O(T)$-invariant cone which contains curvature tensor of product metric space $S^2 \times \mathbb{R}^{n-2}$.

The dual cone G_+^* (see 1.D) consists of curvature tensors with positive sectional curvature. (Again we will write $K_p > 0$ or $K(R_p) > 0$ meaning that R_p belongs to this cone.)

By the way, the cones G_+ and G_+^* are the smallest and biggest $GL(T)$-invariant cones in $A^4(T)$. Other $GL(T)$-invariant cones will lie between G_+ and G_+^*, in particular the cone of the curvature tensors with positive curvature operator i.e.

$$Q_+ = \{R \in A^4(T) \subset S^2(\Lambda^2(T)); R = \sum_i \phi_i^2 \text{ for } \phi_i \in \Lambda^2\},$$

is one of them and if the dimension is big enough then $Q_+ \neq G_+$. Namely if the dimension is equal 2 or 3 then $G_+ = G_+^*$, in particular, cone $Q_+ = G_+ = G_+^*$. In dimension equal to 4 we have $G_+ = Q_+$ and $G_+^* = Q_+^*$.†

In dimension ≥ 5 all inclusions $G_+ \subset Q_+ \subset Q_+^* \subset G_+^*$ are strict. Indeed: it is obvious that $Q_+ \subset Q_+^*$ is strict, the fact that $G_+ \subset Q_+$ is strict is equivalent to the fact that $Q_+^* \subset G_+^*$ is strict. The last fact was shown by example, see [Zol].‡

$S^4(T)$. Consider cone C_+ which we will call the cone of *positive forms*. It consists of all forms $E \in S^4(T)$ such that $E = \sum s_i^{\circ 2}$, where $s_i^{\circ 2}$ is symmetric square of a positively definite quadratic form s_i. We will write $E > 0$ if $E \in C_+ \subset S^4(T)$. Again, tensor E is positive if it is a symmetric part of a positive Φ-tensor in $S^2(S^2(T))$.§

†By the way this fact gives the Thorpe's characterization of curvature tensors with positive sectional curvature, namely, if M is positively curved 4-manifold then there is a function f on M such that $R_x + f(x)\omega \in S^2(\Lambda^2(T_x))$ is a section of positive quadratic forms on $\Lambda^2(T)$. Here ω denotes the volume form, a section of $\Lambda^4(T) \subset S^2(\Lambda^2(T))$, see [Zol] for details.

‡Note that there is an inaccuracy in [Grom]$_{SGMC}$ *"The closer of this cone [i.e. $\bar{Q}_+$] (given by $Q \geq 0$) can be defined as the minimal closed convex $O(n)$-invariant cone which contains the curvature of the product metric on $S^2 \times \mathbb{R}^{n-2}$ [i.e. $\bar{C}_+$]."...*

§By the way C_+ is also the smallest $GL(T)$-invariant cone in $S^4(T)$. The biggest such cone C_+^* consists of all symmetric 4-form E such that

$$E(X,X,X,X) > 0$$

for any non zero $X \in T$. Gromov in [Grom]$_{\mathrm{PDR}}$ 3.1.4 states that a symmetric form in $E \in S^{2k}(T)$ is positive if and only if corresponding quadratic form $E(S^2(T^k)) \to \mathbb{R}$ is positively defined. In our notations this fact is equivalent to the fact that $C_+ = Q_+$, and this is equivalent to $C_+^* = Q_+^*$. The cone C_+^* is nothing but set of all positively defined forms in $S^{2k}(T)$, equivalently it is set of positively defined homogeneous degree $2k$ polynomials on T. Analogously the cone Q_+^* is the set of homogeneous degree

2. Proofs

I will not give here a proof of the first part of the local theorem by two reasons: first it is real pain to write and read, and second, it is not really mine. The proof I have is just a modification of one unpublished Perelman result. Since this result of Perelman was never published and (as far as I know) was never written, I put it in the Appendix B (It is more fun to look at the original proof than at my compilations).

Proof of the second part of Local Theorem.

Let us prove first that if (M, g) is a Riemannian manifold with $G(M) > 0$ then (M, g) is isometric to a convex submanifold in $\mathbb{R}^q$. This is equivalent to the fact that there is an isometric embedding $i : M \to \mathbb{R}^q$, such that $\Phi(i) > 0$ (see 1.D).

In general, a smooth isometric embedding of (M, g) with $G(M) > 0$ may have undefined Φ-tensor, but there is a way to make it positive.

Consider any smooth free isometric embedding $i : (M, g) \to \mathbb{R}^q$, then by Theorem [Grom]$_{\mathrm{PDR}}$ 3.1.5(A) for any tensor field $E \subset S^4(T)$ such that $E_x > 0$ (see 1.E) at all $x \in M$ one can find a C^1-close isometric embedding $i' : (M, g) \to \mathbb{R}^q$, such that $E(i') = E(i) + E$. In particular, one may choose $E = cg^{\circ 2}$ for any $c > 0$. Since $G(M) > 0$ the number c can be chosen big enough so that at $\Phi(i') > 0$.¶

Now, since $M \subset \mathbb{R}^q$ is a convex submanifold there is an open set $U \subset M$ which is an intersection of open convex hyper-surfaces F_i with angles between each pair of outward normals $> \pi/2$ everywhere on U. Then one can approximate each F_i as a convex polyhedral hyper-surface F_i^ϵ such that the condition on angles will be satisfied. The intersection U^ϵ of all F_i^ϵ is obviously a polyhedral submanifold and it has curvature ≥ 0 (see B). Cutting subdomains from U^ϵ if necessary one gets the needed approximation. □

Proof of Global Theorem. Let us first assume that $T(M)$ is stably trivial. Once we represent our submanifold M as an intersection of (open) convex hyper-surfaces with angles between any pair of outward normals $> \pi/2$ everywhere on M we can repeat the construction from the proof of Local Theorem and it will finish the proof.

$2k$ polynomials on T which can be expressed as sum of squares. Therefore this statement is equivalent to the fact that *each positively defined polynomial is a sum of squares of polynomials*, and this was shown to be wrong in general, namely Hilbert [Hil] showed that this statement is true ONLY in the following three cases: (i) dim $T \leq 2$ and any k, (ii) $k = 1$ and any dim T, (iii) $k = 2$ and dim $T = 3$. This does not effect the rest of the book, except that reader must use (practically always) the C_+-sense for positiveness.

¶There is a simpler way to construct i' with $E(i') = E(i) + E$ for $E = cg \circ g$. First construct an isometric embedding $j : \mathbb{R}^q \to \mathbb{R}^{q'}$, such that $E(j) = ch^{\circ 2}$, where $h = \sum_{i=1}^q (dx_i)^2$ is the unit 2-form on $\mathbb{R}^q$ and then take $i' = j \circ i$. One can construct j on the following way: first choose a collection of linear functions $l_i : \mathbb{R}^q \to \mathbb{R}$ such that $h^{\circ 2} = \sum_i dl_i^4$, then take diagonal of product of the following mappings: a linear mapping $L : \mathbb{R}^q \to \mathbb{R}^q$ and twists $\tau_i : \mathbb{R}^q \to \mathbb{R}^2$, $\tau_i(x) = \big(a_i \sin(b_i l_i(x)), a_i \cos(b_i l_i(x))\big)$ with appropriately chosen L, a_i and b_i.

Existence of such a representation is obviously equivalent to the existence of a smooth section of orthonormal bases $\{e_i\}$ in $N(M)$ such that $s = \sum_i s_i e_i$ with positively defined $s_i \in S^2(T)$.

Consider a cover U_k, $k \in \{1, 2, .., n\}$ of M such that on each U_k there is a smooth section of orthonormal bases $\{e_{i,k}\} \subset N(M)$ with the above properties. Since $T(M)$ is stably trivial we can assume that $N(M)$ is a trivial bundle. Therefore we can extend these bases to all M, and get n bases $\{e_{i,k}\}$ for all $N(M)$. Therefore at each point we have an isometric rotation $E_{k,k'} \in O(q-m)$ which sends $\{e_{i,k}\}$ to $\{e_{i,k'}\}$. Without loss of generality we can assume that corresponding mapping $E_{k,k'} : M \to O(q-m)$ is homotopic to a trivial one. Now let us take a smooth partition of unit $u_k : M \to [0,1]$, $u_k|_{M\setminus U_k} \equiv 0$ and $\sum_k u_k(x) \equiv 1$ for all $x \in M$. At each point $x \in U_k \subset M$ we have $\Phi_x \equiv \sum_i s_{i,k}^2$. Therefore for each $x \in M$ we have $\Phi_x \equiv \sum_{i,k} u_k(x) s_{i,k}^2$. Consider $nN(M) = N_1(M) \oplus N_2(M) \oplus ... \oplus N_n(M)$, the sum of n copies of the normal bundle, take the basis $\{e_{i,k}\}$ for N_k, and consider the subbundle $N_\Delta(M)$, which is spanned by $(\sqrt{u}_1 e_{i,1}, \sqrt{u}_2 E_{12} e_{i,1}, ..., \sqrt{u}_n E_{1n} e_{i,1})$. It is obviously a trivial subbundle with trivial orthogonal subbundle. Therefore, if one considers $\mathbb{R}^{(n-1)(q-m)} \times N(M)$ then there is a bundle isomorphism $i : nN(M) \to \mathbb{R}^{(n-1)(q-m)} \times N(M)$ which is an isometry on each fiber, which sends $N_\Delta(M)$ to $N(M)$ and moreover if $p_\Delta : N^n \to N_\Delta$ is the orthogonal projection then $i \circ p_\Delta(e_{i,k}) = \sqrt{u}_k e_{i,k}$.

Therefore, we get a smooth section of orthonormal bases $\{e_{i,k}\} \subset N'(M) = \mathbb{R}^{(n-1)dim(N_x)} \times N(M)$, i.e. if we had $N(M)$ as a normal bundle of $M \subset \mathbb{R}^q$ then $N'(M)$ is a normal bundle of $M \subset \mathbb{R}^q \times 0 \subset \mathbb{R}^q \times \mathbb{R}^{(n-1)(q-m)}$. Now for each pair of indexes i, k we have a nonnegative quadratic form $s_{i,k} = \langle s, e_{i,k} \rangle$, $\Phi_x \equiv \sum_{i,k} u_k(x) s_{i,k}^2$ and at each point we have at least one quadratic form which is strictly positive. It is not hard to rotate basis $e_{i,k}$ a little to get a new smooth section of bases in $N'(M)$ with representation $s = \sum_{i,k} e_{i,k}(s'_{i,k})$ where each $s'_{i,k}$ is strictly positive.

If $T(M)$ is not stably trivial one still can find an embedding $M \to \mathbb{R}^q$ which has positive Φ-curvature at each point. Take a small tubular neighborhood U of M. Let $\tilde{M}$ be a finite cover of M such that $T(\tilde{M})$ is stably trivial. We can assume that $N(\tilde{M})$ is trivial, therefore $N(M)$ is equivalent to a flat bundle. From the above we get existence of flat bundle $U' \to U$ such that the new induced normal bundle of M with respect to U' is trivial. U' is an open flat manifold and that makes possible to repeat the same construction as above. □

Problem section

The opposite question, i.e. which polyhedral metrics could be smoothed to a Riemannian manifold with positive curvature, is still open. All examples I know so far meet the following

Conjecture. *Any polyhedral metric with curvature $\geq k$ can be smoothed in to a Riemannian orbifold with geometric curvature $\geq k - \epsilon$.*

In case of dimension equal to 2 the above Conjecture is a trivial corollary of Alexandrov's embedding theorem [Al]. In dimension in dimension 3, one can smooth each vertex using the same Alexandrov's embedding theorem (this time we need embedding of surface of curvature ≥ 1 into S^3) and one can do the smoothing in such a way that the only singular points left will be "midpoints of edges" and these singular points are conic. Then one can smooth the remaining points in the same way.

The conjecture would imply, in particular, that any simply connected manifold with positive geometric curvature is diffeomorphic to a sphere. Indeed the Corollary 0.3 implies that if M is a Riemannian manifold with $G(M) \geq 1$ then it can be approximated by polyhedral spaces X_n with curvature ≥ 1. Therefore the spherical suspension $\Sigma(M)$ is approximated by $\Sigma(X_n)$ (cf. [GW]). Now from the conjecture it would follow that $\Sigma(X_n)$ is smoothable into Riemannian orbifold and therefore M is a quotient of a sphere.

Another question is whether the condition of stably trivial tangent bundle can be removed from Theorem 0.2. So far, I can not even construct an approximation of $(\mathbb{C}P^2, \mathrm{can})$ by polyhedral metrics with curvature $\geq -\epsilon$.

One may ask whether it is possible to construct an approximation of $(\mathbb{C}P^2, \mathrm{can})$ by polyhedral metrics with curvature ≥ 0. This is already a rigid question, in particular, from Cheeger's results [Ch] it is easy to see that any nonnegatively curved polyhedral metric on $\mathbb{C}P^2$ carries complex structure. As it was pointed out by Dmitry Panov $\mathbb{C}P^2$ carries polyhedral metrics with curvature ≥ 0 [for example, take nonnegatively curved polyhedral metric on S^2 then the space of all pairs of points in S^2 homeomorphic to $\mathbb{C}P^2$ and naturally comes with nonnegatively curved polyhedral metric] although it is not clear whether such metrics can approximate canonical metrics on $\mathbb{C}P^2$, (see [Pan] for more examples and general discussion of polyhedral spaces with complex structure).

Is there any way to generalize Alexandrov embedding theorem? For example, is it possible to characterize Riemannian manifolds which are isometric to a complete convex hyper-surface in a complete convex hyper-surface in ... in $\mathbb{R}^q$? Is it true that any simply connected Riemannian manifold with $G > 0$ is isometric to one of those? Again, if this manifold is compact it would immediately give that any such manifold is diffeomorphic to standard sphere.

Appendix A: Example of positive curvature tensor which is not geometrically positive

Here I present calculations of J.Eschenburg, which show that curvature tensor R of $SU(3)$ with bi-invariant metric has non-negative curvature operator but it is not true that $G \geq 0$. This gives an example for dimension ≥ 8, from the work of Zoltek [Zol] it follows that such examples exist for dimension ≥ 5 but the calculations below are much simpler and I hope that it will be useful for a reader who wants to quickly convince himself that such monsters do live.

Consider Lie algebra $su(3)$, and adjoint representation ad: $su(3) \to \Lambda^2(su(3))$. The curvature operator of $SU(3)$ with bi-invariant metric has curvature operator $R : \Lambda^2(su(3)) \to \Lambda^2(su(3))$ which coincide with projection on $\mathrm{Im}(ad)$.

Now if one can prove that $Im(ad)$ has no simple bi-vector inside then it will follow that curvature operator of $SU(3)$ with bi-invariant metric is not geometrically positive.

Therefore we only have to show that if $0 \neq x \in su(3)$ then $ad_x \in \Lambda^2(su(3))$ is not a simple bi-vector, i.e $\neq v \wedge w$.

It is sufficient to prove it for ad_x, where x is tangent to a maximal torus of diagonal elements in a matrix representation. Therefore in the matrix representation it looks like $x = diag\{ai, bi, ci\}$ with $a + b + c = 0$. Take the standard real basis in $su(3)$, which comes from matrix form, i.e. take $A_1 = diag\{i, 0, -i\}, A_2 = diag\{0, i, -i\}$, take $F_1 = e_2 \wedge e_3, F_2 = e_3 \wedge e_1, F_3 = e_1 \wedge e_2$ be real and $E_1 = ie_2 \circ e_3, E_2 = ie_3 \circ e_1, E_3 = ie_1 \circ e_2$ imaginary parts of basis. Here e_1, e_2, e_3 is a basis of $\mathbb{C}^3$ where $SU(3)$ acts.

Then by direct calculation we have $\mathrm{ad}_x = (c-b)F_1 \wedge E_1 + (a-c)F_2 \wedge E_2 + (b-a)F_3 \wedge E_3$. Now the fact that bi-vector $\phi \in \Lambda^2(T)$ is simple is equivalent to $\phi \wedge \phi = 0$, and

$$ad_x \wedge ad_x =$$

$$= (c-b)(a-c)F_1 \wedge E_1 \wedge F_2 \wedge E_2 + (a-c)(b-a)F_2 \wedge E_2 \wedge F_3 \wedge E_3 +$$

$$+ (b-a)(c-b)F_3 \wedge E_3 \wedge F_1 \wedge E_1$$

Therefore if ad_x is simple then at least two of numbers $(c-b), (a-c), (b-a)$ are zeros and since $a + b + c = 0$ we have that $a = b = c = 0$, i.e. $x = 0$. □

Appendix B: A theorem of Perelman and why I need it.

In this appendix I will present the proof of an unpublished result of G.Perelman, on whose idea I build the proof of the first part of the Local Theorem.

Let M be an Alexandrov m-space and $U \subset M$ be an open subset. Let $F : U \to \mathbb{R}^m$ be a chart $F(p) = (x_1(p), x_2(p), ..., x_m(p))$. We say that F is convex if each of the co-ordinate functions x_i is convex. The proof of the following claim easily follows from Proposition 3 [Per]

B.1 Claim. *Let g be a convex function on U and for some convex chart $F : U \to \mathbb{R}^m$ we have $\partial g/\partial x_i < 0$ then $g \circ F^{-1}$ is a convex function on $F(U)$. Moreover for any $p \in U$ and $v \in T_p$ we have*

$$\nabla_v^2 g \leq \nabla_{dF(v)}^2 \big(g \circ F^{-1}\big).$$

In particular, if S is a level surface of g and F be a short chart then for any $p \in S$

$$II_S(X,X) \leq II_{F(S)}(dF(X), dF(X)).$$

B.2 Definition. *Let $X_n \overset{GH}{\to} X$ be a converging sequence of metric spaces and $f_n : X_n \to X$ be corresponding sequence of Hausdorff approximations. We say that a sequence of measures μ_n on X_n weakly converge to measure μ on X if for any continuous function α with compact support on X we have $\int_{X_n} \alpha \circ f_n d\mu_n \to \int_X \alpha d\mu$.*

The proof of the following result I heard from G. Perelman about seven years ago. The proof below should be close to the original but some ideas might differ.

B.3 Theorem. *Let M_n be a sequence of Riemannian m-manifolds with curvature $\geq k$ which Lipschitz converge to a closed Riemannian manifold M. Then scalar curvature on M_n converges weakly to the scalar curvature on M. (i.e. $Sc_{g_n} dv_{g_n}$ converges weakly to $Sc_g dv_g$).*

Let me note that if one has no lower bound for curvature then there are examples when limit of scalar curvatures is smaller than scalar curvature of the limit, and it is unknown whether it could also be bigger.

Let us prepare the following Lemma (which is in fact a partial case of the theorem):

B.4 Lemma. *Let F_n be a sequence of smooth convex hyper-surfaces in $\mathbb{R}^{m+1}$ which Hausdorff converges to a smooth convex hyper-surface F. Let $Sc(F)$ and $Sc(F_n)$ denote scalar curvatures of F and F_n and $h(F)$, $h(F_n)$ denote m-Hausdorff measure of the corresponding hyper-surface. Then $Sc(F_n)dh(F_n)$ converges weakly to $Sc(F)dh(F)$*

Proof of Lemma B.4. Let α be a continuous function with compact support in $\mathbb{R}^{m+1}$. Let us denote by $C_r(F)$ the set of points in $\mathbb{R}^{m+1}$ which lie on outgoing normal rays to the hyper-surface F on the distance $< r$ to the hyper-surface. Let us define $\alpha_F : C_\infty(F) \to \mathbb{R}$, $\alpha_F(x) = \alpha(y)$ where $y \in F$ is a closest point on the hyper-surface.

Now, it is well known and easy to see that $\int_{C_r(F)} \alpha_F dv$ is a polynomial of degree m on r, moreover, the quadratic term is exactly $r^2 \int_F \alpha Sc(F) dh(F)$

Now if we have $F_n \to F$, then $C_r(F_n)$ converge to $C_r(F)$ and α_{F_n} converge to α_F. Therefore, $\int_{C_r(F_n)} \alpha_{F_n} dv \to \int_{C_r(F)} \alpha_F dv$ and the coefficient with r^2 of corresponding polynomials also converges. That finishes the proof. □

Proof of Theorem B.3. We first want to construct special distance-like charts in a neighborhood of any point in M together with some nice approximating charts on M_n.

B.5 Lemma. *Given $p \in M$, $v \in T^*_p(M)$ and $\epsilon > 0$ there is $\delta > 0$ and sequence $M_n \ni p_n \to p \in M$ and sequence of convex functions $f_n : B_\delta(p_n) \subset M_n \to \mathbb{R}$ which converges to a convex function $f : B_\delta(p) \subset M \to \mathbb{R}$ such that $d_p f = v$, $|f''| < \epsilon$ everywhere on $B_\delta(p)$.*

Proof of Lemma B.5. Consider an orthonormal basis $\{e_i\}$ in $T_p(M)$ such that $\sum_i e_i = cv$. Take $r > 0$ an let $a_i = \exp_p re_i$. Now $f = \sum_i \phi \circ \mathrm{dist}_{a_i}$, where $\phi(x) = \alpha \log x - \beta x^2$ if dim $(M) = 2$ and $\phi(x) = \alpha \frac{1}{x^{n-2}} - \beta x^2$ if dim $(M) > 2$. The same arguments as in [PP] 4.3 show that f satisfy the conditions in the Lemma for appropriately chosen α and β, in a small ball $B_\delta(p)$.

Now to construct an approximation of this function construct a sequence $a_{i,n} \to a_i$ for each i and take $f_n = \sum_i \phi \circ \mathrm{dist}_{a_{i,n}}$. Again the same reasoning as in [PP] 4.3 proves that there is $\epsilon > 0$ such that for large n the function is convex in a δ-neighborhood of p_n. $\square$

Now we may take any orthonormal basis $v_i \subset T^*_p(M)$ and construct a function f_i : $B_\delta(p)$ together with an approximations $f_{i,n} : B_\delta(p_n)$. In addition to the above properties these functions will be almost orthogonal for a small enough δ, i.e. one can assume that angle between level surfaces lies in between $\pi/2 \pm \epsilon$.

Now we start the induction by dimension, we can take dim $= 2$ as a base, in which case convergence follows from Gauss-Bonnet formula. Now assume we have already proved it for all dimensions $< m$.

To save space/time on the notation let us agree that extra index n will always denote corresponding babe for M_n.

Let $p \in M$ and $S_1, S_2, ..., S_m$ be one-parameter families of co-ordinate surfaces $f_i = c$. Let us denote by Sc_i is the "scalar" curvature of directions tangent to S_i, in other words $Sc_i = Sc - Ricc(u_i)$ where u_i is unit vector field normal to S_i. Note that from lower curvature bound we have $|\mathrm{Rm}| < c_1 + c_2 Sc$ and therefore $(1+\alpha)(m-2)Sc = Sc_1 + Sc_2 + ... + Sc_m$ where α depend on angles between these co-ordinate surfaces and $\alpha \to 0$ as all these angles converge to $\pi/2$, in particular as $\epsilon \to 0$.

Let $Sc(S_i)$ be the scalar curvature of the intrinsic metric of the corresponding co-ordinate surface. Since the Jacobian of our charts converges to the Jacobian of the limit chart from the induction hypothesis we have $Sc(S_{i,n})dv_{g_n}$ converges weakly to $Sc(S_i)dv_g$.

From Gauss formula we have $Sc_i + G(S_i) = Sc(S_i)$, where $G(S_i) = \sum_{i \neq j} k_i k_j$ where k_i are the principal curvatures of S_i. Since each S_i is convex

$$Sc_i \leq Sc(S_i) \geq G(S_i).$$

Therefore, after passing to a subsequence $Sc_{i,n}$ should converge weakly to some $\bar{S}c_i \leq Sc(S_i) \leq Sc_i + n(n-1)\epsilon^2$.

Now let us prove the following lower bound: $\bar{S}c_i \geq Sc_i - C\epsilon^2$ for some $C = C(m)$.

The inequality above also insures that after passing to a subsequence $G(S_{i,n})$ converges to some measure $\bar{G}(S_i)$ and obviously $\bar{S}c_i = Sc(S_i) - \bar{G}(S_i) \geq Sc_i - \bar{G}(S_i)$ therefore it is enough to show that $\bar{G}(S_i) \leq C\epsilon^2$ for some fixed C.

To give this last estimate let us construct a new chart similar to the one before, $H = (h_1, h_2, ..., h_m)$ with the approximations $H_n = (h_{1,n}, h_{2,n}, ..., h_{m,n})$ such that $\partial f/\partial h_i < -1/10m$. From the Claim above we have that $G(S_n) \leq cG(H(S_n))$ as well as $G(S) \leq cG(H(S)) \leq C\epsilon^2$

Now $H(S_n)$ converges to $H(S)$ as convex hyper-surfaces in $\mathbb{R}^m$ and applying Lemma we get $G(H(S_n))$ converges weakly to $G(H(S))$.

Since for any $\epsilon > 0$ there is a finite covering of M by charts as in Lemma B.5 we obtain the Theorem. □

Along the same lines one can prove stronger statements:

Smooth Proposition. *Let (M_n, g_n) be a sequence of Riemannian m-manifolds with curvature $\geq \kappa$ which GH-converges to a Riemannian manifold (M, g) of the same dimension $= m$. Then there is a sequence of reparametrizations (diffeomorphisms) $f_n : M \to M_n$, such that curvature tensor of $df_n^*(g_n)$ weakly converges to the curvature tensor of g on M.*

Corollary. *Let R be an $SO(T)$ invariant convex set in $A^4(T)$. Assume that there is a lower bound $\kappa > -\infty$ for sectional curvature in R. Let M_n be a sequence of Riemannian manifolds with curvature tensor from R at each point which converges to a Riemannian manifold M of the same dimension. Then the curvature tensor at any point of M is from R.*

For example, smooth limit of manifolds with positive curvature operator must have positive curvature operator.

Note that this corollary can not hold for general $SO(T)$ invariant convex set in $A^4(T)$, for example as it shown in [Loh], [Loh] it is not true for sets $R = \{r \in A^4 : Ricci(r) \leq c\}$ curvature and for $R' = \{r \in A^4 : c \leq Sc(r) \leq c + \epsilon\}$. Although I believe it should be still true with much more relaxed limitations on R.

Finally one can give a singular version of this result which we need in our paper:

First let me describe the singular curvature tensor of a polyhedral, assume we have a $1 \pm \epsilon$-bi-Lipschitz parametrization of polyhedral P by smooth Riemannian manifold $f : M \to P$, such that f^{-1} is smooth on each simplex. One can think about P as (M, d) where d is a singular metric. Now let us define the curvature tensor of d as following: its support is the image of $(n-2)$-skeleton of P and on each $(n-2)$ simplex it is defined as $h_{n-2}(2\pi - \omega)\alpha$ where h_{n-2} is the Hausdorff measure of this image, ω is the total angle around this simplex Δ and $\alpha = dx \wedge dy$ is a bi-vector field with the following properties: $|\alpha| = 1$ everywhere on the image of simplex and $\alpha|_{f^{-1}(\Delta)} = 0$.

Singular Proposition. *Let P_n be a polyhedral m-spaces with curvature $\geq \kappa$ which GH-converges to a Riemannian manifold (M, g) of the same dimension $= m$. Then there is a sequence of smooth parametrizations $f_n : M \to P_n$, such that the described singular curvature tensor weakly converges to the curvature tensor of g on M.*

As in the corollary above, since the described curvature tensor on (M, d_n) is geometrically positive we get that the curvature tensor on the limit (M, g) is geometrically positive and that proves the first part of the Local Theorem 0.1.

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