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On Lightlike Hypersurfaces of a Semi-Riemannian Space form

Rıfat Güneş, Bayram Şahin and Erol Kılıç

Abstract

In this paper, we study a Lightlike hypersurface of a semi-Riemann manifold. We show that a lightlike hypersurface is totally geodesic if and only if it is locally symmetric. Also, we show that a lightlike Hypersurface of $IR^{4m}_{4q}(m, q > 1)$ is totally geodesic under some restrictions. Finally, we give some results on Ricci curvature of a lightlike hypersurface to be symmetric.

1. Introduction

The general theory of lightlike (or, null) hypersurfaces is one of the most important topics of differential geometry. A few authors have studied lightlike (null) hypersurfaces (or submanifolds) of semi-Riemannian manifold [1], [2], [3], [4], and others. In [1], the authors have constructed the vector bundles related to a degenerate submanifold in a semi-Riemann manifold and obtained many properties about these submanifolds.

In the present paper, we consider real lightlike hypersurfaces of a semi-Riemann manifold. We show that M is totally geodesic in a locally symmetric semi-Riemannian manifold if and only if M is locally symmetric. Also, it is shown that M is totally geodesic in a semi-Euclidean space if $(\nabla_X \phi_a) = 0, a = 1, 2, 3$. We give some corollaries on screen distribution and induced metric depend upon the above results.

2. Preliminaries

Firstly, we note that the notations and fundamental formulas used in this study are the same as [3]. Let $\overline{M}$ be a $(m+2)-$ dimensional semi-Riemannian manifold with index $q \in \{1, ..., m+1\}$. Let M be a hypersurface of $\overline{M}$. Denote by g the induced tensor field by $\overline{g}$ on M. M is called a lightlike hypersurface if g is of constant rank m. Consider the vector bundle $TM^{\perp}$whose fibres are defined by

$$T_xM^{\perp} = \left\{Y_x \in T_x\overline{M} \mid \overline{g}_x\left(Y_x, X_x\right) = 0, \forall X_x \in T_xM\right\}$$

for any $x \in M$. Thus, a hypersurface M of $\overline{M}$ is lightlike if and only if $TM^{\perp}$is a distribution of rank 1 on M.

The fundamental difference of the theory of lightlike (or, degenerate) hypersurfaces and the classical theory of hypersurfaces of a semi-Riemannian Manifold $\overline{M}$ comes from the fact that, in the first case, the normal bundle $TM^{\perp}$ lies in the tangent bundle of a lightlike hypersurface.

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is nondegenerate subbundle of TM called the screen distribution on M and denoted $S(TM)$. We have the following splitting into orthogonal direct sum:

$$TM = S(TM) \perp TM^{\perp}. \tag{2.1}$$

The subbundle $S(TM)$ being non-degenerate, so is $S(TM)^{\perp}$ and the following holds:

$$T\overline{M} = S(TM) \perp S(TM)^{\perp}, \tag{2.2}$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\overline{M}\mid_M$. In fact, $TM^{\perp}$ is a subbundle of $S(TM)^{\perp}$. Let $ltr(TM)$ denote its complementary vector bundle in $S(TM)^{\perp}$. Then we have

$$S(TM)^{\perp} = TM^{\perp} \oplus ltr(TM). \tag{2.3}$$

Let U be a coordinate neighborhood of M and ξ be a basis of $\Gamma(TM^{\perp}\mid_U)$. Then there exists a basis N of $\Gamma(ltr(TM)\mid_U)$ satisfying the following conditions:

$$g(N,\xi)=1$$

and

$$\overline{g}\,(N,N)=\overline{g}\,(W,W)=0,\forall W\in\Gamma\,(S(TM)\mid_U)\,.$$

The subbundle $ltr(TM)$ is called a lightlike transversal vector bundle of M. We note that $ltr(TM)$ is never orthogonal to TM [3]. From (2.1), (2.2) and (2.3) we have the following decomposition

$$T\overline{M}\mid_M=S(TM)\perp\left(TM^{\perp}\oplus ltr(TM)\right)=TM\oplus ltr(TM).$$

Hence we have a local quasi-orthonormal field $\{\xi,N,W_i\}, i\in\{1,2,3,...,m\}$ of frames of $T\overline{M}$ along M, where $\{W_i\}$ is orthonormal basis of $\Gamma\,(S(TM)\mid_U)$.

Let $\overline{\nabla}$ be Levi-Civita connection on $\overline{M}$. We have

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y), \tag{2.4}$$

and

$$\overline{\nabla}_XV=-A_VX+\nabla^{\perp}_XV, \tag{2.5}$$

for any $X,Y\in\Gamma\,(TM)$ and $V\in\Gamma\,(ltr(TM))$, where $\nabla_XY, A_VX\in\Gamma\,(TM)$ and $h(X,Y),\nabla^{\perp}_XV\in\Gamma\,(ltr(TM))$. ∇ called an induced linear connection, is a symmetric linear connection on M, $\nabla^{\perp}$ is a linear connection on the vector bundle $ltr(TM)$, h is a $\Gamma\,(ltr(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Locally, suppose $\{\xi,N\}$ is a pair of sections on $U\subset M$. Then define a symmetric $F(U)$-bilinear form B and a 1-form τ on U by

$$B(X,Y)=\overline{g}(h(X,Y),\xi),\forall X,Y\in\Gamma\left(TM\mid_{U}\right)$$

and

$$\tau\left(X\right)=\overline{g}\left(\nabla_{X}^{\perp}N,\xi\right).$$

Thus (2.4) and (2.5) locally become

$$\overline{\nabla}_{X}Y=\nabla_{X}Y+B(X,Y)N, \tag{2.6}$$

and

$$\overline{\nabla}_{X}N=-A_{N}X+\tau\left(X\right)N, \tag{2.7}$$

respectively.

Let denote P as the projection of TM on $S(TM)$. We consider decomposition

$$\nabla_{X}PY=\nabla_{X}^{*}PY+C(X,PY)\xi \tag{2.8}$$

and

$$\nabla_{X}\xi=-A_{\xi}^{*}X+\epsilon\left(X\right)\xi, \tag{2.9}$$

where $\nabla_{X}^{*}PY$, $A_{\xi}^{*}X$ belong to $S(TM)$ and C is a 1-form on $U\cdot$. From (2.7) and (2.9) it is easy to check that $\epsilon=-\tau$. Thus we can write

$$\nabla_{X}\xi=-A_{\xi}^{*}X-\tau\left(X\right)\xi. \tag{2.10}$$

Thus we have the equations [3]

$$g\left(A_{N}X,PY\right)=C(X,PY),\overline{g}(A_{N}X,N)=0 \tag{2.11}$$

$$g(A_\xi^* X, PY) = B(X, PY), \overline{g}(A_\xi^* X, N) = 0 \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$.

We denote the curvature tensors associated with $\overline{\nabla}$ and ∇ by $\overline{R}$ and R, respectively. Then we have

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &+(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \end{aligned} \tag{2.13}$$

We note that the induced connection on M satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Z) \tag{2.14}$$

for any $X, Y, Z \in \Gamma(TM\mid_U)$[3].

Now, we give some definitions used in this paper. A vector field X on a lightlike submanifold is called a Killing vector field if $L_X g = 0$, where L is the Lie derivative. A distribution D on a lightlike submanifold is called a Killing distribution if each vector field belonging to D is a Killing vector field. A distribution D is called a parallel distribution if $\nabla_X Y \in \Gamma(D)$, for $X, Y \in \Gamma(D)$. A manifold M is called locally symmetric if $\nabla R = 0$, where ∇ is the linear connection on M and R is the curvature tensor field on M. Geometrically, M is locally symmetric if and only if at each point the geodesic symmetry is a connection-preserving transformation[5].

3. Lightlike Hpersurfaces of a Semi-Riemannian Space Form

First, we start the following lemma whose proof follows from (2.13).

Lemma 3.1 *Let $\overline{M}$ be a semi-Riemann manifold and M be a lightlike hypersurface of $\overline{M}$. Then we have*

$$\overline{R}(X,Y)Z = R(X,Y)Z + B(X,Z)AY - B(Y,Z)AX$$
$$+(\nabla_X B)(Y,Z)N + B(Y,Z)\tau(X)N - (\nabla_Y B)(X,Z)N,$$
$$-B(X,Z)\tau(Y)N \tag{3.15}$$

where $\overline{R}$ and R are curvature tensors of $\overline{M}$ and M, respectively.

Lemma 3.2 *Let $\overline{M}$ be a semi-Riemann manifold and M be a lightlike hypersurface of $\overline{M}$. Then we have*

$$\begin{aligned}(\overline{\nabla}_W\overline{R})(X,Y,Z) = (\nabla_W R)(X,Y,Z) + B\left(W,R(X,Y)Z\right)N + \left(\nabla_W B\right)(X,Z)AY\\
-\left(\nabla_W B\right)(X,Z)\tau\left(Y\right)N + B(X,Z)\left(\nabla_W A\right)Y + B(X,Z)B(W,AY)N\\
-\left(\nabla_W B\right)(Y,Z)AX - B(Y,Z)\left(\nabla_W A\right)X - B(Y,Z)B(W,AX)N\\
+\left(\nabla_W\left(\nabla_X B\right)\right)\right)(Y,Z)\,N - \left(\nabla_W\left(\nabla_Y B\right)\right)\right)(X,Z)\,N\\
+B(Y,Z)\left(\nabla_W\tau\right)(X)N - B(Y,Z)\tau\left(X\right)AW + \tau\left(X\right)\tau\left(W\right)B(Y,Z)N\\
+\left(\nabla_Y B\right)(X,Z)AW - \left(\nabla_Y B\right)(X,Z)\tau\left(W\right)N - \left(\nabla_X B\right)(Y,Z)AW\\
+\left(\nabla_X B\right)(Y,Z)\tau\left(W\right)N - B(X,Z)\left(\nabla\tau\right)\left(Y\right)N + B(X,Z)\tau\left(Y\right)AW\\
+B(X,Z)\tau\left(Y\right)\tau\left(W\right)N - \left(\nabla_{\nabla_W X}B\right)(Y,Z)N + \left(\nabla_{\nabla_W Y}B\right)(X,Z)N\\
-\overline{R}\left(h(W,X),Y\right)Z - \overline{R}(X,h(W,Y))Z - \overline{R}\left(X,Y\right)h(W,Z)\\
+\left(\nabla_W B\right)(Y,Z)\tau\left(X\right)N\end{aligned}$$

for any $X,Y,Z,W \in \Gamma TM)$ and $N \in \Gamma\left(ltr(TM)\right)$.

Proof. By the definition of covariant derivation of $\overline{R}$, we have

$$\begin{aligned}\left(\overline{\nabla}_W\overline{R}\right)(X,Y,Z) &= \overline{\nabla}_W\overline{R}(X,Y,Z) - \overline{R}\left(\overline{\nabla}_W X,Y\right)Z - \overline{R}\left(X,\overline{\nabla}_W Y\right)Z\\
&\quad \overline{R}\left(X,Y\right)\overline{\nabla}_W Z.\end{aligned}$$

In this equation, using (2.6), (2.7) and (3.15) we obtain the assertion of the lemma. □

Theorem 3.1 *Let $\overline{M}$ be a locally symmetric semi-Riemann manifold and M be a lightlike hypersurface of $\overline{M}$ such that $A\xi$ is not a null vector field. Then M is locally symmetric if and only if M is totally geodesic.*

Proof. By the definition of lightlike hypersurface, M is locally symmetric if and only if

$$\overline{g}\left(\left(\nabla_X R\right)(Y,Z,W),T\right)=0$$

and

$$\overline{g}\left(\left(\nabla_X R\right)(Y,Z,W),N\right)=0$$

for any $X,Y,Z,W\in\Gamma\left(TM\right),T\in\Gamma(S(TM))$ and $N\in\Gamma\left(ltr(TM)\right).$ From Lemma 3.2. and (2.11) we get

$$\begin{aligned}
-\overline{g}\left(\left(\nabla_X R\right)(Y,Z,W),T\right) &= \left(\nabla_W B\right)(X,Z)C(Y,T)-\left(\nabla_W B\right)(Y,Z)C(X,T)\\
&\quad +B(X,Z)g((\nabla_W A)\,Y,T)-B(Y,Z)g((\nabla_W A)\,X,T)\\
&\quad -B(Y,Z)\tau\left(X\right)C(W,T)+B(X,Z)\tau\left(Y\right)C(W,T)\\
&\quad +\left(\nabla_Y B\right)(X,Z)C(W,T)-\left(\nabla_X B\right)(Y,Z)C(W,T)\\
&\quad -\overline{g}\left(\overline{R}(Z,T)h(W,X),Y\right)\\
&\quad -\overline{g}(\overline{R}(X,h(W,Y)Z,T)-\overline{g}(\overline{R}(X,Y)h(W,Z),T) \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
-\overline{g}\left(\left(\nabla_W R\right)(X,Y)\,Z,N\right) &= g(\nabla_W AY,N)B(X,Z)-g(\nabla_W AX,N)B(Y,Z)\\
&\quad -B(W,X)\overline{R}\left(N,Y,Z,N\right)-B(W,Y)\overline{R}(X,N,Z,N)\\
&\quad -B(W,Z)\overline{R}(X,Y,N,N)\\
&= g(\nabla_W AY,N)B(X,Z)-g(\nabla_W AX,N)B(Y,Z)\\
&\quad -B(W,X)\overline{R}\left(N,Y,Z,N\right)\\
&\quad -B(W,Y)\overline{R}(X,N,Z,N). \qquad (3.17)
\end{aligned}$$

Now, we suppose that M is totally geodesic, then from (3.16) and (3.17) we have $\nabla R = 0$. i.e. M is locally symmetric. Conversely, suppose M is locally symmetric, then from (3.17), for $W = \xi$, we have

$$g(\nabla_\xi AY, N)B(X,Z) - g(\nabla_\xi AX, N)B(Y,Z) = 0.$$

Hence we get

$$\begin{aligned} 0 &= g(\overline{\nabla}_\xi AY, N)B(X,Z) - g(\overline{\nabla}_\xi AX, N)B(Y,Z) \\ &= \xi g(AY, N)B(X,Z) - g(AY, \overline{\nabla}_\xi N)B(X,Z) \\ &\quad -\xi g(AX, N)B(Y,Z) + g(AX, \overline{\nabla}_\xi N)B(Y,Z) \\ &= \xi g(AY, N)B(X,Z) + g(AY, A\xi)B(X,Z) \\ &\quad -\xi g(AX, N)B(Y,Z) - g(AX, A\xi)B(Y,Z). \end{aligned}$$

For $X = \xi$ we obtain

$$\begin{aligned} 0 &= g(AY, A\xi)B(\xi, Z) - g(A\xi, A\xi)B(Y,Z) \\ &= -g(A\xi, A\xi)B(Y,Z), \end{aligned}$$

which proves assertion of this theorem. □

Theorem 3.2 *Let M be a lightlike hypersurface of semi-Euclidean space $IR^{4m}_{4q}, (q > 1, m > 1)$. If $(\nabla_X \phi_a) Y = 0, a = 1,2,3$, then M is totally geodesic, where $\phi_a, a = 1,2,3$ are types of (1,1) tensor fields.*

Proof. Let $J_a, a = 1,2,3$ be almost quaternion Hermitian structures of IR^{4m}_{4q}. Then we can write

$$J_a Y = \phi_a Y + F_a Y \tag{3.18}$$

for any $Y \in \Gamma(TM)$, where $\phi_a Y \in \Gamma(TM)$ and $F_a Y \in \Gamma(ltr(TM))$. Since $\dim(ltr(TM)) = 1$ we have

$$J_a Y = \phi_a Y + \eta_a(Y)N, \tag{3.19}$$

where $\eta_a(Y) = \overline{g}(Y, J_a\xi)$. On the other hand , since J_a are parallel in IR^{4m}_{4q}, we obtain

$$\overline{\nabla}_X J_a Y - J_a \overline{\nabla}_X Y = 0.$$

Using (2.6), (2.7) and (3.19) we derive

$$\begin{aligned} 0 &= \overline{\nabla}_X(\phi_a Y + \eta_a(Y)N) - J_a \overline{\nabla}_X Y \\ &= \nabla_X \phi_a Y + B(X, \phi_a Y) + X(\eta_a(Y))N - \eta_a(Y)AX + \tau(X)\eta_a(Y)N \\ &\quad - J_a(\nabla_X Y + h(X,Y)) \\ &= \nabla_X \phi_a Y + B(X, \phi_a Y) + X(\eta_a(Y))N - \eta_a(Y)AX + \tau(X)\eta_a(Y)N \\ &\quad - \phi_a \nabla_X Y - \eta_a(\nabla_X Y)N - B(X,Y)J_a N. \end{aligned}$$

Hence we have

$$(\nabla_X \phi_a) Y = \eta_a(Y)AX + B(X,Y)J_a N. \tag{3.20}$$

Now we suppose that $(\nabla_X \phi_a) Y = 0$, then we have

$$\eta_a(Y)AX = B(X,Y)U_a, \tag{3.21}$$

where $U_a = -J_a N$. Thus from (3.21) we get

$$\begin{aligned} \eta_1(Y)AX &= B(X,Y)U_1 \\ \eta_2(Y)AX &= B(X,Y)U_2 \\ \eta_3(Y)AX &= B(X,Y)U_3. \end{aligned}$$

Since U_1, U_2 and U_3 linearly independent we have $B(X,Y)=0$. □

From the Theorem 3.2 and a theorem of Duggal-Bejancu(cf. [3] Theorem 2.2, P.88) we can give the following corollaries.

Corollary 3.1 *Let M be a lightlike hypersurface of semi-Euclidean space $IR_{4q}^{4m}, (q>1, m>1)$. If $(\nabla_X \phi_a) Y = 0, a=1,2,3,$ we have the following assertions;*

a) A_ξ^* vanishes identically on M.

b) There exists a unique torsion-free metric connection ∇ induced by $\overline{\nabla}$ on M.

c) $TM^\perp$ is a parallel distribution with respect to ∇.

d) $TM^\perp$ is a Killing distribution on M.

Corollary 3.2 *Let M be a totally geodesic lightlike hypersurface of semi-Euclidean space $IR_{4q}^{4m}, (q>1, m>1)$. Then screen distribution of M is parallel if and only if $(\nabla_X \phi_a) Y = 0, a=1,2,3.$*

Proof. Since M is totally geodesic, from (3.20) we have

$$(\nabla_X \phi_a) Y = \eta_a(Y) AX$$

for any $X, Y \in \Gamma(TM)$. Thus we get

$$\overline{g}((\nabla_X \phi_a) Y, N) = 0.$$

On the other hand, from (2.11) we obtain

$$\overline{g}((\nabla_X \phi_a) Y, T) = \eta_a(Y) C(X,T).$$

Thus $C(X,T)=0 \Longleftrightarrow \overline{g}((\nabla_X \phi_a) Y, T) = 0$. This complete the proof. □

From the semi-Riemann (Also Riemann) we know that mean curvature of a submanifold is $\alpha = trace\ A$. Thus we can give definition of mean curvature of lightlike hypersurface as $\alpha = trace\ A$. By the definition of the lightlike hypersurface in a semi-Riemann manifold we have $\alpha = \sum_{i=1}^{m-1} \epsilon_i g(A_N w_i, w_i) + \overline{g}(A_N \xi, N)$. From (2.11), we have $\alpha = \sum_{i=1} \epsilon_i g(A_N w_i, w_i)$, where $\{w_i\}$ $i \in \{1, 2, ..., m-1\}$ are the orthonormal basis of screen distribution.

Proposition 3.1 *Let M be a lightlike hypersurface of an $(m+2)$-dimensional semi-Riemann manifold $\overline{M}$. Then we have*

$$\alpha = \sum_{i=1}^{m} \epsilon_i C(w_i, w_i)$$

Proof. From (2.11), proof is trivial. □

Theorem 3.3 *Let M be a lightlike hypersurface of an $(m+2)$-dimensional semi-Riemann space form $\overline{M}(c)$. Then we have*

$$Ric(X, Y) = mcg(PX, PY) - B(X, Y)\alpha + \sum_{i=1}^{m} \epsilon_i B(w_i, Y)C(X, w_i) \tag{3.22}$$

for any $X, Y \in \Gamma(TM)$.

Proof. By the definition of lightlike hypersurface, we have

$$Ric(X, Y) = \sum_{i=1}^{m} \epsilon_i g(R(X, w_i)Y, w_i) + \overline{g}(R(X, \xi)Y, N).$$

Thus, from (2.13) we get

$$Ric(X, Y) = mcg(PX, PY) - \sum_{i=1}^{m} \epsilon_i C(w_i, w_i)B(X, Y) + \sum_{i=1}^{m} \epsilon_i B(w_i, Y)C(X, w_i)$$

or

$$Ric(X,Y) = mcg(PX,PY) - \alpha B(X,Y) + \sum_{i=1}^{m} \epsilon_i B(w_i,Y)C(X,w_i).$$

□

Proposition 3.2 *The Ricci tensor of a lightlike hypersurface in a semi-Riemann space form is degenerate.*

From (2.14) we can easily see that the induced connection is not a metric connection. Moreover, as the tansversal bundle is not orthogonal to the tangent bundle of a lightlike submanifold, we conclude that the shape operator of a lightlike submanifold is not self-adjoint. Therefore the Ricci tensor field is not symmetric in a lightlike submanifold in general. A. Bejancu ([2])showed that the Ricci tensor of a lightlike hypersurface in a semi-space form is symmetric if and only if $d\tau = 0$. Now, we give an another necessary and sufficient condition on the Ricci tensor field of a lightlike submanifold to be symmetric.

Proposition 3.3 *The Ricci tensor of lightlike hypersurface in a semi-Riemann space form $\overline{M}(c)$ is symmetric if and only if the shape operator of a lightlike hypersurface of $\overline{M}(c)$ is symmetric with respect to the second fundamental form of M.*

Proof. From (3.22) we have

$$Ric(X,Y) - Ric(Y,X) = \sum_{i=1}^{m} \epsilon_i B(w_i,Y)C(X,w_i) - B(w_i,X)\epsilon_i C(Y,w_i).$$

On the other hand, using equations (2.11) and (2.12) we arrive at

$$\sum_{i=1}^{m}\epsilon_i B(w_i,Y)C(X,w_i) = \sum_{i=1}^{m}\epsilon_i g\left(A_N X, w_i\right) g(A^*_\xi Y, w_i)$$
$$= g(A^*_\xi Y, \sum_{i=1}^{m}\epsilon_i g\left(A_N X, w_i\right) w_i)$$
$$= g(A^*_\xi Y, A_N X)$$
$$= B(Y, AX).$$

Thus we derive

$$Ric(X,Y) - Ric(Y,X) = B(Y,AX) - B(X,AY).$$

□

Corollary 3.3 *The Ricci tensor of lightlike hypersurface in a semi-Riemann space form $\overline{M}(c)$ is symmetric if and only if $C(X, A^*_\xi Y) = C(Y, A^*_\xi X)$*

Theorem 3.4 *Let M be a lightlike hypersurface of a semi-Riemann space form $\overline{M}(c)$. If M is totally geodesic, then the Ricci tensor of M is parallel with respect to ∇. Conversely, if the Ricci tensor of M is parallel with respect to ∇ then $C(A^*_\xi Z, AX) = C(A^*_\xi X, AZ)$*

Proof. First, we compute derivative of Ricci tensor. We define $\left(\nabla_Z Ric\right)(X,Y) = \nabla_Z Ric(X,Y) - Ric(\nabla_Z X, Y) - Ric(X, \nabla_Z Y)$.

Then from (2.14) and (3.22) we have

$$\begin{aligned}\left(\nabla_Z Ric\right)(X,Y) &= -(m)c\left\{B(Z,X)\eta\left(Y\right) + B(Z,Y)\eta\left(X\right)\right\} \\ &\quad - \left(\nabla_Z B\right)(X,Y)\alpha - B(X,Y)(Z(\alpha)) + \sum_{i=1}^{m-1}\epsilon_i\{B(\nabla_Z w_i, Y)C(X,w_i) \\ &\quad + \left(\nabla_Z B\right)(w_i,Y)C(X,w_i) + B(w_i,Y)C(X,\nabla_Z w_i) \\ &\quad + B(w_i,Y)\left(\nabla_Z C\right)(w_i,X)\}. \qquad (3.23)\end{aligned}$$

Thus from (3.23) , we obtain that if M is totally geodesic, then $(\nabla_Z Ric)(X,Y) = 0$. Conversely we suppose that $(\nabla_Z Ric)(X,Y) = 0$. Then for $Y = \xi$, we get

$$0 = -(m-1)cB(Z,X) + B(X, \nabla_Z \xi)\alpha - \sum_{i=1}^{m} \epsilon_i B(w_i, \nabla_Z \xi) C(X, w_i)$$

by the using (2.10) we derive

$$0 = -(m-1)cB(Z,X) - B(X, A^*_\xi Z)\alpha - \sum_{i=1}^{m} \epsilon_i B(w_i, A^*_\xi Z) C(X, w_i). \tag{3.24}$$

Interchanging Z and X in (3.24) and subtracting, we get

$$-\sum_{i=1}^{m} \epsilon_i B(w_i, A^*_\xi Z) C(X, w_i) + \sum_{i=1}^{m} \epsilon_i B(w_i, A^*_\xi X) C(Z, w_i) = 0,$$

and in a similar way to the proof of Proposition 3.3, we have

$$-g(A^*_\xi A^*_\xi Z, AX) + g(A^*_\xi A^*_\xi X, AZ) = 0.$$

Thus from (2.11) we conclude that

$$C(A^*_\xi Z, AX) = C(A^*_\xi X, AZ),$$

which proves assertion of the theorem. □

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Rıfat GÜNEŞ, Bayram ŞAHİN, Erol KILIÇ
Received 29.04.2002
İnönü University
Faculty of Science and Art
Department of Mathematics
Malatya-TURKEY
e-mail: rgunes@inonu.edu.tr
e-mail: bsahin@inonu.edu.tr
e-mail: ekilic@inonu.edu.tr