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On locally m-pseudoconvex A^*-algebras

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Abstract

We consider classical A^*-algebras in the context of locally pseudoconvex algebras. Results concerning the auxiliary topology and A^*-algebras of the first kind are given.

Key words and phrases: Locally m-pseudoconvex A^*-algebra, auxiliary topology, Q-algebra, pre-C^*-algebra, A^*-algebra of the first kind.

Introduction

This paper is concerned with a natural extension of the classical (Banach) A^*-algebras (cf. [1], Definition 1., p. 214) in the general context of locally m-pseudoconvex algebras. We consider a locally m-pseudoconvex algebra $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$, $0 < p_\lambda \leq 1$, endowed with a generalized involution $x \longmapsto x^*$, on which there is defined a second locally pseudoconvex topology, called the auxiliary topology, given by a family $(\|\cdot\|_\alpha)_{\alpha\in\Gamma}$ of q_α-seminorms, $0 < q_\alpha \leq 1$, with C^*-properties. We call the resulting algebra a locally m-pseudoconvex A^*-algebra. Such an algebra will be denoted by $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$. We show that the auxiliary topology, of every locally m-pseudoconvex A^*-algebra $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$, is necessarily locally m-convex and hence $\left(E, \left(\|\cdot\|_\alpha^{\frac{1}{q_\alpha}}\right)_{\alpha\in\Gamma}\right)$ is a locally pre-C^*-algebra. Furthermore, if $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$ is a Q-algebra, then $\|\cdot\| = \sup\left\{\|\cdot\|_\alpha^{\frac{1}{q_\alpha}} : \alpha \in \Gamma\right\}$ is a pre-C^*-algebra norm such that $\|\cdot\|_\alpha \leq \|\cdot\|^{q_\alpha}$, for every $\alpha \in \Gamma$. This last norm is the

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coarsest among algebra norms which are stronger than $(\|\cdot\|_\alpha)_{\alpha\in\Gamma}$. Moreover, $(E, \|\cdot\|)$ is a C^*-algebra if $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is pseudo-complete (i.e., every bounded and closed idempotent disk is Banach). On the other hand, we consider A^*-algebras of the first kind. In this case, we prove that the auxiliary topology is unique. We also obtain that if $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is of the first kind, then $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is (modulo a topological algebra isomorphism) topologically and algebraically isomorphic to $(E, \|\cdot\|)$.

1. Preliminaries

All algebras in this paper are complex and associative. An involutive antimorphism on an algebra E is a vector involution $x \longmapsto x^*$ ([1]) such that $(xy)^* = x^*y^*$, for every $x, y \in E$. A vector space involution $x \longmapsto x^*$ is said to be a generalized involution if either it is an algebra involution (i.e. $(xy)^* = y^*x^*$, for every $x, y \in E$) or an involutive antimorphism. Let E be a vector space and $\|\cdot\|_p$, $0 < p \leq 1$, a p-seminorm, on E, (i.e., non negative function $x \longmapsto \|x\|_p$ such that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ and $\|\lambda x\|_p = |\lambda|^p \|x\|_p$, for all x, y in E and $\lambda \in C$). If, in addition, $\|x\|_p = 0$ implies $x = 0$, then $\|\cdot\|_p$ is a p-norm. By a p-normed space, we mean a space endowed with a p-norm. A complete p-normed space is called a p-Banach space. Moreover, if E is an algebra and $\|\cdot\|_p$ is submultiplicative (i.e., $\|xy\|_p \leq \|x\|_p \|y\|_p$, for all $x, y \in E$), then $\|\cdot\|_p$ is called an algebra p-norm. A p-normed algebra is an algebra endowed with an algebra p-norm. A complete p-normed algebra is called a p-Banach algebra. Let (E, τ) be a locally pseudoconvex space ([10], [14]) the topology of which is given by a family $\{|\cdot|_\lambda : \lambda \in \Lambda\}$ of p_λ-seminorms, $0 < p_\lambda \leq 1$. If E is endowed with an algebra structure such that $|xy|_\lambda \leq |x|_\lambda |y|_\lambda$ for every $x, y \in E$ and $\lambda \in \Lambda$, we say that $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is a locally m-pseudoconvex algebra. If, in addition, E is endowed with an involution $x \longmapsto x^*$ such that $|x|_\lambda = |x^*|_\lambda$, for any $x \in E$ and $\lambda \in \Lambda$, then $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$ is called a locally m-pseudoconvex.$*$-algebra. An element a of E is said to be hermitian (resp. normal) if $a = a^*$ (resp. $aa^* = a^*a$). We designate by $H(E)$ (resp. $N(E)$) the set of hermitian (resp. normal) elements of E. We denote Ptak's function on E by P_E, that is, for every $a \in E$, $P_E(a) = \rho_E(aa^*)^{\frac{1}{2}}$, where ρ_E is the spectral radius, i.e. $\rho_E(a) = \sup\{|\lambda| : \lambda \in Sp(a)\}$.

Let $\left(E, \|\cdot\|_p\right)$, $0 < p \leq 1$, be a p-Banach algebra with a generalized involution. If $\left(E, \|\cdot\|_p\right)$ is hermitian (i.e., the spectrum of every hermitian element is real), we show, as in the Banach case ([9]), that P_E is an algebra seminorm such that $\rho_E \leq P_E$ and $P_E(a)^2 = P_E(aa^*)$, for every $a \in E$. Moreover $\mathrm{Rad}E = \{x \in E : P_E(x) = 0\}$. Taking into account the fact that, in any p-Banach algebra $\left(E, \|\cdot\|_p\right)$, we have $\rho_E(a)^p = \lim_n \|a^n\|_p^{\frac{1}{n}}$, for every $a \in E$, we prove, as in [9], the following result.

Proposition 1.1 *Let $\left(E, \|\cdot\|_p\right)$, $0 < p \leq 1$, be a p-Banach algebra with a generalized involution $x \longmapsto x^*$. The following assertions are equivalent.*

1) *E is hermitian.*

2) *There is $c > 0$ such that $\rho_E(a) \leq cP_E(a)$, for every $a \in N(E)$.*

3) *$\rho_E(a) \leq P_E(a)$, for every $a \in E$.*

Using Theorem 3 of [15] and the fact that the quotient of a p-Banach algebra by a primitive ideal is a primitive p-Banach algebra, we extend theorem 4.8 of Kaplansky ([5]) to the p-Banach case as follows.

Theorem 1.2 *Any real semi-simple p-Banach algebra, $0 < p \leq 1$, in which every square is quasi-invertible, is necessarily commutative.*

2. Locally m-pseudoconvex A^*-algebras.

We introduce locally m-pseudoconvex A^*-algebras as a natural extension of the classical Banach A^*-algebras (cf. [1], Definition 1., p. 214) as follows.

Definition 2.1 *A locally m-pseudoconvex A^*-algebra is a locally m-pseudoconvex $*$-algebra $\left(E, (|\cdot|_\lambda)_{\lambda \in \Lambda}\right)$, $0 < p_\lambda \leq 1$, on which there is defined a second locally pseudoconvex Hausdorff topology, called the auxiliary topology, given by a family $(\|\cdot\|_\alpha)_{\alpha \in \Gamma}$ of q_α-seminorms, $0 < q_\alpha \leq 1$, such that*

$$\|x^* x\|_\alpha = \|x\|_\alpha^2, \text{ for every } x \in E \text{ and } \alpha \in \Gamma. \tag{1}$$

We do not pose either submultiplicativity or preservation of the involution for q_α-seminorms $\|\cdot\|_\alpha$, $(\alpha \in \Gamma)$. These properties occur automatically from [2] which is an extension of Sebestyén's result ([11]). To make the paper self-contained, we give the following result.

Proposition 2.2 *Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$ and $(\|\cdot\|_\alpha)_{\alpha\in\Gamma}$ be as described in definition 2.1. Then*

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha, \text{ for every } x, y \in E \text{ and } \alpha \in \Gamma.$$

In particular,

1) $\|x\|_\alpha = \|x^*\|_\alpha$, *for every $x \in E$ and $\alpha \in \Gamma$,*

2) $\|x\|_\alpha = \sup\{\|xy\|_\alpha : \|y\|_\alpha \leq 1\} = \sup\{\|yx\|_\alpha : \|y\|_\alpha \leq 1\}$, *for every $x \in E$ and $\alpha \in \Gamma$.*

Proof. By considering the algebraic identity

$$\begin{aligned} 4ab &= (b + a^*)^*(b + a^*) + i(b + ia^*)^*(b + ia^*) \\ &\quad -(b - a^*)^*(b - a^*) - i(b - ia^*)^*(b - ia^*) \end{aligned}$$

is valid for every $a, b \in E$, we obtain that

$$\|ab\|_\alpha \leq 4^{2-q_\alpha} \|a^*\|_\alpha \|b\|_\alpha, \text{ for every } a, b \in E. \tag{2}$$

Thus by (1) we get

$$\|a^*\|_\alpha \leq 4^{1-\frac{q_\alpha}{2}} \|a\|_\alpha, \text{ for every } a \in E. \tag{3}$$

As a consequence of (2) and (3), we get

$$\|ab\|_\alpha \leq 4^{3-\frac{3q_\alpha}{2}} \|a\|_\alpha \|b\|_\alpha, \text{ for every } a, b \in E.$$

Consider on $E/Ker\,\|\cdot\|_\alpha$ the q_α-norm, also denoted by $\|\cdot\|_\alpha$, defined by

$$\|\pi_\alpha(x)\|_\alpha = \|x\|_\alpha;\ x \in E,$$

where π_α is the natural quotient map of E onto $E/Ker\,\|\cdot\|_\alpha$. Let $\overset{\wedge}{F_\alpha}$ be the completion of $F_\alpha = E/Ker\,\|\cdot\|_\alpha$ with respect to the q_α-norm $\|\cdot\|_\alpha$. The q_α-norm in $\overset{\wedge}{F_\alpha}$ will also designated by $\|\cdot\|_\alpha$. Then we have

$$\|a^*a\|_\alpha = \|a\|_\alpha^2;\ \text{for every } a \in \overset{\wedge}{F_\alpha} \tag{4}$$

and also

$$\|ab\|_\alpha \leq 4^{3-\frac{3q_\alpha}{2}} \|a\|_\alpha \|b\|_\alpha, \text{ for every } a, b \in \hat{F}_\alpha . \tag{5}$$

For $a \in \hat{F}_\alpha$, put

$$\|a\|'_\alpha = \sup\{\|ab\|_\alpha : \|b\|_\alpha \leq 1\}.$$

We get an algebra q_α-norm, on $\hat{F}_\alpha$, such that

$$4^{\frac{q_\alpha}{2}-1} \|a\|_\alpha \leq \|a\|'_\alpha \leq 4^{3-\frac{3q_\alpha}{2}} \|a\|_\alpha, \text{ for every } a \in \hat{F}_\alpha .$$

Moreover, one get from the above that

$$\rho_{\hat{F}_\alpha}(a) = P_{\hat{F}_\alpha}(a), \text{ for every } a \in N(\hat{F}_\alpha), \tag{6}$$

which yields

$$\rho_{\hat{F}_\alpha}(a)^{q_\alpha} = \|a\|_\alpha, \text{ for every } a \in N(\hat{F}_\alpha). \tag{7}$$

By Proposition 1.1, the algebra $\left(\hat{F}_\alpha, \|\cdot\|_\alpha\right)$ is hermitian and so

$$\rho_{\hat{F}_\alpha}(a) \leq P_{\hat{F}_\alpha}(a), \text{ for every } a \in \hat{F}_\alpha . \tag{8}$$

We consider first that $x \longmapsto x^*$ is an algebra involution. In this case, we get by (7) and (8), for every $n \in N^*$,

$$\|ab\|^2_\alpha \leq \left\|(bb^*)^{2^{n-1}}(a^*a)^{2^n}(bb^*)^{2^{n-1}}\right\|^{2^{-n}}_\alpha, \text{ for every } a, b \in \hat{F}_\alpha .$$

It then follows from (5) and (4) that

$$\|ab\|^2_\alpha \leq (4^{6-3q_\alpha})^{2^{-n}} \|a\|^2_\alpha \|b\|^2_\alpha; \text{ for every } n \in N^* \text{ and } a, b \in \hat{F}_\alpha .$$

Therefore taking limit for $n \longrightarrow \infty$, we conclude that

$$\|ab\|_\alpha \leq \|a\|_\alpha \|b\|_\alpha, \text{ for every } a, b \in \hat{F}_\alpha$$

and in particular

$$\|ab\|_\alpha \leq \|a\|_\alpha \|b\|_\alpha \,, \text{ for every } a, b \in E.$$

Suppose now that $x \longmapsto x^*$ is an involutive antimorphism. We will show that, in this case, the algebra $\overset{\wedge}{F_\alpha}$ is commutative. It is sufficient to consider the real q_α-Banach algebra $H(\overset{\wedge}{F_\alpha})$. By (7), we have $\mathrm{Rad}(H(\overset{\wedge}{F_\alpha})) = \{0\}$. $\overset{\wedge}{F_\alpha}$ is Hermitian and by Theorem 1.2 $H(\overset{\wedge}{F_\alpha})$ is commutative. The second statement of the proposition follows immediately from (1) and the submultiplicativity of q_α-seminorms $\|\cdot\|_\alpha$, $(\alpha \in \Gamma)$. This completes the proof. □

For the rest of the paper, $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ will denote a locally m-pseudoconvex A^*-algebra $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$, $0 < p_\lambda \leq 1$, with auxiliary topology given by $(\|\cdot\|_\alpha)_{\alpha\in\Gamma}$, $0 < q_\alpha \leq 1$.

The following result shows that the auxiliary topology is necessarily locally m-convex.

Proposition 2.3 *Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ be a locally m-pseudoconvex A^*-algebra. Then, for every $\alpha \in \Gamma$, $\|\cdot\|_\alpha^{\frac{1}{q_\alpha}}$ is an algebra seminorm.*

Proof. Let us first notice that if $x \longmapsto x^*$ is an involutive antimorphism, then the algebra $\overset{\wedge}{F_\alpha}$ is commutative. So there is no loss in assuming that $x \longmapsto x^*$ is an algebra involution. On the other hand, the algebra $\overset{\wedge}{F_\alpha}$ is hermitian and $P_{\overset{\wedge}{F_\alpha}}$ is an algebra seminorm such that

$$P_{\overset{\wedge}{F_\alpha}}(a)^2 = P_{\overset{\wedge}{F_\alpha}}(a^*a), \text{ for every } a \in \overset{\wedge}{F_\alpha}.$$

But, by (7),

$$P_{\overset{\wedge}{F_\alpha}}(a)^{q_\alpha} = \|a\|_\alpha \,, \text{ for every } a \in \overset{\wedge}{F_\alpha}.$$

Whence $\|\cdot\|_\alpha^{\frac{1}{q_\alpha}}$ is an algebra seminorm for $P_{\overset{\wedge}{F_\alpha}}$ is so. This completes the proof. □

As a consequence, we obtain the following results.

Corollary 2.4 *Let* $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ *be a locally m-pseudoconvex* A^**-algebra. Then, for every* $\alpha \in \Gamma$,

$$\|a\|_\alpha^2 \leq \rho(a^*a)^{q_\alpha}, \textit{ for every } a \in E.$$

In particular, E is semi-simple.

Proof. Observe first that one checks that,

$$\rho_{\widehat{E}}(a) \leq \rho(a), \text{ for every } a \in E,$$

where $\widehat{E}$ is the completion of $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$. But

$$\rho_{\widehat{E}}(a) = \sup\left\{\lim_n \|a^n\|_\alpha^{\frac{1}{nq_\alpha}} : \alpha \in \Gamma\right\}.$$

So

$$\sup\left\{\lim_n \|a^n\|_\alpha^{\frac{1}{nq_\alpha}} : \alpha \in \Gamma\right\} \leq \rho(a), \text{ for every } a \in E.$$

On the other hand, we have

$$\|h\|_\alpha = \left\|h^{2^n}\right\|_\alpha^{\frac{1}{2^n}}, \text{ for every } h \in H(E).$$

and $n = 1, 2, ...$ This implies that

$$\|h\|_\alpha \leq \rho(h)^{q_\alpha}, \text{ for every } h \in H(E).$$

We consider first that $x \longmapsto x^*$ is an algebra involution. In this case, we get, for every $a \in E$,

$$\|a\|_\alpha^2 = \|aa^*\|_\alpha \leq \rho(aa^*)^{q_\alpha}.$$

Suppose now that $x \longmapsto x^*$ is an involutive antimorphism. In this case, the algebra $\overset{\wedge}{F_\alpha}$ is commutative by Theorem 1.2 and hence $E/Ker\,\|\cdot\|_\alpha$ is also commutative. So we have

$$\|a\|_\alpha^2 = \|\pi_\alpha(a)\|_\alpha^2 \leq \rho\left(\pi_\alpha(aa^*)\right)^{q_\alpha} \leq \rho\left(aa^*\right)^{q_\alpha}.$$

The second claim follows from standard arguments. □

Corollary 2.5 *Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ be a locally m-pseudoconvex A^*-algebra. If $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$ is Q-algebra, then there is $\lambda_0 \in \Lambda$ such that for every $\alpha \in \Gamma$, we have*

$$\|x\|_\alpha^{\frac{1}{q_\alpha}} \leq |x|_{\lambda_0}^{\frac{1}{p_{\lambda_0}}} \text{, for every } x \in E.$$

Proof. By an analogous result of ([12], Corollary 4.1, p. 551), there is $\lambda_0 \in \Lambda$ such that

$$\rho(x)^{p_{\lambda_0}} \leq |x|_{\lambda_0}, \text{ for every } x \in E.$$

Then, by Corollary 2.4, we have

$$\|x\|_\alpha^{\frac{2}{q_\alpha}} \leq \rho(x^*x) \leq |x^*x|_{\lambda_0}^{\frac{1}{p_{\lambda_0}}} \leq |x|_{\lambda_0}^{\frac{2}{p_{\lambda_0}}}, \text{ for every } \alpha \in \Gamma \text{ and } x \in E.$$

Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ be a locally m-pseudoconvex A^*-algebra such that $\left(E, (|.|_\lambda)_{\lambda\in\Lambda}\right)$ is Q-algebra. Put

$$\|x\| = \sup\left\{\|x\|_\alpha^{\frac{1}{q_\alpha}} : \alpha \in \Gamma\right\}.$$

Then $\|\cdot\|$ is a pre-C^*-algebra norm such that $\|x\|_\alpha \leq \|x\|^{q_\alpha}$, for every $x \in E$ and $\alpha \in \Gamma$. Furthermore we have the following. □

Proposition 2.6 *Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ be a locally m-pseudoconvex A^*-algebra such that $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$ is Q-algebra.*

1) *If $\left(E, \left(\|\cdot\|_\alpha^{\frac{1}{q_\alpha}}\right)_{\alpha\in\Gamma}\right)$ is pseudo-complete, then $(E, \|\cdot\|)$ is a C^*-algebra.*

2) *If $\left(E, \left(\|\cdot\|_\alpha^{\frac{1}{q_\alpha}}\right)_{\alpha\in\Gamma}\right)$ is M-complete (i.e., every bounded and closed disk is Banach), then $(E, \|\cdot\|)$ and $\left(E, \left(\|\cdot\|_\alpha^{\frac{1}{q_\alpha}}\right)_{\alpha\in\Gamma}\right)$ have the same bounded sets.*

Proof. 1) Completeness of $(E, \|\cdot\|)$ follows from the fact that the unit ball $B_{\|\cdot\|} = \{x \in E : \|x\| \leq 1\}$, of $(E, \|\cdot\|)$, is a bounded and closed idempotent disk in $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$.

2) It is due to the fact that any barrel in M-complete locally convex space is bornivorous. □

Remark 2.7. If $\left(E,(\|\cdot\|_{\alpha})_{\alpha\in\Gamma}\right)$ is barreled, then $\left(E,(\|\cdot\|_{\alpha})_{\alpha\in\Gamma}\right)$ is a pre-C^*-algebra. Moreover one can easily verify that $\|\cdot\|$ is the coarsest among algebra norms which are stronger than $\|\cdot\|_{\alpha}$ for each $\alpha\in\Gamma$.

3. A^*-algebras of the first kind.

Definition 3.1 *A locally m-pseudoconvex A^*-algebra $\left(E,(|\cdot|_{\lambda})_{\lambda\in\Lambda},(\|\cdot\|_{\alpha})_{\alpha\in\Gamma}\right)$ is said to be of the first kind if, for every λ, there exists a constant $c_\lambda>0$ and $\alpha\in\Gamma$ (depending on λ) such that*

$$\max(|xy|_{\lambda}^{\frac{1}{p_\lambda}},|yx|_{\lambda}^{\frac{1}{p_\lambda}})\leq c_\lambda\,|x|_{\lambda}^{\frac{1}{p_\lambda}}\,\|y\|_{\alpha}^{\frac{1}{q_\alpha}},\ \textit{for all } x,y\in E. \tag{9}$$

Throughout this section, we suppose that $(|\cdot|_{\lambda})_{\lambda\in\Lambda}$ and $(\|\cdot\|_{\alpha})_{\alpha\in\Gamma}$ are p_λ-norms and q_α-norms respectively and $E_\lambda=(E,|\cdot|_{\lambda})$ is a Q-algebra, for every $\lambda\in\Lambda$.

Remarks 3.2

1) By Corollary 2.4, we have

$$\|a\|_{\alpha}^{\frac{1}{q_\alpha}}\leq\rho_{E_\lambda}(aa^*)^{\frac{1}{2}}\leq|aa^*|_{\lambda}^{\frac{1}{2p_\lambda}}\leq|a|_{\lambda}^{\frac{1}{p_\lambda}},\ \text{for every } a\in E. \tag{10}$$

Denote by $\widehat{E_\lambda}$ the completion of the p_λ-normed algebra E_λ and the p_λ-norm in $\widehat{E_\lambda}$ by $|\cdot|_{\lambda}$. By (10), the q_α-norm $\|\cdot\|_{\alpha}$ can be extended to $\widehat{E_\lambda}$. So $\left(\widehat{E_\lambda},|\cdot|_{\lambda}\right)$ is a p_λ-Banach $*$-algebra on which there is defined a second algebra C^*-norm $\|\cdot\|_{\alpha}^{\frac{1}{q_\alpha}}$. Let $\widehat{F_\alpha}$ be the completion of $\widehat{E_\lambda}$ with respect to the auxiliary norm $\|\cdot\|_{\alpha}^{\frac{1}{q_\alpha}}$. Then, by (9), we have

$$\max(|xy|_{\lambda}^{\frac{1}{p_\lambda}},|yx|_{\lambda}^{\frac{1}{p_\lambda}})\leq c_\lambda\,|x|_{\lambda}^{\frac{1}{p_\lambda}}\,\|y\|_{\alpha}^{\frac{1}{q_\alpha}},\ \text{for all } x,y\in\widehat{E_\lambda}. \tag{11}$$

This implies that $\widehat{E_\lambda}$ is a two-sided ideal of $\widehat{F_\alpha}$. Indeed let $a\in\widehat{F_\alpha}$. Then there exists a sequence $(a_n)_n$ of $\widehat{E_\lambda}$ such that $\lim_n\|a_n-a\|_{\alpha}=0$. Moreover, by (11), we have

$$\lim_n[\max(|xa_n-xa|_{\lambda}^{\frac{1}{p_\lambda}},|a_nx-ax|_{\lambda}^{\frac{1}{p_\lambda}})]=0,\ \text{for every } x\in\widehat{E_\lambda}.$$

It follows that ax and xa are in $\widehat{E_\lambda}$. Conversely, if $\widehat{E_\lambda}$ is a two-sided ideal of $\widehat{F_\alpha}$, then using the closed graph and uniform boundedness theorems, we prove that (11) is satisfied. For this last fact the proof, being straighforward, is omitted. So a locally m-pseudoconvex A^*-algebra $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is for the first kind if, and only if, for every λ there exists $\alpha \in \Gamma$, such that $\widehat{E_\lambda}$ is a two-sided ideal of $\widehat{F_\alpha}$. In this case, we also have

$$\max(|ax|_\lambda^{\frac{1}{p_\lambda}}, |xa|_\lambda^{\frac{1}{p_\lambda}}) \leq c_\lambda |a|_\lambda^{\frac{1}{p_\lambda}} \|x\|_\alpha^{\frac{1}{q_\alpha}}, \text{ for all } a \in \widehat{E_\lambda} \text{ and } x \in \widehat{F_\alpha}. \tag{12}$$

2) For every $x \in \widehat{F_\alpha}$, put

$$|x|_{\lambda,1} = \sup\left\{\max(|ax|_\lambda, |xa|_\lambda) : a \in \widehat{E_\lambda} \text{ and } |a|_\lambda \leq 1\right\}.$$

By (12), it is easy to see that $|x|_{\lambda,1}^{\frac{1}{p_\lambda}} \leq c_\lambda \|x\|_\alpha^{\frac{1}{q_\alpha}}$ for every $x \in \widehat{F_\alpha}$. This together with the fact that $\widehat{E_\lambda}$ is dense in $\widehat{F_\alpha}$ implies that $|\cdot|_{\lambda,1}$ is an algebra p_λ-norm on $\widehat{F_\alpha}$. On the other hand, we also have

$$\|x\|_\alpha^{\frac{2}{q_\alpha}} = \|xx^*\|_\alpha^{\frac{1}{q_\alpha}} = \rho_{\widehat{E_\lambda}}(xx^*), \text{ for every } x \in \widehat{F_\alpha}.$$

But $\rho_{\widehat{E_\lambda}}(xx^*) = \rho_{\widehat{F_\alpha}}(xx^*)$ for $xx^* \in N(E_\lambda)$. Hence

$$\|x\|_\alpha^{\frac{2}{q_\alpha}} \leq |x^*|_{\lambda,1}^{\frac{1}{p_\lambda}} |x|_{\lambda,1}^{\frac{1}{p_\lambda}} \leq c_\lambda \|x\|_\alpha^{\frac{1}{q_\alpha}} |x|_{\lambda,1}^{\frac{1}{p_\lambda}}, \text{ for every } x \in \widehat{F_\alpha}.$$

This implies that $\|x\|_\alpha^{\frac{1}{q_\alpha}} \leq c_\lambda |x|_{\lambda,1}^{\frac{1}{p_\lambda}}$, for every $x \in \widehat{F_\alpha}$. So, for each $\lambda \in \Lambda$, $|.|_{\lambda,1}$ defines an algebra p_λ-norm on $\widehat{F_\alpha}$ which is equivalent to $\|\cdot\|_\alpha$.

If E is an A^*-algebra of the first kind, then the auxiliary norm on E is unique ([6], Lemma 3.1. p. 508). In a more general context of locally m-pseudoconvex A^*-algebras, we have the following proposition.

Proposition 3.3 *Let* $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\lambda)_{\lambda\in\Lambda}\right)$ *be a locally* m*-pseudoconvex* A^**-algebra of the first kind. If* $\left(\|\cdot\|_{1,\lambda}\right)_{\lambda\in\Lambda}$ *is a family of* r_λ*-norms,* $0 < r_\lambda \leq 1$*, defining another auxiliary locally* m*-pseudoconvex topology on* E*, then* $(\|\cdot\|_\lambda)_{\lambda\in\Lambda}$ *and* $\left(\|\cdot\|_{1,\lambda}\right)_{\lambda\in\Lambda}$ *are equivalent.*

Proof. For $\lambda \in \Lambda$, define

$$\|x\|_{2,\lambda} = \max\left(\|x\|_{\lambda}^{\frac{1}{q_\lambda}}, \|x\|_{1,\lambda}^{\frac{1}{r_\lambda}}\right), \ x \in E.$$

It is clear that the family $\left(\|\cdot\|_{2,\lambda}\right)_{\lambda\in\Lambda}$ of norms defines a locally m-convex topology on E such that

$$\|x^*x\|_{2,\lambda} = \|x\|_{2,\lambda}^2 \text{ for all } x \in E \text{ and } \lambda \in \Lambda.$$

Let $\widehat{F_\lambda}$, $\widehat{F'_\lambda}$, $\widehat{F''_\lambda}$ be the completions of $\widehat{E_\lambda}$ with respect to $\|\cdot\|_\lambda$, $\|\cdot\|_{1,\lambda}$ and $\|\cdot\|_{2,\lambda}$, respectively. By Remark 3.2, there exists a constant $k_\lambda > 0$ so that, for $a, b \in \widehat{E_\lambda}$,

$$\max(|ab|'_\lambda, |ba|'_\lambda) \leq k_\lambda |a|'_\lambda \|b\|_\lambda \leq k_\lambda |a|'_\lambda \|b\|_{2,\lambda}$$

Thus, $\widehat{E_\lambda}$ is a two-sided ideal of $\widehat{F''_\lambda}$. Now since the identity mapping is a continuous $*$-isomorphism of $\left(\widehat{E_\lambda}, \|\cdot\|'_{2,\lambda}\right)$ onto $\left(\widehat{E_\lambda}, \|\cdot\|_\lambda\right)$, it follows from [8, lemma 2] that $\left(\widehat{F_\lambda}, \|\cdot\|_\lambda\right)$ and $\left(\widehat{F''_\lambda}, \|\cdot\|_{2,\lambda}\right)$ are topologically isomorphic. Similarly it can be shown that $\left(\widehat{F'_\lambda}, \|\cdot\|'_\lambda\right)$ is topologically isomorphic to $\left(\widehat{F''_\lambda}, \|\cdot\|''_\lambda\right)$ and so it follows that the auxiliary norms $\|\cdot\|_\lambda$ and $\|\cdot\|'_\lambda$ are equivalent. □

Proposition 3.4 *Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ be a locally m-pseudoconvex A^*-algebra of the first kind. Then $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is topologically and algebraically isomorphic to a pre-C^*-algebra.*

Proof. Since $(E, |\cdot|_\lambda)$ is a Q-algebra, one has $\rho(x)^{p_\lambda} \leq |x|_\lambda$ for every $x \in E$ and $\lambda \in \Lambda$. On the other hand, using (9), we obtain

$$\rho(xy) \leq |xy|_\lambda^{\frac{1}{p_\lambda}} \leq c_\lambda |x|_\lambda^{\frac{1}{p_\lambda}} \|y\|_\alpha^{\frac{1}{q_\alpha}}, \text{ for all } x, y \in E.$$

Writing this for $y = x^k$, with $k = 1, 2, ...,$ and using submultiplicativity of $\|\cdot\|_\alpha$, it follows that $\rho(x) \leq \|x\|_\alpha^{\frac{1}{q_\alpha}}$ for every $x \in E$. Then, using Corollary 2.4, we have

$$\|x\|_\alpha^{\frac{1}{q_\alpha}} = \rho(x), \text{ for every } x \in N(E).$$

Now, for every $x \in E$, we get

$$\|x\|_{\alpha}^{\frac{2}{q_\alpha}} \leq \sup_{\gamma\in\Gamma} \|x\|_{\gamma}^{\frac{2}{q_\gamma}} = \sup_{\gamma\in\Gamma} \|xx^*\|_{\gamma}^{\frac{1}{q_\gamma}} = \rho(x^*x) \leq \|x\|_{\alpha}^{\frac{2}{q_\alpha}} .$$

Thus the topology of $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is equivalent to the pre-C^*-norm

$$\|x\|_{\alpha}^{\frac{1}{q_\alpha}} = \sup\left\{\|x\|_{\gamma}^{\frac{1}{q_\gamma}} : \gamma \in \Gamma\right\} = \|x\| \text{, for every } x \in E.$$

This completes the proof. □

Remark 3.5. In the previous proposition, the algebra $\left(E, (\|\cdot\|_\lambda)_{\lambda\in\Lambda}\right)$ becomes topologically and algebraically isomorphic to a C^*-algebra under a weaker notion of completion. More precisely, one has that $(E, \|\cdot\|)$ is a C^*-algebra if and only if $\left(E, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ is a pseudo-complete algebra.

4. Examples

To illustrate the above results, we give the following examples.

1) Let $0 < p_0 < 1$ and define E to be the set of all complex sequences $x = (x_n)_n$ such that

$$|x|_p = \sum_{n=1}^{\infty} |x_n|^p < +\infty, \text{ for every } p \in \left]p_0, 1\right[. \tag{13}$$

One can easily verify that the formula (13) defines, on E, a family of p-seminorms. Endow E with the usual pointwise operations and the involution $((x_n)_n)^* = (\overline{x_n})_n$. Then $\left(E, \left(|\cdot|_p\right)_p\right)$ is a complete locally m-pseudoconvex (not locally convex) $*$-algebra. For every $k \in N$, put

$$\|x\|_k = \sup\left\{|x_n| : n \leq k\right\}.$$

Then $\left(E, \left(|\cdot|_p\right)_p, (\|\cdot\|_k)_k\right)$ is a locally m-pseudoconvex A^*-algebra. It is not of the first kind. Notice that the pre C^*-algebra norm $\|\cdot\|$, given by Proposition 2.6, is $\|x\| =$

$\sup\{|x_n| : n \in N\}$. Furthermore, for every $0 < p < 1$, $\left(E, |\cdot|_p, \|\cdot\|\right)$ is a p-Banach (not Banach) A^*-algebra of the first kind.

2) Let Ω be a nonempty open set of R (real field) and $k \in N^*$. Consider $E = \mathcal{C}^k(\Omega)$ the set of all complex -valued $\mathcal{C}^k$-functions on Ω, provided with the pointwise operations and the involution $f^* = \overline{f}$. For every compact subset K of Ω, put

$$p_{K,k}(f) = \max_{j\leq k} \sup_{x\in K} \left|f^{(j)}(x)\right|, \text{ for every } f \in \mathcal{C}^k(\Omega).$$

Let K be a compact subset of Ω. Applying Leibniz's rule, it easy to see that there is $\alpha(k)$ (depending on k) such that, for every $f, g \in \mathcal{C}^k(\Omega)$, we have

$$p_{K,k}(fg) \leq \alpha(k) p_{K,k}(f) p_{K,k}(g).$$

Put

$$|f|_{K,k} = \alpha(k) p_{K,k}(f), \text{ for every } f \in \mathcal{C}^k(\Omega).$$

Then $\left(E, \left(|\cdot|_{K,k}\right)_K\right)$ is a metrizable and complete locally m-convex $*$-algebra. For every compact subset K of Ω, put

$$\|f\|_K = \sup\{|f(t)| : t \in K\}.$$

Then $\left(E, \left(|\cdot|_{K,k}\right)_K, (\|\cdot\|_K)_K\right)$ is a locally m-convex A^*-algebra. It is not of the first kind.

3) Let $(A, \|\cdot\|, *)$ be an H^*-algebra in the spirit of F. F. Bonsall and J. Duncun (cf. [1], definition 6., p. 182). Then $(A, \|\cdot\|, |\cdot|)$, where $|x| = \sup\{\|xy\| : \|y\| \leq 1\}$, for every $x \in A$, is an A^*-algebra of the first kind. Now let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}\right)$ be a locally m-convex H^*-algebra (cf. [2]) that is a complete locally m-convex $*$-algebra on which there is defined a family $(\langle ., . \rangle_\lambda)_{\lambda\in\Lambda}$ of positive semi-definite pseudo-inner products such that $|x|_\lambda^2 = \langle x, x\rangle_\lambda$, $\langle xy, z\rangle_\lambda = \langle y, x^*z\rangle_\lambda$ and $\langle yx, z\rangle_\lambda = \langle y, zx^*\rangle_\lambda$, for all x, y, $z \in E$ and $\lambda \in \Lambda$. Put

$$\|a\|_\lambda = \sup\{|ab|_\lambda : |b|_\lambda \leq 1\}, \text{ for every } a \in E.$$

Then $(\|\cdot\|_\lambda)_{\lambda\in\Lambda}$ is a family of seminorms in E such that $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\lambda)_{\lambda\in\Lambda}\right)$ is locally m-convex A^*-algebra of the first kind. The reader is referred to [2] for all details.

4) Let $\left(E, (|\cdot|_\lambda)_{\lambda\in\Lambda}, (\|\cdot\|_\alpha)_{\alpha\in\Gamma}\right)$ be a locally m-pseudoconvex A^*-algebra of the first kind. Recall that a mapping $T : E \longrightarrow E$ is called a multiplier on E if $T(ab) = T(a)b = aT(b)$, for all a, $b \in E$. It is obvious that T is necessarily linear and, by the closed graph theorem, it is also continuous on $\left(\widehat{E_\lambda}, |\cdot|_\lambda\right)$, for every $\lambda \in \Lambda$. So $|T(x)|_\lambda \leq |T|_\lambda |x|_\lambda$, for every $x \in E$, where $|T|_\lambda = \sup\{|T(x)|_\lambda : |x|_\lambda \leq 1\}$. Now consider the algebra $M_d(E)$ of all double multipliers (S, T) on E (here a double multiplier is a pair (S, T) of multipliers such that $xS(y) = T(x)y$, for every $x, y \in E$). Endow $M_d(E)$ with the involution $(S, T)^* = (T^*, S^*)$, where $T^*(x) = T(x^*)^*$, $S^*(x) = S(x^*)^*$, for every $x \in E$, (cf. [13]) and the locally m-pseudoconvex topology given by the following family of p_λ-norms

$$|(S, T)|_\lambda = \max(|S|_\lambda, |T|_\lambda), \text{ for every } (S, T) \in M_d(E).$$

The algebra $\left(M_d(E), (|(.,.)|_\lambda)_{\lambda\in\Lambda}\right)$ becomes a locally m-pseudoconvex A^*-algebra. For this, it remains only to define an auxiliary topology on $M_d(E)$. Let $x \in E$ and $(S, T) \in M_d(E)$. Using **2)** of Remarks 3.2, we get

$$|S(x)|_{\lambda,1}^{\frac{1}{p_\lambda}} \leq c_\lambda |S|_\lambda^{\frac{1}{p_\lambda}} \|x\|_\alpha^{\frac{1}{q_\alpha}} \text{ and } |T(x)|_{\lambda,1}^{\frac{1}{p_\lambda}} \leq c_\lambda |T|_\lambda^{\frac{1}{p_\lambda}} \|x\|_\alpha^{\frac{1}{q_\alpha}}$$

But also, by **2)** of Remarks 3.2, $|\cdot|_{\lambda,1}$ and $\|\cdot\|_\alpha$ are equivalent. This implies that S and T are continuous on $(E, \|\cdot\|_\alpha)$. Put $\|S\|_\alpha = \sup\{\|S(x)\|_\alpha : \|x\|_\alpha \leq 1\}$ and $\|T\|_\alpha = \sup\{\|T(x)\|_\alpha : \|x\|_\alpha \leq 1\}$. Then, by 2) of proposition 2.2, we have for each $x \in \widehat{E_\lambda}$,

$$\|T(x)\|_\alpha \leq \|x\|_\alpha \|S\|_\alpha \text{ and } \|S(x)\|_\alpha \leq \|x\|_\alpha \|T\|_\alpha.$$

This implies that $\|T\|_\alpha = \|S\|_\alpha$. Thus, for each $(S, T) \in M_d(E)$, define $\|(S, T)\|_\alpha = \|S\|_\alpha$. It is obvious that $\|(.,.)\|_\alpha$ is an algebra q_α-norm. In order to complete the proof, it will be sufficient to show that $\|(.,.)\|_\alpha$ satisfies the C^*-property. Since $\|S\|_\alpha = \|S^*\|_\alpha$and $\|T\|_\alpha = \|T^*\|_\alpha$, one gets from the above that

$$\left\|(S, T)^* (S, T)\right\|_\alpha \leq \|(S, T)\|_\alpha^2.$$

On the other hand

$$\|(S, T)\|_\alpha^2 = \|S\|_\alpha^2 = \sup\{\|S(x)^* S(x)\|_\alpha : \|x\|_\alpha \leq 1\}.$$

But

$$S(x)^*S(x) = S^*(x^*)S(x) = x^*(T^*S)(x).$$

Hence

$$\|(S,T)\|_\alpha^2 = \sup\{\|x^*(T^*S)(x)\|_\alpha : \|x\|_\alpha \leq 1\} \leq \|T^*S\|_\alpha = \left\|(S,T)^*(S,T)\right\|_\alpha.$$

Thus

$$\left\|(S,T)^*(S,T)\right\|_\alpha = \|(S,T)\|_\alpha^2, \text{ for every } (S,T) \in M_d(E).$$

and the desired result follows.

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