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Automorphism Groups of (M−1)-surfaces with the M-property

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Abstract

A compact Riemann surface X of genus g is called an (M−1)-surface if it admits an anticonformal involution that fixes g simple closed curves, the second maximum number by Harnack's theorem. If X also admits an automorphism of order g which cyclically permutes these g curves, then we shall call X an (M−1)-surface with the *M-property*. In this paper we investigate the automorphism groups of (M−1)-surfaces with the M-property.

Key Words: Riemann surface, (M−1)-surface.

1. Introduction

Let X be a compact Riemann surface of genus $g > 1$. X is said to be *symmetric* if it admits an anticonformal involution $T\colon X \to X$ which we call a *symmetry* of X. The fixed point set of T is either empty or consists of k simple closed curves, each of which is called a *mirror* of T. Here k is a positive integer and by Harnack's theorem $1 \leq k \leq g+1$. If T has $g+1$ mirrors, then it is called an *M-symmetry* and X is called an *M-surface*. In [10] and [11], we defined an M-surface to have the *M-property* if it admits an automorphism of order $g+1$ which cyclically permutes the mirrors of an *M-symmetry*, and we worked out the automorphism groups of M-surfaces with the M-property. Now let X be a Riemann surface of genus $g > 2$ and T a symmetry of X. If T has g mirrors, then it is called an *(M−1)-symmetry* and X is called an *(M−1)-surface*. Similarly, if X admits

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an automorphism of order g which cyclically permutes the mirrors of T, then we shall call it an (M$-$1)-surface with the *M-property.* In this paper we study the automorphism groups of (M$-$1)-surfaces with the M-property.

The design of this paper is as follows. Sections 2 and 3 are devoted to the background material. In Sections 4 and 5, we work out the full automorphism groups of nonhyperelliptic and hyperelliptic (M$-$1)-surfaces with the M-property, respectively. Theorems 4.1, 4.2 and 5.1 are the main results of these sections and we can state them as follows:

Theorem 1.1. *Let $X = \mathcal{U}/K$ be an (M$-$1)-surface of genus $g > 2$ with the M-property, where $\mathcal{U}$ is the hyperbolic plane and K is a torsion free Fuchsian group. Then K is always contained as a normal subgroup of index N in an NEC group Δ and*

- (i) *If X is nonhyperelliptic, then $N = 4g$, Δ has signature $(0; +; [2]; \{(2, 2, g)\})$ if g is odd or $(0; +; [\,]; \{(2^{(4)}, g)\})$ if g is even, and $Aut^{\pm}X (\cong \Delta/K)$ is isomorphic to $C_2 \times D_g$, where C_2 and D_g denote the cyclic group of order 2 and the dihedral group of order $2g$ respectively;*
- (ii) *If X is hyperelliptic, then $N = 8g$, Δ has signature $(0; +; [\,]; \{(2^{(3)}, 2g)\})$ and Δ/K is isomorphic to $C_2 \times D_{2g}$ and contained in $Aut^{\pm}X$.* □

In Section 6, we discuss the Wiman surfaces of type II, showing that these surfaces have the M-property.

2. Non-Euclidean Crystallographic Groups

Let $\mathcal{U}$ denote the hyperbolic plane and $\mathcal{L}$ denote the group of isometries of $\mathcal{U}$. (Throughout the paper we shall use these notations). A *non-Euclidean crystallographic (NEC) group* is a discrete subgroup Γ of $\mathcal{L}$ and in this paper we shall assume that $\mathcal{U}/\Gamma$ is compact. Let $\mathcal{L}^+$ be the subgroup of $\mathcal{L}$ consisting of conformal isometries. An NEC group contained in $\mathcal{L}^+$ is called a *Fuchsian group*, otherwise it is called a *proper NEC group.* Each NEC group has a signature

$$(g; \pm; [m_1, m_2, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}) \tag{2.1}$$

which determines and is determined by the algebraic and geometric structure of the group. If Γ has signature (2.1), then $\mathcal{U}/\Gamma$ is a compact surface of genus g with k holes which is orientable if a $+$ sign is used and nonorientable if a $-$ sign is used. The integers

$m_1, m_2, \ldots, m_r$ are called the *proper periods* and represent the branching over interior points of $\mathcal{U}/\Gamma$ in the natural projection from $\mathcal{U}$ to $\mathcal{U}/\Gamma$. The brackets $(n_{i1}, \ldots, n_{is_i})$ are called the *period cycles* and the integers $n_{i1}, \ldots, n_{is_i}$ are called the *link periods* and represent the branching around the ith hole. Here each proper and link period is a positive integer and greater than 1. The subgroup Γ^+ of Γ consisting of conformal isometries is called the *canonical Fuchsian group* of Γ.

Associated with signature (2.1) is a presentation for Γ with canonical generators:

(i) $x_1, \ldots, x_r$ (elliptic elements),

(ii) $c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$ (reflections),

(iii) $e_1, \ldots, e_k$ (usually hyperbolic elements, but sometimes elliptic),

(iv) $a_1, b_1, \ldots, a_g, b_g$ (hyperbolic elements), if $\mathcal{U}/\Gamma$ is orientable or
$a_1, \ldots, a_g$ (glide reflections), if $\mathcal{U}/\Gamma$ is nonorientable;

and relations

(a) $x_i^{m_i} = 1$, for $i = 1, \ldots, r$;

(b) $c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1$, for $i = 1, \ldots, k$ and $j = 1, \ldots, s_i$;

(c) $e_i c_{i0} e_i^{-1} = c_{is_i}$ for $i = 1, \ldots, k$ and

(d) $x_1 x_2 \ldots x_r e_1 e_2 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1$, if $\mathcal{U}/\Gamma$ is orientable or
$x_1 x_2 \ldots x_r e_1 e_2 \ldots e_k a_1^2 a_2^2 \ldots a_g^2 = 1$, if $\mathcal{U}/\Gamma$ is nonorientable.

The hyperbolic area of a fundamental region for Γ is given by

$$\mu(\Gamma) = 2\pi\left(\alpha g - 2 + k + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}(1 - \frac{1}{n_{ij}})\right),$$

where $\alpha = 2$ if there is a $+$ sign and $\alpha = 1$ if there is a $-$ sign. If Θ is a subgroup of Γ of finite index, then the Riemann-Hurwitz formula holds:

$$|\Gamma : \Theta| = \frac{\mu(\Theta)}{\mu(\Gamma)}.$$

A Fuchsian group with signature $(0; +; [l, m, n]; \{\quad\})$ (which we abbreivate $[l, m, n]$) is called a *triangle* group.

Sometimes we shall use an abbreviated notation for signatures. For example, the abbreviated form of

$$(4;+;[2,2,2,5,5];\{(3,3,3,6),(\ ),(\ ),(\ ),(\ )\})$$

will be

$$(4;+;[2^{(3)},5^{(2)}];\{(3^{(3)},6),(\ )^{(4)}\}).$$

In this paper, in calculating the signature of a subgroup Ω of a given NEC group Γ, we shall use Hoare's theorem (see Theorem 1 of [7]). It gives us a procedure for calculating the signature of Ω once we are given the action of the canonical generators of Γ on the cosets of Ω. If Γ is a Fuchsian group, then Theorem 1 of Singerman [14] is sufficient to calculate the signature of Ω. See Hoare [7] and Singerman [14] for details.

3. Automorphisms of Riemann and Klein Surfaces

Let X be a compact Riemann surface of genus $g>1$. Then X can be represented as $\mathcal{U}/K$, where K is a Fuchsian group without elliptic elements. Since X is compact, K contains no parabolic elements. Such Fuchsian groups are called *surface groups.* An automorphism of X is a conformal or anticonformal homeomorphism $f\colon X\to X$. All automorphisms of X, including the anticonformal ones, form a group under composition of maps and we shall denote it by $Aut^{\pm}X$ and the subgroup consisting of conformal automorphisms by $Aut^{+}X$. A finite group G acts as a group of automorphisms of X if and only if G is isomorphic to the factor group Γ/K, where Γ is an NEC group containing K as a normal subgroup. So we can find an epimorphism from Γ to G with kernel K. Such an epimorphism is called a *surface kernel* (*smooth*) *epimorphism.* It is known that $Aut^{+}X$ and $Aut^{\pm}X$ are isomorphic to $N^{+}(K)/K$ and $N(K)/K$ respectively, where $N^{+}(K)$ and $N(K)$ denote the normalisers of K in $\mathcal{L}^{+}$ and $\mathcal{L}$, respectively.

Definition 3.1. A compact Riemann surface X of genus $g>1$ is called *hyperelliptic* if it admits a conformal involution fixing $2g+2$ points. This involution is central in $Aut^{\pm}X$ and is called the *hyperelliptic involution.*

The following theorem is given in [13]; see also [3].

Theorem 3.1. *Let X be an (M−1)-surface of genus $g>2$. If X is nonhyperelliptic, then*

(i) *X admits exactly one (M−1)-symmetry; and*

(ii) $Aut^{\pm}X = C_2 \times Aut^{+}X$, *where C_2 is generated by the (M−1)-symmetry and $Aut^{+}X$ is isomorphic to a finite subgroup of the rotation group of the 2-sphere.*

If X is hyperelliptic, then X admits exactly two (M−1)-symmetries and their product is the hyperelliptic involution. □

It is known that every Klein surface S can be represented as $\mathcal{U}/\Gamma$, where Γ is an NEC group without elliptic elements. By a Klein surface we mean a compact surface with a dianalytic structure, (see [1]). If Γ contains reflections, then S has boundaries. Let H be a finite group acting on S by automorphisms. Then H is isomorphic to Δ/Γ, where Δ is an NEC group and $\Gamma \triangleleft \Delta$. Let $Aut(S)$ denote the group of all automorphisms of S. Then $Aut(S)$ is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ denotes the normaliser of Γ in $\mathcal{L}$. Let Γ^+ be the subgroup of Γ consisting of orientation preserving elements. Then $S^+ = \mathcal{U}/\Gamma^+$ is a Riemann surface known as the *complex double* of S, (see [1]). S is isomorphic (dianalytically equivalent) to $S^+/\langle T\rangle$, where T is a symmetry of S^+. If S^+ is hyperelliptic, then S is called *hyperelliptic*.

We now give the following theorem; see Theorem 4.5. (iii) of [5].

Theorem 3.2. *Let $S = \mathcal{U}/\Gamma$ be a nonorientable Klein surface of genus p with k boundaries. Then S is hyperelliptic if and only if there exists an NEC group that contains Γ as a subgroup of index 2 and has signature* $(0; +; [2^{(p)}]; \{(2^{(2k)})\})$. □

Finally, we give the following theorem which appears in [1]. For details on Klein surfaces see [1]and [4].

Theorem 3.3. *Let X be a Riemann surface and $T\colon X \to X$ a symmetry of X. Then the automorphisms of the Klein surface $X/\langle T\rangle$ consist of the conformal automorphisms of X commuting with T.* □

4. Nonhyperelliptic (M−1)-Surfaces with the M-property

Let X be an (M−1)-surface of genus $g > 2$ with the M-property and $T\colon X \to X$ an (M−1)-symmetry. Then $S = X/\langle T\rangle$ is a nonorientable Klein surface of genus 1 with g boundary components (see [4]). So S can be represented as $\mathcal{U}/\Gamma$, where Γ is an NEC group with signature

$$(1; -; [\,]; \{(\quad)^{(g)}\}). \tag{4.1}$$

Let $Aut(S)$ denote the group of all automorphisms of S. It follows from Theorem 3.3 that $Aut(S)$ consists of the elements of $Aut^{+}X$ commuting with T. Also, $Aut(S)$ is isomorphic to Δ/Γ, where Δ is the normaliser of Γ in $\mathcal{L}$. Since X has the M-property, $Aut(S)$ has a cyclic subgroup of order g whose generators cyclically permute the boundary components of S (mirrors of T). Thus, Δ has a subgroup Λ which contains Γ as a normal subgroup of index g and $\Lambda/\Gamma \cong C_g$. So there is an epimorphism from Λ to C_g whose kernel has signature (4.1).

We shall now find the signature of Λ. Since the generators of C_g cyclically permute the boundaries of S, the quotient surface $S/\langle C_g \rangle$ will have at least one smooth boundary component. Therefore, the signature of Λ will contain at least one empty period cycle and possibly some nonempty period cycles. So the signature of Λ will be of the form

$$(h; \pm; [m_1, m_2, \ldots, m_n]; \{(\quad)^{(k)}, (n_{11}, \ldots, n_{1s1}), \ldots, (n_{r1}, \ldots, n_{rsr})\}) \tag{4.2}$$

with $k \geq 1$. As Γ is contained in Λ with index g, by the Riemann-Hurwitz formula, we get

$$\frac{g-1}{g} = \delta h - 2 + k + r + \sum_{i=1}^{n}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{j=1}^{r}\sum_{\mu=1}^{s_j}(1 - \frac{1}{n_{j\mu}}), \tag{4.3}$$

where $\delta = 2$ if there is a $+$ sign and $\delta = 1$ if there is a $-$ sign in (4.2). Note that the left-hand side of (4.3) is less than 1. So is the right-hand side and hence we have the following restrictions on h, k, r and n: $0 \leq h \leq 1$, $1 \leq k \leq 2$, $0 \leq r \leq 1$ and $n \leq 3$.

Under these restrictions let us now find possible signatures for Λ.

Case 1: $h = 0$, $k = 1$, $r = 0$. It follows from (4.3) that $n \neq 0, 1$. Let $n = 2$. Then (4.3) gives

$$\frac{1}{g} = \frac{1}{m_1} + \frac{1}{m_2}, \tag{4.4}$$

and we see that unless $m_1, m_2 > g$, (4.4) cannot be solved. However, if $m_1, m_2 > g$, then we cannot define an epimorphism from Λ to C_g whose kernel has signature (4.1). This is because there is no element of C_g whose order is greater than g. Now let $n = 3$. Then (4.3) gives

$$1 + \frac{1}{g} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}. \tag{4.5}$$

It follows from (4.5) that we cannot have all $m_i > 2$, $(i = 1, 2, 3)$. Assume therefore that $m_1 = 2$ for the rest of this case. Thus, (4.5) becomes

$$\frac{1}{2} + \frac{1}{g} = \frac{1}{m_2} + \frac{1}{m_3}. \tag{4.6}$$

Since the left-hand side of (4.6) is greater than 1/2, we see that one of m_2 and m_3 must be less than 4. Suppose that $m_2 = 3$. Then by (4.6), m_3 can only be 3, 4 or 5. If we use these values of m_3 in (4.6), then we find the following signatures for Λ:

$(0; +; [2, 3, 3]; \{(\quad)\})$ $(g = 6)$

$(0; +; [2, 3, 4]; \{(\quad)\})$ $(g = 12)$

$(0; +; [2, 3, 5]; \{(\quad)\})$ $(g = 30)$.

However, we can show that there are no epimorphisms from the corresponding NEC groups to the finite groups C_6, C_{12} and C_{30} whose kernels have signature (4.1).

Now let $m_2 = 2$. Then we have $m_3 = g$. Therefore, in this case Λ has signature

$$(0; +; [2, 2, g]; \{(\quad)\}).$$

Case 2: $h = 0$, $k = 1$, $r = 1$. In this case (4.3) gives

$$\frac{g-1}{g} = \sum_{i=1}^{n}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{\mu=1}^{s}(1 - \frac{1}{n_\mu}). \tag{4.7}$$

If $n \neq 0$, then n has to be 1, otherwise (4.7) cannot be solved. Let $n = 1$, then (4.7) becomes

$$\frac{g-1}{g} = 1 - \frac{1}{m} + \frac{1}{2}\sum_{\mu=1}^{s}(1 - \frac{1}{n_\mu}). \tag{4.8}$$

Since $\frac{1}{2}\sum_{\mu=1}^{s}(1 - \frac{1}{n_\mu}) \geq \frac{1}{4}$, m can only be 2 or 3. Let $m = 2$. It is clear that s must be 1. Then (4.8) gives

$$1 - \frac{2}{g} = 1 - \frac{1}{n^*}, \tag{4.9}$$

and solving (4.9) we find that $n^* = g/2$, where g is even. Thus, Λ has signature

$$(0; +; [2]; \{(\quad), (g/2)\}).$$

Now let $m = 3$. Then (4.8) can be solved only when $s = 1$ and $g = 12$. So we find the following signature for Λ:

$$(0; +; [3]; \{(\quad), (2)\}).$$

However, it can easily be shown that there is no epimorphism from Λ to C_{12} whose kernel has signature (4.1).

Let $n = 0$. Then (4.8) becomes

$$\frac{g-1}{g} = \frac{1}{2}\sum_{\mu=1}^{s}(1 - \frac{1}{n_\mu}). \tag{4.10}$$

It follows from (4.10) that s can only be 2 or 3. Let $s = 3$. Then (4.10) gives

$$1 + \frac{2}{g} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}. \tag{4.11}$$

Since the left-hand side of (4.11) is greater than 1, we cannot have all $n_i > 2$, $(i = 1, 2, 3)$. Suppose that $n_1 = 2$. Then we get

$$\frac{1}{2} + \frac{2}{g} = \frac{1}{n_2} + \frac{1}{n_3}. \tag{4.12}$$

Since the left-hand side of (4.12) is greater than 1/2, we cannot have both n_2 and n_3 greater than 3. Let $n_2 = 3$. Then by (4.12) we see that n_3 can only be 3, 4 or 5. Using these values in (4.12), we find the following signatures for Λ:

$$(0; +; [\;]; \{(\quad), (2, 3, 3)\}) \quad (g = 12)$$
$$(0; +; [\;]; \{(\quad), (2, 3, 4)\}) \quad (g = 24)$$
$$(0; +; [\;]; \{(\quad), (2, 3, 5)\}) \quad (g = 60).$$

However, we can show that there are no epimorphisms from the corresponding NEC groups to the finite groups C_{12}, C_{24} and C_{60} whose kernels have signature (4.1).

Now let $n_2 = 2$. Then from (4.12) it follows that $n_3 = g/2$, where g is even. Thus, in this case Λ has signature

$$(0; +; [\;]; \{(\quad), (2, 2, g/2)\}).$$

Now let $s = 2$. Then (4.10) gives

$$\frac{2}{g} = \frac{1}{n_1} + \frac{1}{n_2}. \tag{4.13}$$

It follows from (4.13) that we cannot have both n_1 and n_2 greater or less than g. If one of n_1 and n_2 is less than g, then the other is greater than g. In that case we cannot define an epimorphism from Λ to C_g whose kernel has signature (4.1). This is because there is no element of C_g whose order is greater than g. Therefore, we have $n_1 = n_2 = g$ and so Λ has signature

$$(0; +; [\,]; \{(\quad), (g, g)\}).$$

Case 3: $k = 2$. It follows from (4.3) that h and r must be 0, and therefore we have

$$\frac{g-1}{g} = \sum_{i=1}^{n}(1 - \frac{1}{m_i}). \tag{4.14}$$

Since the left-hand side of (4.14) is less than 1, we must have $n = 1$. Thus, (4.14) gives

$$\frac{g-1}{g} = 1 - \frac{1}{m}, \tag{4.15}$$

and solving (4.15) we find that $m = g$. So in this case Λ has signature

$$(0; +; [g]; \{(\quad)^{(2)}\}).$$

Case 4: $h = 1$. It follows from (4.3) that we must have $\delta = 1$, $r = 0$ and $k = 1$. Then (4.3) becomes

$$\frac{g-1}{g} = \sum_{i=1}^{n}(1 - \frac{1}{m_i}). \tag{4.16}$$

As in the previous case, we see that $n = 1$. So (4.16) becomes

$$\frac{g-1}{g} = 1 - \frac{1}{m}, \tag{4.17}$$

and solving (4.17) we find that $m = g$. So in this case Λ has signature

$$(1; -; [g]; \{(\quad)\}).$$

As a result, we have the following possible signatures left for Λ:

$$\begin{aligned}
\Lambda_1 &: (0;+;[g];\{(\ )^{(2)}\})\\
\Lambda_2 &: (1;-;[g];\{(\ )\})\\
\Lambda_3 &: (0;+;[2,2,g];\{(\ )\})\\
\Lambda_4 &: (0;+;[\ ];\{(\ ),(g,g)\})\\
\Lambda_5 &: (0;+;[\ ];\{(\ ),(2,2,g/2)\})\\
\Lambda_6 &: (0;+;[2];\{(\ ),(g/2)\}).
\end{aligned}$$

Lemma 4.1. *There are no epimorphisms from Λ_3, Λ_4, Λ_5 and Λ_6 to C_g whose kernels have signature* (4.1).

Proof. Let us consider the group Λ_3. It has a presentation

$$\langle x_1, x_2, x_3, c, e \mid x_1^2 = x_2^2 = x_3^g = c^2 = ece^{-1}c = x_1x_2x_3e = 1\rangle.$$

If g is odd, then there is no element of order 2 in C_g and so there is no epimorphism from Λ_3 onto C_g whose kernel has signature (4.1). Let g be even and assume that $\theta\colon \Lambda_3 \to C_g$ is an epimorphism as required. In order to get g empty period cycles for the signature of the kernel of θ (for short $Ker\theta$), we must have $\theta(c) = \theta(e) = 1$. Since $\alpha^{\frac{g}{2}}$ is the only element of order 2 in C_g, the elliptic generators x_1 and x_2 must map to $\alpha^{\frac{g}{2}}$, otherwise there will be elliptic elements in the kernel, where α is a generator for C_g. By the relation $x_1x_2x_3e = 1$ in the presentation of Λ_3, it follows that x_3 must map to the identity. This means that there is an element of order g in $Ker\theta$, and this is not allowed. Thus, there is no epimorphism from Λ_3 to C_g whose kernel has signature (4.1). Similarly, we can show that there are no such epimorphisms from the groups Λ_4, Λ_5 and Λ_6 to C_g either. □

Lemma 4.2. *Let g be even. Then there is an epimorphism from Λ_1 to C_g whose kernel has signature* (4.1).

Proof. The group Λ_1 has a presentation

$$\langle x, c_1, c_2, e_1, e_2 \mid x^g = c_1^2 = c_2^2 = e_1c_1e_1^{-1}c_1 = e_2c_2e_2^{-1}c_2 = xe_1e_2 = 1\rangle. \tag{4.18}$$

In order to define an epimorphism from Λ_1 onto C_g whose kernel has signature (4.1), we need to consider the following points. Under such an epimorphism one of the reflection generators of Λ_1 (for example c_1) must map to the identity of C_g, otherwise there will be no reflections in the kernel. Also, the hyperbolic generator e_1 must map to the identity and so we can get g empty period cycles for the signature of the kernel. By the relation $xe_1e_2 = 1$ in (4.18), x can map to an element of order g and e_2 to its inverse. However, c_2 cannot map to the identity, otherwise there will be more than g empty period cycles in the signature of the kernel and this is not allowed. Thus, c_2 must map to an element of order 2 and this is not possible unless g is even.

By considering these points the epimorphism $\theta_1 \colon \Lambda_1 \to C_g$ has to be as follows:

$$\theta_1 : \begin{cases} x & \mapsto \alpha = (1, 2, \ldots, g-1, g) \\ c_1 & \mapsto 1 = (1)(2)\ldots(g-1)(g) \\ e_1 & \mapsto 1 = (1)(2)\ldots(g-1)(g) \\ c_2 & \mapsto \alpha^{\frac{g}{2}} = (1, \frac{g}{2}+1)(2, \frac{g}{2}+2)\ldots(\frac{g}{2}, g) \\ e_2 & \mapsto \alpha^{-1} = (g, g-1, \ldots, 2, 1) \end{cases}$$

where α is a generator of C_g.

If we had begun with the reflection c_2 and considered the same points as above, we would have got the epimorphism $\theta_1^* \colon \Lambda_1 \to C_g$ given by interchanging the subscripts. It is clear that the kernels of θ_1 and θ_1^* are distinct. However, as we shall see below, they have signature (4.1). If $f \colon C_g \to C_g$ is an automorphism of C_g, then $f \circ \theta_1$ is another epimorphism from Λ_1 to C_g and $Ker\theta_1 = Ker(f \circ \theta_1)$. Therefore, there are only two epimorphisms from Λ_1 to C_g (up to automorphism of C_g) whose kernels have signatures (4.1), namely θ_1 and θ_1^*.

By [7] section 4, x induces no proper periods for $Ker\theta_1$ other than 1. The reflection generator c_1 fixes all the cosets. Hence it induces g reflection generators for $Ker\theta_1$. We call these reflections $c_{11}, c_{12}, \ldots, c_{1g}$. The orbits of the dihedral group $\langle e_1c_1e_1^{-1}, c_1\rangle \cong D_1$ are $\{1\}, \{2\}, \{3\}, \ldots, \{g\}$ and they induce the links $c_{11} \sim c_{11}$, $c_{12} \sim c_{12}$, $\ldots$, $c_{1g} \sim c_{1g}$ respectively. Each of these links is a chain and so we have g chains. Therefore, there are g empty period cycles in the signature of $Ker\theta_1$. These g cosets cannot be partitioned as required in Hoare's theorem and hence $\mathcal{U}/Ker\theta_1$ is nonorientable. By the Riemann-Hurwitz formula $\mu(Ker\theta_1) = g\mu(\Lambda_1)$ we find that the genus of $\mathcal{U}/Ker\theta_1$ is 1 and so $Ker\theta_1$ has signature (4.1). Similarly, we can show that $Ker\theta_1^*$ has signature (4.1). □

Remark 4.1. The calculations above tell us that $\mathcal{U}/Ker\theta_1$ is a nonorientable Klein surface of genus 1 with g boundary components. It admits an automorphism Φ of order

g cyclically permuting the boundary components. Its complex double X is an (M−1)-surface of genus g. By Theorem 3.3, Φ is a conformal automorphism of X and cyclically permutes the mirrors of T, where T is an (M−1)-symmetry of X such that $\mathcal{U}/Ker\theta_1$ and $X/\langle T\rangle$ are dianalytically equivalent. Thus, X has the M-property and C_g is contained in Aut^+X. Similar discussions apply to $\mathcal{U}/Ker\theta_1^*$. Although θ_1 and θ_1^* have different kernels, we can show that the automorphism groups of $\mathcal{U}/Ker\theta_1$ and $\mathcal{U}/Ker\theta_1^*$ are isomorphic. Thus, the difference between θ_1 and θ_1^* will not effect our results and so from now on we shall consider only θ_1.

Theorem 4.1. *Let $X = \mathcal{U}/K$ be a nonhyperelliptic (M−1)-surface of genus $g > 2$ (g even) with the M-property. Then K is always contained as a normal subgroup of index $4g$ in a maximal NEC group Δ, where Δ has signature $(0; +; [\ ]; \{(2^{(4)}, g)\})$, and $Aut^{\pm}X(\cong \Delta/K)$ is isomorphic to $C_2 \times D_g$.*

Proof. It follows from Lemma 4.2 and Remark 4.1 that K is normal in an NEC group Λ_1 with index $2g$, where Λ_1 has signature $(0; +; [g]; \{(\ \ )^{(2)}\})$. Since X has the M-property, Λ_1/K has a subgroup isomorphic to C_g. Also X admits the (M−1)-symmetry $T\colon X \to X$, which is anticonformal. By Theorem 3.1, T is central in $Aut^{\pm}X$ and C_g is contained in Aut^+X. Therefore, Λ_1/K is isomorphic to $C_2 \times C_g$. In order to find $Aut^{\pm}X$ we shall look for possible extensions of θ_1. First, we need to find NEC and finite groups that contain Λ_1 and C_g (with the same index) respectively. It follows from [2] that Λ_1 is always contained as a normal subgroup of index 2 in an NEC group Δ_1 with signature

$$(0; +; [\ ]; \{(2^{(4)}, g)\}). \tag{4.19}$$

Let G be a finite group containing C_g with index 2 and $\mu_1\colon \Delta_1 \to G$ an extension of θ_1 with the same kernel. It follows from the nonhyperelliptic case in Theorem 3.1 that G is isomorphic to a finite subgroup of the rotation group of the 2-sphere. As is well-known, any finite rotation group of the 2-sphere is cyclic, dihedral, or isomorphic to A_4, S_4 or A_5. As the groups A_4, S_4 and A_5 have no cyclic subgroups of index 2, G can only be isomorphic to C_{2g} or D_g. Since there is no element of order $2g$ in Δ_1, we cannot define an epimorphism from Δ_1 to C_{2g} whose kernel has signature (4.1). Thus, the only possibility is that $G \cong D_g$.

Having found the groups containing Λ_1 and C_g, we can now define the epimorphism.

The group Δ_1 has a presentation

$$\langle d_0, d_1, d_2, d_3, d_4 \mid d_0^2 = d_1^2 = d_2^2 = d_3^2 = d_4^2 = (d_0d_1)^2 = (d_1d_2)^2 = (d_2d_3)^2 = (d_3d_4)^2 = (d_4d_0)^g = 1\rangle,$$

where Λ_1 is generated by $x = d_4d_0$, $c_1 = d_1$, $c_2 = d_3$, $e_1 = d_0d_2$, $e_2 = d_2d_4$ and D_g has a presentation

$$\langle \alpha, \beta \mid \beta^2 = \alpha^g = (\alpha\beta)^2 = 1\rangle.$$

Then the epimorphism $\mu_1\colon \Delta_1 \to D_g$ has to be defined as follows:

$$\mu_1 : \begin{cases} d_0 & \mapsto \beta \\ d_1 & \mapsto 1 \\ d_2 & \mapsto \beta \\ d_3 & \mapsto \alpha^{\frac{g}{2}} \\ d_4 & \mapsto \beta\alpha. \end{cases}$$

As before, we can show that $Ker\mu_1$ has signature (4.1) and θ_1 is the restriction of μ_1 by construction. Moreover, θ_1 and μ_1 have the same kernel and hence $X = \mathcal{U}/K$ is the complex double of $\mathcal{U}/Ker\mu_1$. It follows from [2] that the NEC group Δ_1 is maximal and so μ_1 has no further extensions. Therefore, the automorphism group of the nonhyperelliptic Klein surface $\mathcal{U}/Ker\mu_1$ is isomorphic to D_g and by Theorem 3.1 $Aut^{\pm}X$ is isomorphic to $C_2 \times D_g$.

Finally, we show that X is nonhyperelliptic. As g is even, the only central involution in D_g is $\alpha^{\frac{g}{2}}$, where α is a generator of D_g with $\alpha^g = 1$. However, using Hoare's theorem we can show that the signature of $\mu_1^{-1}(\langle \alpha^{\frac{g}{2}}\rangle)$ is different from $(0; +; [2]; \{(2^{(2g)})\})$, where $\langle \alpha^{\frac{g}{2}}\rangle$ denotes the cyclic group of order 2 generated by $\alpha^{\frac{g}{2}}$. Thus, the Klein surface $\mathcal{U}/Ker\mu_1$ is not hyperelliptic by Theorem 3.2 and neither is its complex double. □

Lemma 4.3. *Let g be odd. Then there is an epimorphism from Λ_2 to C_g whose kernel has signature* (4.1).

Proof. The group Λ_2 has signature $(1; -; [g]; \{(\ )\})$ and a presentation

$$\langle a, x, c, e \mid x^g = c^2 = ece^{-1}c = xea^2 = 1\rangle. \tag{4.20}$$

Let $\theta_2\colon \Lambda_2 \to C_g$ be an epimorphism whose kernel has signature (4.1). As before, we can show that $\theta_2(c)$ and $\theta_2(e)$ must be the identity and $\theta_2(x)$ must be a generator of C_g.

By the relation $xea^2 = 1$ in (4.20), we see that $(\theta_2(a))^2$ is a generator of C_g which is impossible unless g is odd. Thus the epimorphism $\theta_2\colon \Lambda_2 \to C_g$ acts as follows:

$$\theta_2 : \begin{cases} x & \mapsto \alpha^2 = (1, 3, \ldots, g, 2, 4, \ldots, g-1) \\ c & \mapsto 1 = (1)(2)\ldots(g) \\ e & \mapsto 1 = (1)(2)\ldots(g) \\ a & \mapsto \alpha^{-1} = (g, g-1, \ldots, 2, 1). \end{cases}$$

As before, we can show that $Ker\theta_2$ has signature (4.1). Note that θ_2 is the unique epimorphism up to automorphism of C_g whose kernel has signature (4.1). □

Theorem 4.2. *Let $X = \mathcal{U}/K$ be a nonhyperelliptic (M−1)-surface of genus $g > 2$ (g odd) with the M-property. Then K is always contained as a normal subgroup of index $4g$ in a maximal NEC group Δ, where Δ has signature $(0; +; [2]; \{(2, 2, g)\})$, and $Aut^{\pm}X (\cong \Delta/K)$ is isomorphic to $C_2 \times D_g$.*

Proof. Since X has the M-property, it follows from Lemma 4.3 that K is normal in an NEC group Λ_2. Following similar argument in the proof of Theorem 4.1, we can show that Λ_2/K is isomorphic to $C_2 \times C_g$. Since we want to determine $Aut^{\pm}X$, we shall look for possible extensions of θ_2. As in the proof of Theorem 4.1, we can extend θ_2 to an epimorphism $\mu_2\colon \Delta_2 \to D_g$. Here Δ_2 is a maximal NEC group by [2] and contains Λ_2 as a subgroup of index 2. It has signature $(0; +; [2]; \{(2, 2, g)\})$ and a presentation

$$\langle u, c_0, c_1, c_2, c_3 \mid u^2 = c_0^2 = c_1^2 = c_2^2 = c_3^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^g = uc_0uc_3 = 1\rangle.$$

Then the epimorphism $\mu_2\colon \Delta_2 \to D_g$ has to be defined as follows:

$$\mu_2 : \begin{cases} u & \mapsto \beta \\ c_0 & \mapsto \beta\alpha \\ c_1 & \mapsto 1 \\ c_2 & \mapsto \beta\alpha \\ c_3 & \mapsto \beta\alpha^{-1}, \end{cases}$$

where α and β generate D_g and obey $\beta^2 = \alpha^g = (\beta\alpha)^2 = 1$. Similarly, using Hoare's theorem we can show that $Ker\mu_2$ has signature (4.1). Note that θ_2 is the restriction of μ_2

by construction. It follows that θ_2 and μ_2 have the same kernel and therefore $X = \mathcal{U}/K$ is the complex double of $\mathcal{U}/Ker\mu_2$. As Δ_2 is maximal, μ_2 has no further extensions. Thus, the group of automorphisms of $\mathcal{U}/Ker\mu_2$ is isomorphic to D_g and by Theorem 3.1 $Aut^{\pm}X$ is isomorphic to $C_2 \times D_g$.

Since g is odd, there is no central involution in D_g and hence $\mathcal{U}/Ker\mu_2$ is not hyperelliptic and neither is its complex double. □

5. Hyperelliptic (M−1)-Surfaces with the M-property

Theorem 5.1. *Let $X = \mathcal{U}/K$ be a hyperelliptic (M−1)-surface of genus $g > 2$ with the M-property. Then K is always contained as a normal subgroup of index $8g$ in an NEC group Ω with signature $(0; +; [\ ]; \{(2^{(3)}, 2g)\})$ and with Ω/K isomorphic to $C_2 \times D_{2g}$ and contained in $Aut^{\pm}X$ with index at most 2.*

Proof. Since X is hyperelliptic, it follows from Theorem 3.1 that X has two (M−1)-symmetries. Let $T\colon X \to X$ be one of the (M−1)-symmetries of X. Then $S = X/\langle T\rangle$ is a nonorientable Klein surface of genus 1 with g boundary components, where $\langle T\rangle$ denotes the cyclic group of order 2 generated by T. We know that S can be represented as $\mathcal{U}/\Gamma$, where Γ is an NEC group with signature (4.1). Since X is hyperelliptic, S is a hyperelliptic Klein surface, see [5]. Let $Aut(S)$ denote the group of automorphisms of S. Then $Aut(S)$ is isomorphic to Ω/Γ, where Ω is an NEC group containing Γ as a normal subgroup of finite index. It follows from [4] that Ω has signature $(0; +; [\ ]; \{(2^{(3)}, 2g)\})$ and $Aut(S)$ is isomorphic to D_{2g}, see Theorem 6.3.3 of [4].

Let us now prove this result. Since Ω/Γ is isomorphic to D_{2g}, we can find an epimorphism $\phi\colon \Omega \to D_{2g}$ whose kernel has signature (4.1). The group Ω has a presentation

$$\langle c_0, c_1, c_2, c_3 \mid c_0^2 = c_1^2 = c_2^2 = c_3^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^{2g} = 1\rangle. \quad (5.1)$$

Then we can define the epimorphism $\phi\colon \Omega \to D_{2g}$ as follows:

$$\phi : \begin{cases} c_0 & \mapsto xy \\ c_1 & \mapsto 1 \\ c_2 & \mapsto y^g \\ c_3 & \mapsto x, \end{cases}$$

where x and y generate D_{2g} and obey $x^2 = y^{2g} = (xy)^2 = 1$.

Using Hoare's theorem we can show that $Ker\phi$ has signature (4.1). Thus, the automorphism group of the Klein surface $\mathcal{U}/Ker\phi$ is isomorphic to D_{2g}.

Here the automorphism y^g has order 2 and is central in D_{2g}. We can show that $\phi^{-1}(\langle y^g \rangle)$ has signature $(0;+;[2];\{(2^{(2g)})\})$. Thus, the Klein surface $\mathcal{U}/Ker\phi$ is hyperelliptic by Theorem 3.2, and so is its complex double.

Let $X = \mathcal{U}/K$ be the complex double of $\mathcal{U}/Ker\phi$. Then $Ker\phi$ contains K as a subgroup of index 2 and we know that Ω contains $Ker\phi$ as a normal subgroup of index $4g$. We shall now show that Ω contains K as a normal subgroup of index $8g$ and Ω/K is isomorphic to $C_2 \times D_{2g}$. We can easily do this by defining an epimorphism $\Phi\colon \Omega \to C_2 \times D_{2g}$ by means of the epimorphism ϕ as follows, where $Ker\Phi = K$.

The group $C_2 \times D_{2g}$ has a presentation

$$\langle k, x, y \mid k^2 = x^2 = y^{2g} = (xy)^2 = 1, kx = xk, ky = yk \rangle,$$

and we can define the epimorphism $\Phi\colon \Omega \to C_2 \times D_{2g}$ as follows:

$$\Phi : \begin{cases} c_0 & \mapsto kxy \\ c_1 & \mapsto k \\ c_2 & \mapsto ky^g \\ c_3 & \mapsto kx, \end{cases}$$

where c_0, c_1, c_2 and c_3 are the generators of Ω in (5.1). Using Hoare's theorem we can show that $Ker\Phi$ is a Fuchsian surface group, i.e. $Ker\Phi$ has signature $(g;-)$. We can also show that $\Phi^{-1}(\langle k \rangle)$ has signature $(1;-;[\ ];\{(\quad)^{(g)}\})$. Thus, k is an (M$-$1)-symmetry and $Ker\Phi = K$. As a result, Ω contains K as a normal subgroup of index $8g$ and Ω/K, which is isomorphic to $C_2 \times D_{2g}$, is contained in $Aut^{\pm}X$. It follows from [2] that Ω is maximal and therefore $Aut^{\pm}X$ is isomorphic to $C_2 \times D_{2g}$. □

6. Wiman Surfaces of Type II

It was shown by Wiman [16] that the largest possible order of an automorphism of a Riemann surface of genus $g > 1$ is $4g + 2$ and the second largest possible order is $4g$. (Also see [6]). The corresponding Riemann surfaces are obtained as the quotients of the hyperbolic plane by the kernels of smooth homomorphisms from triangle groups with signatures $[2, 2g+1, 4g+2]$ and $[2, 4g, 4g]$ onto C_{4g+2} and C_{4g}, respectively. Following Kulkarni [8] we call these surfaces *Wiman surfaces of type I and II*, respectively. It is known that these surfaces are hyperelliptic. However, only the Wiman surfaces of type II are (M−1)-surfaces; see [12] and [13].

Let $X = \mathcal{U}/K$ be the Wiman surface of type II of genus $g > 2$. Then K is contained as a normal subgroup of index $8g$ in a triangle group of signature $[2, 4, 4g]$. The group of conformal automorphisms of X, Aut^+X, consists of $8g$ automorphisms and has a presentation

$$\langle x, y \mid x^2 = y^{4g} = 1,\ xyx^{-1} = y^{2g-1} \rangle, \tag{6.1}$$

see Kulkarni [8].

It is also known that X can be formed as follows. Consider a regular $4g$-sided hyperbolic polygon and label its sides by the numbers 1, 2, 3, ..., $2g$, 1, 2, 3, ..., $2g$, where $g > 2$. By identifying the opposite sides of this polygon we obtain the Wiman surface of type II of genus g.

We illustrate the Wiman surface of type II of genus 3 in the Figure, which consists of 48 triangles and each triangle has internal angles $\pi/2$, $\pi/4$ and $\pi/12$. Here the dotted and the dashed line segments are also considered as the sides of the triangles. After identification the dotted and the dashed line segments become the mirrors of the (M−1)-symmetries, and the vertices become a single point on the surface.

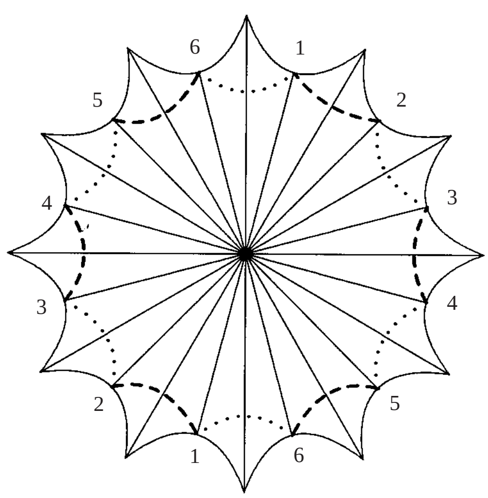

Figure. The Wiman surface of type II of genus 3.

If we draw the same figure for X (the Wiman surface of type II of genus g), then we can observe that y^4 is the rotation of order g about the centre of the polygon, where y is the same generator of Aut^+X in (6.1). From the identification of the sides it follows that y^4 cyclically permutes the mirrors of the (M−1)-symmetries. Thus, X has the M-property.

Finally, we give the following presentation for $Aut^{\pm}X$, which can be obtained from (6.1):

$$\langle\, p, q, r \,|\, p^2 = q^2 = r^2 = (pq)^2 = (pr)^{4g} = (qr)^2(pr)^{2g} = 1\rangle, \qquad (6.2)$$

where p, q and r are the reflections on the sides of a triangle. q is an (M−1)-symmetry and so it has g mirrors. p has one mirror and r has two mirrors. The presentation (6.2) can also be deduced from the fourth row, $(4, m, n)$, of Table 1 of Natanzon [13], where we choose $m = 1$ and $n = 2g$.

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