Turk J Math
27 (2003) , 273 – 281.

Boundary Points of Self-Affine Sets in $\mathbb{R}$

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Abstract

Let A be an $n\times n$ expanding matrix with integer entries and $D = \{0, d_1, \cdots , d_{N-1}\} \subseteq \mathbb{Z}^n$ be a set of N distinct vectors, called an N-digit set. The unique non-empty compact set $T = T(A, D)$ satisfying $AT = T + D$ is called a *self-affine set.* If T has positive Lebesgue measure, it is called a *self-affine region.* In general, it is not clear how to determine a point to be on the boundary of a self-affine region. In this note, we consider one-dimensional self-affine regions T and present a simple approach to get increasing subsets of the boundary of T. This approach also gives a characterization of strict product-form digit sets introduced by Odlyzko.

Key Words: Self-affine sets, boundary points, strict product-form digits.

1. Introduction

Let $M_n(\mathbb{Z})$ denote the set of $n \times n$ matrices with entries in $\mathbb{Z}$, and let $A \in M_n(\mathbb{Z})$ be expanding, that is, each eigenvalue of A has modulus > 1. Suppose $|\det A| = q$, we let $D = \{d_0 = 0, d_1, \cdots , d_{N-1}\} \subseteq \mathbb{Z}^n$ be a set of N distinct vectors, and call it an N-*digit set.* It is well known that there exists a unique compact set $T = T(A, D)$, called an *(integral) self-affine set*, satisfying the set-valued equation

$$T = A^{-1} \bigcup_{d\in D} (T + d)$$

2000 *Mathematics Subject Classification:* Primary 52C22

(see e.g. [1]). The family $\{S_i(x) = A^{-1}(x + d_i)\}_{i=0}^{N-1}$ is called an *iterated function system* (IFS). The set T can be expressed explicitly by

$$T = \big\{ \sum_{i=1}^{\infty} A^{-i} x_i : \; x_i \in \mathcal{D} \big\}. \tag{1.1}$$

T is called an *(integral) self-affine region* if T has nonvoid interior. A self-affine region is called an *(integral) self-affine tile* if $N = q$. If T is a self-affine region, then it is equal to the closure of its interior, i.e., $T = \overline{T^o}$ and the boundary ∂T of T has Lebesgue measure zero. This fact was first proved in [5] for self-affine tiles and it was later generalized to self-affine regions in [3]. For such a tile, there exists a subset $\mathcal{J} \subseteq \mathbb{Z}^n$, such that

$$T + \mathcal{J} = \mathbb{R}^n \quad \text{and} \quad (T + t)^o \bigcap (T + t')^o = \emptyset, \quad t \neq t', \; t, t' \in \mathcal{J}.$$

In wavelet theory, self-affine regions are the supports of scaling functions [2]

$$f(x) = \sum_{d_i \in D} c_i f(Ax - d_i).$$

In number theory, self-affine tiles give the generalized decimal expansions. Recently, the geometric and algebraic properties of self-affine tiles have been studied extensively in literature. However, the knowledge on the identification of boundary points is still limited. In this note, the emphasis will be on the identification of boundary points.

If $T^o = \emptyset$, then $T = \partial T$ and there is little to study about the boundary points. Hence we only need to study self-affine regions. We note that, in general, it is a complicated problem to characterize D so that $T(A, D)$ is a self-affine region as mentioned in [6] and [7]. In general, even in the one-dimensional case, it is not known which pair (q, D) makes the set $T(q, D) \subseteq \mathbb{R}$ a tile. When q is a prime power, the problem is solved in [6]. If $N > q$, the problem is even more difficult, as has been stated by Odlyzko [7]: "the task of classifying them seems hopeless."

We use the convex hull of T, denoted by K, to approximate the boundary of T. In this case, we apply the IFS defining T to K successively. This approach is elementary and simple in one-dimensional case. In particular, we use it to find the boundary points of disconnected tiles in $\mathbb{R}$ which are infinite unions of intervals. It also gives a characterization of strict product-form digits. But it is not clear how to find the initial set for the iteration in higher dimensions.

In section 2, we present the method to find increasing subsets of the boundary of self-affine tiles. We illustrate the idea in examples. In section 3, we discuss the difficulties arising in the use of this method in higher dimensions.

2. Getting the boundary points: simple arithmetic in the service of self-affine sets

Throughout the section, T will denote a one-dimensional set. Let $q \geq 2$ be an integer, and $D = \{0, d_1, d_2, ..., d_{q-1}\} \subset \mathbb{Z}$, $0 < d_1 < d_2 < \cdots < d_{q-1}$, be a q-digit set. Let $T := T(q, D)$ be the self-affine tile defined by q and D. Then we have the following result.

Proposition 2.1 [4] *The following two statements holds true and are equivalent.*
(i) T is a connected tile if and only if $D = \{0, a, 2a, \cdots, (q-1)a\}$ for some $a > 0$.
(ii) T is a connected tile if and only if $T = [0, a]$ for some $a > 0$.

The above proposition shows that disconnected tiles are of interest. Suppose that E gives a complete set of coset representatives of Z_q, and suppose that it has a factorization $E = E_1 + E_2 + ... + E_r$, $|E| = |E_1||E_2|...|E_r| = q$, where $0 \in E_i$ for all i. Then for any integers $0 \leq f(1) \leq f(2) \leq ... \leq f(r)$, set

$$D = q^{f(1)}(E_1) + q^{f(2)}(E_2) + \cdots + q^{f(r)}(E_r).$$

Then D is called a *product-form digit set.* If we have the extra condition that $E = \{0, 1, 2, ..., q-1\}$, D is called a *strict product-form digit set* [6]. For such digit sets, the following result is well-known.

Theorem 2.2 [6], [7] *An integral self-affine tile T(A,D) is a finite union of intervals if and only if $D = jD'$, where $j \geq 1$ is an integer and D' a strict product-form digit set.*

Remark. For $N > q$, there are no results in the literature concerning conditions as in Theorem 2.2.

It follows from Theorem 2.2 that, for a prime number q, any q-digit set of strict-product form will be of the form $D = \{0, a, 2a, \cdots, (q-1)a\}$ for some $a > 0$. We will see in the examples below that for q=4, $D = \{0, 1, 8, 9\}$ is a strict product-form digit set and $T(4, D) = [0, 1] \cup [2, 3]$ using our technique.

Thus the most interesting case is when $T(A,D)$ is an infinite union of disjoint intervals. For q=4, $D=\{0,1,8,25\}, T(4,D)$ is not of strict product form and consists of an infinite number of intervals [6]. Using our technique, we will obtain boundary points of $T(4,D)$ in the examples.

Let $\mathcal{C}_n$ denote the space of all non-empty compact subsets of $\mathbb{R}^n$. Let $||\cdot||$ be a norm on $\mathbb{R}^n$. We define the *Hausdorff metric* on $\mathcal{C}_n$ with respect to $||\cdot||$ by

$$d_H(D,D') := \max\{\sup_{x\in D}\inf_{x'\in D'}\|x-x'\|\,,\ \sup_{y'\in D'}\inf_{y\in D}\|y-y'\|\}$$

It is well known that $(\mathcal{C}_n, d_H)$ is a complete metric space. In this paper, the convergence of sequences of compact sets will be with respect to the Hausdorff metric.

The way we approximate the boundary of T will be independent of the connectedness. For this, we consider the convex hull of T, which is of course a closed set in dimension 1, the simplest case. We denote it by K. We let

$$S_j = \frac{1}{q}(x+d_j),\ \ j=0,1,2,...,N-1, \text{where } d_0=0,$$

and for $J=(j_1,...,j_l)$, we let $S_J = S_{j_1}\circ ...\circ S_{j_l}$. Then $S_j(K)\subseteq K$, $j=0,1,2,...,N-1$. We let $K_0=K$ and $K_l=\cup_{|J|=l}S_J(K_0)$ for $l\geq 1$. Hence $K_{l+1}\subseteq K_l$ and

$$T(q,D)=T=\cup_{j=0}^{N-1}S_j(T)=\cap_{l=1}^{\infty}K_l=\lim_{l\to\infty}K_l.$$

If we choose K_0 to be any non-empty compact set, we still have $T(q,D)=T=\lim_{l\to\infty}K_l$ [1]. But we don't necessarily have $T\subseteq K_l$ or $\partial K_l\subseteq \partial T$ (see Section 3). If T has an empty interior, then we can take $K_0=\{0\}\subseteq\partial T$ and $K_l\subseteq\partial T=T$.

We denote the boundary of T by ∂T. We set $\partial_0 T=\partial K_0=\{0, d_{N-1}/(q-1)\}\subseteq\partial T$ (by (1.1)) and $\partial_l T=\partial K_l\subseteq\partial T$ for $l\geq 1$. Letting $I_l=K\setminus K_l$, we see that I_l is increasing in l and $K\setminus T=\cup_{l=1}^{\infty}I_l=\lim_{l\to\infty}I_l$. We note that I_l consists of a finite number of open intervals. Therefore, $\partial_l T=\partial_0 T\cup\partial I_l$ is increasing in l.

We call an increasing sequence of sets $\{U_l\}_{l=1}^{\infty}$ (i.e. $U_l\subseteq U_{l+1}$ for all l) *finitely increasing* if $U_m=U_{m+1}=U_{m+2}=\cdots$ for some $m\in\mathbb{N}$. Therefore, using Theorem 2.2, we have the following characterization of strict product-form digit sets.

Theorem 2.3 *Let $T \subseteq \mathbb{R}$ be a self-affine region. Then*

(i) $\partial T = \lim_{l\to\infty} \partial_l T$ holds true. Here, the convergence is with respect to the Hausdorff metric.

(ii) A self-affine region T is a finite union of intervals if and only if $\partial_l T$ is finitely increasing (or K_l is finitely decreasing). Moreover, in such a case, we can find T explicitly and if T is a tile, then D is a strict product-form digit set.

Proof. (i) We first note that the limit $\lim_{l\to\infty} \partial_l T$ exists since $T = \lim_{l\to\infty} K_l$. It is easy to see that $\partial K_l \subseteq \partial T$. Conversely, every point x of ∂T is a limit of a sequence of points in ∂K_l since $T = \overline{T^o}$ and T^o is non-empty and $T = \lim_{l\to\infty} K_l$. Since $(\mathcal{C}_n, d_H)$ is a complete metric space, $\lim_{l\to\infty} \partial_l T$ is a compact set. Thus $x \in \lim_{l\to\infty} \partial_l T$ so that $\partial T \subseteq \partial K_l$. Therefore, $\partial T = \lim_{l\to\infty} \partial_l T$.
(ii) The proof follows from the observations before the statement of the theorem. □

Although the above theorem is very simple, it is quite useful. In practice, Theorem 2.3 tells us how to get as many points of ∂T as we like. Not only that, but it also characterizes the strict product-form digits. In the first example below, the tile T is an infinite union of intervals as mentioned above. However, the self-affine region T in the second example, which contains the tile of the first example, is a union of two intervals.

Example 1. Let $q = 3$, $D = \{0, 1, 5\}$. Here $S_0(x) = x/3$, $S_1(x) = (x + 1)/3$, $S_2(x) = (x + 5)/3$. Then $K = K_0 = [0, 5/2]$,

$$\begin{aligned} K_1 &= S_0(K) \cup S_1(K) \cup S_2(K) \\ &= [0, 5/6] \cup [1/3, 7/6] \cup [5/3, 15/6] \\ &= [0, 7/6] \cup [5/3, 5/2] \end{aligned}$$

and

$$\begin{aligned} K_2 &= S_0(K_1) \cup S_1(K_1) \cup S_2(K_1) \\ &= [0, 7/18] \cup [5/9, 5/6] \cup [1/3, 13/18] \cup [8/9, 7/6] \\ &\quad \cup [5/3, 37/18] \cup [20/9, 15/6] \\ &= [0, 5/6] \cup [8/9, 7/6] \cup [5/3, 37/18] \cup [20/9, 15/6]. \end{aligned}$$

Hence $\partial_1 T = \{0, 7/6, 5/3, 5/2\}$, $\partial_2 T = \{0, 5/6, 8/9, 7/6, 5/3, 37/18, 20/9, 5/2\}$.

Example 2. Let $q = 3$, $D = \{0, 1, 5, 6\}$. Let $S_0(x) = x/3$, $S_1(x) = (x + 1)/3$, $S_2(x) = (x + 5)/3$, $S_3(x) = (x + 6)/3$. Then $K = K_0 = [0, 3]$

$$\begin{aligned} K_1 &= S_0(K) \cup S_1(K) \cup S_2(K) \cup S_3(K) \\ &= [0, 1] \cup [1/3, 4/3] \cup [5/3, 8/3] \cup [2, 3] \\ &= [0, 4/3] \cup [5/3, 3] \end{aligned}$$

and

$$\begin{aligned} K_2 &= S_0(K_1) \cup S_1(K_1) \cup S_2(K_1) \cup S_3(K_1) \\ &= [0, 4/9] \cup [5/9, 1] \cup [1/3, 7/9] \cup [8/9, 4/3] \\ &\quad \cup [5/3, 19/9] \cup [20/9, 8/3] \cup [2, 2 + 4/3] \cup [2 + 5/9, 3]. \end{aligned}$$

Hence $T(3, D) = K_1 = K_2 = [0, 4/3] \cup [5/3, 3]$ and $\partial T = \partial_1 T = \partial_2 T = \{0, 4/3, 5/3, 3\}$.

For an integral self-affine tile T, it is known that the Lebesgue measure of T is an integer [6]. The second example is given [3] to illustrate that the Lebesgue measure of a self-affine region T does not have to be an integer. But it is known to be a rational number [3]. We included Example 2.2 to illustrate that our method works for self-affine regions which are not tiles. For the next two examples, we consider 4-digit tiles. In the third example, we apply Theorem 2.3 to a disconnected tile to show that it is a finite union of intervals (hence its digit set is in strict-product form). In the fourth example, we obtain boundary points of a tile which is known to be an infinite union of pairwise disjoint intervals; but it is not of product form [3], [6].

Example 3. Let $q = 4$, $D = \{0, 1, 8, 9\}$. We set $S_0(x) = x/4$, $S_1(x) = (x + 1)/4$, $S_2(x) = (x + 8)/4$, $S_3(x) = (x + 9)/4$ and $K_0 = K = [0, 3]$. By computation, we get

$$\begin{aligned} K_1 &= S_0(K) \cup S_1(K) \cup S_2(K) \cup S_3(K) \\ &= [0, 1] \cup [2, 3] \\ &= S_0(K_1) \cup S_1(K_1) \cup S_2(K_1) \cup S_3(K_1) \\ &= K_2. \end{aligned}$$

Hence $[0,1]\cup[2,3]=T(4,D)=K_1=K_2=...$ and $\{0,1,2,3\}=\partial T=\partial_1 T=\partial_2 T=...$, i.e., $\partial_l T$ is finitely increasing. Therefore, D is a strict product-form digit set by Theorem 2.3.

Example 4. Let $q=4$, $D=\{0,1,8,25\}$. Here $S_0(x)=x/4$, $S_1(x)=(x+1)/4$, $S_2(x)=(x+8)/4$, $S_3(x)=(x+25)/4$ and $K_0=K=[0,25/3]$. A tedious computation gives

$$\begin{aligned} K_1 &= S_0(K)\cup S_1(K)\cup S_2(K)\cup S_3(K) \\ &= [0,49/12]\cup[25/4,25/3]. \end{aligned}$$

Hence $\partial_1 T=\partial K_1=\{0,49/12,25/4,25/3\}\subseteq \partial T$.

$$\begin{aligned} K_2 &= S_0(K_1)\cup S_1(K_1)\cup S_2(K_1)\cup S_3(K_1) \\ &= [0,61/48]\cup[25/16,2+49/48]\cup[2+25/16,2+25/12] \\ &\quad \cup[6+1/4,6+61/48]\cup[6+29/16,6+7/3]. \end{aligned}$$

Hence $\partial_1 T\subseteq \partial_2 T=\partial K_2=\{0,61/48,25/16,2+49/48,2+25/16,2+25/12,6+1/4,6+61/48,6+29/16,6+7/3\}\subseteq \partial T$ and $\partial T=\lim_{l\to\infty}\partial_l T$.

3. Discussion

We first note that the method of the previous section also applies to self-affine sets T with $D=\{0,d_1,\cdots,d_{N-1}\}\subseteq\mathbb{R}$. As the title of this paper suggests, the identification of the boundary points of T in higher dimensions is not the subject of this paper. However, we like to talk about complication occurring in higher dimensions. For $n\geq 2$, we do not necessarily have $\partial K_l\subseteq\partial T$. For example, if $A\in M_2(\mathbb{Z})$ is a similarity, then ∂K is a polygon (polytope for $n\geq 2$) when A^m is a multiple of the identity for some m; otherwise, in the generic case, ∂K is a C^1 curve [8].

For a concrete example, we let $A=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D=\{(0,0),(1,0)\}$. Then T is the well-known twin dragon tile. For this tile, it is known that ∂T is a Jordan curve and has Hausdorff dimension ≈ 1.523627 [8]. But ∂K is an octagon. Thus, the method of the previous section does not apply in higher dimensions. A simpler example is $A=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

and $D = \{(0,0),(1,0),(0,1),(1,1)\}$. Then $T = [0,1] \times [0,1]$, the unit square. The convex hull of T is T and a corner point of the convex hull is $(1,1)$ which is a boundary point. But $A^{-1}(1,1) = (1/2,1/2)$, which is an image of $(1,1)$ under a map of the IFS defining T, is not a boundary point. A one-dimensional example is $A = [q]$ and $D = \{0,1,...,q-1\}$.

In fact, it is known that the pieces of the boundary of a tile satisfy a vector iterated function system [8]. Now, let d_H^r be the metric defined on $\mathcal{C}_n^r$, the space of all r-tuples of non-empty compact subsets of $\mathbb{R}^n$, given by

$$d_H^r(\mathbf{D},\mathbf{D}') := \max_{1\le i\le r}\{d_H(D_i,D_i')\}.$$

Then one can show that $(\mathcal{C}_n^r, d_H^r)$ is also a complete space. Let $F = \{\alpha \in \mathcal{J} \setminus \{0\} : T \cap (T+\alpha) \neq \emptyset\}$. Then it is easy to see that F is a finite set. For $\alpha \in F$, we let $T_\alpha = T \cap (T+\alpha)$. Therefore, we have $\partial T = \cup_{\alpha\in F} T_\alpha$. For $\alpha,\beta \in F$, we consider the sets $C(\alpha,\beta) = \{(d,d') \in D \times D : \ \beta = A\alpha + d' - d\} \neq \emptyset$. Using the identity $AT = T + D$, we see that

$$AT_\alpha = (T+D) \cap (T+D+A\alpha) = \bigcup_{\beta} \bigcup_{(d,d')\in C(\alpha,\beta)} A^{-1}(T_\beta + d).$$

Thus the pieces T_α of ∂T satisfies a *vector iterated function system*. Suppose that we have the nonempty compact sets $Q_\alpha^0 \subset T_\alpha$ for all $\alpha \in F$ and Q_α^l is defined iteratively via

$$Q_\alpha^l = \bigcup_{\beta} \bigcup_{(d,d')\in C(\alpha,\beta)} A^{-1}(Q_\beta^{l-1} + d), \ l \geq 1.$$

Then we know that $Q_\alpha^l \subseteq T_\alpha$ increasingly converges to T_α in the Hausdorff metric. For applications, it is not clear how to get $\mathcal{J}$ in the definition of F. Then it is not clear how to get the set F and how to get a point of Q_α^0 in ∂T to start the iteration. If we can do that, all $Q_\alpha^l \subset \partial T$. There are no answers to these questions in the literature nor do we yet have answers to these questions.

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Received 19.10.2001