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On the Weak-Integrity of Trees

Alpay Kırlangıç

Abstract

In this paper the concept of *weak-integrity* is introduced as a new measure of the stability of a graph G and it is defined as $I_w(G) = \min_{S \subset V(G)} \{|S| + m_e(G-S)\}$, where $m_e(G-S)$ denotes the number of edges of a largest component of G-S. We investigate the weak-integrity of trees and compute the weak-integrity of a binomial tree and all the trees with at most 7 vertices. We also give some results about the weak-integrity of graphs obtained from binary operations.

Key Words: Stability; Connectivity; Integrity

1. Introduction

The stability of a communication network composed of processing stations and communication links is of prime importance to network designers. As the network begins losing links or stations, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. To describe the stability of communication networks we have some graph theoretical parameters, e.g., connectivity, thoughness, binding number and integrity. These parameters deal with two fundamental questions about the resulting graph. *How many vertices can still communicate? How diffucult is it to reconnect the graph?* The *integrity* is a measure which deals with the first question. To obtain an answer to the first question, the concepts of *Integrity* and *Edge-Integrity* were introduced by *Barefoot, Entringer and Swart* as a measure of the stability of a graph [3, 4, 7, 8, 10]. The integrity $I(G)$ of a graph G is defined as $I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\}$, where $m(G-S)$ denotes the number of vertices of a largest component of $G-S$. The edge-integrity $I'(G)$ of a graph G is defined as $I'(G) = \min_{S \subseteq E(G)} \{|S| + m(G-S)\}$, where $m(G-S)$ denotes the number of vertices of a largest component of $G-S$. Moreover the *Pure Edge-Integrity* was introduced by Bagga and Deogun[6] and is defined as $I_p(G) = \min_{S \subseteq E(G)} \{|S| + m_e(G-S)\}$, where $m_e(G-S)$ denotes the number of edges of a largest component of $G-S$.

Similiarly, to obtain an answer to the first question stated above, we can need the minumum sum of the *numbers of vertices* being removed and the *numbers of edges* of a largest remaining component. This motivated the author to introduce a *new measure* of stability of a graph G in this sense and it is called *Weak-Integrity*. Formally, the weak-integrity $I_w(G)$ of a graph G is defined as

$$I_w(G) = \min_{S \subset V(G)} \{|S| + m_e(G-S)\},$$

where $m_e(G-S)$ denotes the number of edges of a largest component of $G-S$. Any set S with property that $\mid S \mid + m_e(G-S) = I_w(G)$ is called an $I_w - set$ of G and it is obvious that $I_w(G) \geq I(G) - 1$ for any graph G.

The aim of this paper is to investigate the *weak-integrity* of trees. Let T be a tree of order n and all the trees with $n \leq 7$ vertices are given in Appendix [5]. Let T_i be any tree for $1 \leq i \leq 24$ as in Appendix.

In Section 2 we give several bounds for any tree T and compute the values $I_w(T_i)$ and $I_w(K_2 \times T_i)$ for any T_i in Appendix. In Section 3 the weak-integrity of a binomial tree is given. We also determine the weak-integrity of graphs obtained from binary operations. Moreover, the values $I_w(B_n[T_i])$ and $I_w(T_i \circ B_n)$ for any T_i in Appendix and $2 \leq n \leq 4$ are calculated. In Section 4 we compare the weak-integrity with other stability measures.

2. The Weak-Integrity of Trees

Firstly we can say the following observation for the weak-integrity:

$\triangleright \triangleright$ Let $S \subset V(T)$. Since $m_e(T-S) = m(T-S)-1$ for every S we have $I_w(T) = I(T)-1$ for any tree T.

Now we give the following definitions.

Definition 2.1 *The* connectivity $\kappa(G)$ *of a graph G is the minumum number of vertices whose removal results in a disconnected graph, or trivial graph.*

Definition 2.2 *A subset S of V is called a covering of G if every edge of G has at least one end in S. A covering S is a* minumum covering *if G has no covering S' with* $| S' | < | S |$*. The* covering number $\alpha(G)$ *is the number of vertices in a minumum covering of G.*

Definition 2.3 *A vertex* dominating set *for a graph G is a set S of vertices such that every vertex of G belongs to S or is adjacent to a vertex of S. The minumum cardinality of a vertex dominating set in a graph G is called the* vertex dominating number *of G and it is denoted by* $\sigma(G)$.

Theorem 2.1 *Let S be a dominating set of T. For any tree T,*

$$I_w(T) \leq \sigma(T) + | E(T - S) | .$$

Proof. If S be a dominating set, then each one of vertices in T-S is adjacent to vertices of S and the graph T-S has not any cycle. So if we remove the vertices of S from T, then $m_e(T - S) \leq | E(T - S) |$. This completes the proof. □

The values of $\sigma(T_i)$, $\alpha(T_i)$ and $I_w(T_i)$ for the trees in Appendix are given in Table 1. From these values we make the following observation:
⊳⊳ If $\sigma(T) = \alpha(T)$ for any tree T, then $I_w(T) \leq \sigma(T)$.

Table 1. The values of $\sigma(T_i)$, $\alpha(T_i)$ and $I_w(T_i)$ for the trees in Appendix.

i	$\sigma(T_i)$	$\alpha(T_i)$	$I_w(T_i)$	i	$\sigma(T_i)$	$\alpha(T_i)$	$I_w(T_i)$
1	1	1	1	13	2	2	2
2	1	1	1	14	3	3	3
3	2	2	2	15	2	3	3
4	1	1	1	16	3	3	3
5	2	2	2	17	3	3	2
6	2	2	2	18	2	2	2
7	1	1	1	19	3	3	2
8	3	3	3	20	2	2	2
9	2	2	2	21	2	2	2
10	3	3	2	22	1	1	1
11	2	2	2	23	3	3	3
12	1	1	1	24	2	2	2

Next we give the definition of Cartesian product between two graphs.

Definition 2.4 *The* (Cartesian) product $G_1 \times G_2$ *of graphs* G_1 *and* G_2 *has* $V(G_1) \times V(G_2)$ *as its vertex set and* (u_1, u_2) *is adjacent to* (v_1, v_2) *if either* $u_1 = v_1$ *and* u_2 *is adjacent to* v_2 *or* $u_2 = v_2$ *and* u_1 *is adjacent to* v_1*.*

Now we give an upper bound for $I_w(K_2 \times T)$.

Theorem 2.2 *For any tree T,* $I_w(K_2 \times T) \leq 3I_w(T)$*.*

Proof. Let S be a subset of V(T) such that $\mid S \mid + m_e(T - S) = I_w(T)$. If we remove $2 \mid S \mid$ vertices from graph $K_2 \times T$, then

$$m_e(K_2 \times T - 2 \mid S \mid) = 2m_e(T - S) + m(T - S),$$

where m(T-S) is the number of vertices of a largest component of T-S. Since $m(T - S) = m_e(T - S) + 1$ for any tree T, we have

$$\begin{aligned} I_w(K_2 \times T) &= 2 \mid S \mid + 2m_e(T - S) + m(T - S) \\ &= 2(\mid S \mid + m_e(T - S)) + m_e(T - S) + 1 \\ &= 2I_w(T) + m_e(T - S) + 1 \\ &= 2I_w(T) + I_w(T) - \mid S \mid + 1. \qquad (1) \end{aligned}$$

Since $\mid S \mid \geq \kappa(T) = 1$ for any tree T, we have

$$I_w(T) - \mid S \mid \leq I_w(T) - \kappa(T) = I_w(T) - 1. \qquad (2)$$

If we substitue (2) in (1), then this completes the proof. □

In Table 2 we give the values $I_w(K_2 \times T_i)$ for the trees in Appendix.

Definition 2.5 *The distance d(u,v) between two points u and v in G is the length of a shortest path joining them if any; otherwise* $d(u, v) = \infty$*. A shortest u-v path is often called a geodesic. The* diameter *diam(G) of a connected graph G is the length of any longest geodesic.*

Let deg(x) be the degree of any vertex x in T. In the following theorem we consider a tree T with $diam(T) = 4$(Figure 1). Any tree with $diam(T) = 4$ is isomorphic to the following tree or its induced subgraph T′ with $diam(T') = 4$.

Table 2. The values of $I_w(K_2 \times T_i)$ for the trees in Appendix.

i	$I_w(K_2 \times T_i)$	i	$I_w(K_2 \times T_i)$	i	$I_w(K_2 \times T_i)$
1	2	9	5	17	6
2	3	10	6	18	5
3	4	11	5	19	6
4	3	12	3	20	5
5	5	13	5	21	5
6	5	14	7	22	3
7	3	15	6	23	7
8	6	16	7	24	5

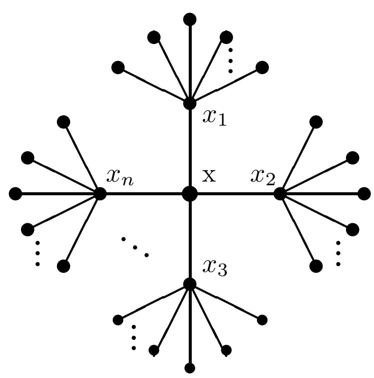

Figure 1. A tree with diam(T) = 4.

Let x be a root vertex of T and let $x_1, x_2, ..., x_n$ be the vertices which are adjacent to x where $deg(x_1) \geq deg(x_2) \geq ... \geq deg(x_n)$(Figure 1).

Theorem 2.3 *Let T be a tree with* $diam(T) = 4$*. Then*

$$I_w(T) = min\{\alpha(T), \min_{0 \leq k < n} \{k + deg(x_{k+1})\}\}.$$

Proof. Consider the tree in Figure 1 and let S be a $I_w - set$ of T. Then we have two cases, depending on S.

Case 1. If S is a cover set, then $m_e(T-S)=0$ and so

$$I_w(T) = \mid S \mid + m_e(T-S) = \alpha(T). \tag{3}$$

Case 2. Let $0 \leq k < n$. Suppose that S contains both root vertex x and k vertices which are $x_1, x_2, x_3, ..., x_{k-1}, x_k$ with order. (It is obvious that if k=0, then S contains only the vertex x.) Then $m_e(T-S)$ is equal to

$$\lceil (\sum_{i=1}^{n} deg(x_i) - \sum_{i=1}^{k} deg(x_i))/(deg(x)-k) \rceil - 1 = deg(x_{k+1}) - 1$$

and

$$I_w(T) = \min_{0 \leq k < n} \{k + deg(x_{k+1})\}. \tag{4}$$

Hence the result follows by (3) and (4). □

3. On the Weak-Integrity of Binomial Trees

In this section we consider the *binomial tree*. The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other (Figure 2). In Figure 2 we call the vertex u *top vertex* of B_n.

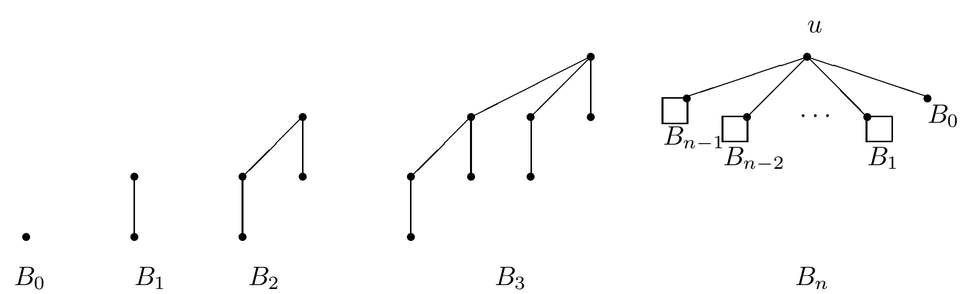

Figure 2. A Binomial Tree.

Now we give the weak-integrity of a binomial tree B_n.

Theorem 3.1 *Let $n \geq 1$ be a positive integer. Then*

$$I_w(B_n) = \begin{cases} 2^{\frac{n+1}{2}} - 1, & \text{if } n \text{ is odd,} \\ \\ 3 \times 2^{\frac{n}{2}-1} - 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let S be a subset of $V(B_n)$ such that $\mid S \mid + m_e(B_n - S) = I_w(B_n)$. If we consider the vertices of S, then we have two cases:

Case 1. First, if we remove top vertex u of B_n, then the remaining graph have the components $B_{n-1}, B_{n-2}, ..., B_0$ and the largest component has $2^{n-1} - 1$ edges. Second, if we remove the top vertex of B_{n-1}, then we have the components $B_{n-2}, B_{n-3}, ..., B_0$ and the largest component has $2^{n-2} - 1$ edges. If we continue to remove the top vertices as mentioned above, the largest component has $2^{n-k} - 1$ edges at kth step. So

$$I_w(B_n) = 2^{n-k} - 1 + k \tag{5}$$

Case 2. First, if we remove any vertex other than top vertex u of B_n, then the largest component has 2^{n-1} edges. Second, if we remove one more vertex from the remaining graph (except any top vertex), then the largest component has 2^{n-2} edges. If we continue to remove the vertices as mentioned above, the largest component has 2^{n-k} edges at the kth step. So

$$I_w(B_n) = 2^{n-k} + k. \tag{6}$$

From (5) and (6) we have $2^{n-k} + k > 2^{n-k} - 1 + k$ for every n,k. That is, iteratively at each step we must remove top vertex.

If we remove r top vertices where $2^i \leq r < 2^{i+1}$ and $0 \leq i \leq n-1$, then one of the remaining connected components has $2^{n-(i+1)} - 1$ edges. Therefore

$$I_w(B_n) = \min_r \{2^r + 2^{n-(r+1)} - 1\}.$$

The function $2^r + 2^{n-(r+1)} - 1$ takes its minimum value at $r = \dfrac{n-1}{2}$ when n is odd and $r = \dfrac{n}{2}$ when n is even. Hence if we substitute the minumum values in the function $2^r + 2^{n-(r+1)} - 1$, the proof is completed. □

Next we give the definition of the *composition* (also known as the lexicographic product) of two graphs. In the following theorem $m(G - S)$ denotes the number of vertices of a largest component of G-S.

Definition 3.1 *The composition $G_1[G_2]$ of two graphs G_1 and G_2 has its vertex set $V(G_1) \times V(G_2)$, with (u_1, u_2) adjacent to (v_1, v_2) if either u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2.*

Theorem 3.2 *Let T be a tree of order m. Then $I_w(B_n[T])$ is equal to*
$min\{\min_{0\leq r<n} \{m2^r + m^2(2^{n-(r+1)} - 1) + 2^{n-(r+1)} \mid E(T) \mid\}, m2^{n-1} + I_w(2^{n-1}T)\}$.

Proof. Let X be a subset of $V(B_n[T])$ such that $\mid X \mid + \ m_e(B_n[T]-X) = I_w(B_n[T])$. The graph $B_n[T]$ has m subgraphs B_n and let S be a set of removing vertices from any graph B_n. Then we have the following two cases, depending on X:

Case 1. Suppose that X contains only vertices of subgraphs B_n in $B_n[T]$. By Theorem 3.1 we know that S must contain the top vertex of B_n at each step. Hence we must remove $X = m \mid S \mid = m2^r$ vertices from $B_n[T]$ and $m_e(B_n[T] - X) = m^2(2^{n-(r+1)} - 1) + 2^{n-(r+1)} \mid E(T) \mid$. So

$$I_w(B_n[T]) = \min_{0\leq r<n} \{m2^r + m^2(2^{n-(r+1)} - 1) + 2^{n-(r+1)} \mid E(T) \mid\}. \quad (7)$$

Case 2. Suppose that X contains the vertices of subgraphs B_n but also the vertices of subgraphs T in $B_n[T]$. Then S must be a cover set of B_n and $m_e(B_n[T]-X) =\mid E(T) \mid$. Moreover, the number of remaining components with edges $\mid E(T) \mid$ is exactly $\beta(B_n)$. Since $\mid S \mid= \alpha(B_n) = 2^{n-1}$ and $\beta(B_n) = 2^{n-1}$, we have

$$I_w(B_n[T]) = m2^{n-1} + I_w(2^{n-1}T) \quad (8)$$

The proof is completed by (7) and (8). □

In Table 3 we give the values $I_w(B_2[T_i])$, $I_w(B_3[T_i])$ and $I_w(B_4[T_i])$ for the trees in Appendix.

Table 3. The values of $I_w(B_2[T_i])$, $I_w(B_3[T_i])$ and $I_w(B_4[T_i])$ for the trees in Appendix.

i	$I_w(B_2[T_i])$	$I_w(B_3[T_i])$	$I_w(B_4[T_i])$	i	$I_w(B_2[T_i])$	$I_w(B_3[T_i])$	$I_w(B_4[T_i])$
1	5	9	14	13	16	29	53
2	8	14	25	14	18	34	62
3	11	19	35	15	19	34	62
4	10	19	35	16	18	34	62
5	13	24	44	17	17	33	62
6	13	24	44	18	18	34	62
7	12	24	44	19	17	33	62
8	16	29	53	20	17	33	62
9	16	29	53	21	18	34	62
10	15	29	53	22	16	32	62
11	15	29	53	23	18	34	62
12	14	28	53	24	18	34	62

Now we consider the corona operation.

Definition 3.2 *The* Corona $G_1 \circ G_2$ *is defined as the graph* G *obtained by taking one copy of* G_1 *of order* n *and* n *copies of* G_2*, and then joining the* i*th vertex of* G_1 *to every vertex in the* i*th copy of* G_2.

Next we give the values of $I_w(T_i \circ B_2)$, $I_w(T_i \circ B_3)$ and $I_w(T_i \circ B_4)$ for the trees in Appendix.

Table 4. The values of $I_w(T_i \circ B_2)$, $I_w(T_i \circ B_3)$ and $I_w(T_i \circ B_4)$ for the trees in Appendix.

i	$I_w(T_i \circ B_2)$	$I_w(T_i \circ B_3)$	$I_w(T_i \circ B_4)$	i	$I_w(T_i \circ B_2)$	$I_w(T_i \circ B_3)$	$I_w(T_i \circ B_4)$
1	5	7	9	13	9	13	19
2	6	9	12	14	10	14	21
3	7	11	15	15	10	14	21
4	7	11	15	16	10	14	21
5	8	12	17	17	10	14	21
6	8	12	17	18	9	14	21
7	8	12	17	19	10	14	21
8	9	13	19	20	9	14	21
9	9	13	19	21	9	14	21
10	9	13	19	22	8	14	21
11	9	13	19	23	10	14	21
12	8	13	19	24	9	14	21

The following theorem gives the value $I_w(B_n \circ T)$ for every n.

Theorem 3.3 *Let T be a tree with m vertices and q edges. Let $r = \lfloor \frac{ln\ (2^{n-1}(m+q+1))}{ln\ 4} \rfloor$. Define $f(r) = \{2^r + 2^{n-(r+1)}(m+q+1) - 1\}$ if $m+q+1 < 2^{n+1}$. Then*

$$I_w(B_n \circ T) = min\{2^n + q, 2^n + I_w(2^n T), f(r)\}.$$

Proof. Let S be a subset of $V(B_n \circ T)$. Then we have two cases, depending on S.

Case 1. If S contains all vertices of graph B_n, then we have

$$I_w(B_n \circ T) = min\{2^n + q, 2^n + I_w(2^n T)\}. \tag{9}$$

Case 2. If S contains 2^r vertices of graph B_n where $0 \leq r \leq n-1$, then we have $m_e(B_n \circ T - S) = 2^{n-(r+1)}m + 2^{n-(r+1)}q + 2^{n-(r+1)} - 1$. Then

$$I_w(B_n \circ T) = \min_r \{2^r + 2^{n-(r+1)}(m+q+1) - 1\}.$$

The function $2^r + 2^{n-(r+1)}(m+q+1) - 1$ takes its minumum value at $r = \lfloor \frac{ln\ (2^{n-1}(m+q+1))}{ln\ 4} \rfloor$ if $(m+q+1) < 2^{n+1}$. Consequently, $I_w(B_n \circ T)$ is equal to

$$f(r) = \{2^r + 2^{n-(r+1)}(m+q+1) - 1\} \text{ if } m+q+1 < 2^{n+1}. \tag{10}$$

Hence the result follows from (9) and (10). □

The following theorems give the weak-integrity of graphs $B_n \times C_m$ and $B_n \times P_m$.

Theorem 3.4 *let $a = \lfloor \frac{n}{2} \rfloor$ and $b = \lfloor \frac{\sqrt{(2^{n+1}-1)m}}{\sqrt{2^n}} \rfloor$, where m and n are positive integers. Then*

$$I_w(B_n \times C_m) = min\{m(2^a + 2^{n-a} - 1), 2^n b + (2^{n+1} - 1)\lfloor \frac{m-b}{b} \rfloor - 2^n\}.$$

Proof. Let S be a subset of $V(B_n \times C_m)$ that achieves the weak-integrity of $B_n \times C_m$. Then we have two cases, depending on S.

Case 1. If we remove $|S| = m2^r$ vertices where $0 \leq r \leq n-1$, then

$$m_e(B_n \times C_m - S) = 2^{n-(r+1)} \mid E(C_m) \mid + (2^{n-(r+1)} - 1)m.$$

Since $\mid E(C_m) \mid = m$, we have

$$I_w(B_n \times C_m) = \min_r \{m2^r + 2m2^{n-(r+1)} - m\}$$

$$= m \min_r \{2^r + 2^{n-r} - 1\}.$$

The function $2^r + 2^{n-r} - 1$ takes its minumum value at $r = \lfloor \frac{n}{2} \rfloor$.

Case 2. If we remove $\mid S \mid = 2^n r$ vertices where $1 \leq r \leq m-1$, then we have

$$m_e(B_n \times C_m - S) = 2^n(\lfloor \frac{m-r}{r} \rfloor - 1) + (2^n - 1)\lfloor \frac{m-r}{r} \rfloor.$$

So

$$I_w(B_n \times C_m) = \min_r \{2^n r + \lfloor \frac{m-r}{r} \rfloor (2^{n+1} - 1) - 2^n\}.$$

The function $2^n r + \lfloor \frac{m-r}{r} \rfloor (2^{n+1} - 1) - 2^n$ takes its minumum value at $r = \lfloor \frac{\sqrt{(2^{n+1}-1)m}}{\sqrt{2^n}} \rfloor$.

If we substitute the minumum values in the functions, the proof is completed. □

Theorem 3.5 *Let* $a = \frac{ln(\frac{(2m-1)(2^{n-1})}{m})}{ln\ 4}$ *and* $b = \lfloor \frac{\sqrt{(2^{n+1}-1)(1+m)}}{\sqrt{2^n}} \rfloor - 1$, *where m and n are positive integers. Let's define f(a) as*

$$f(a) = \begin{cases} m2^{\lfloor a \rfloor} + (2m-1)2^{n-(\lfloor a \rfloor + 1)} - m, & \text{if } n \text{ is odd,} \\ m2^{\lceil a \rceil} + (2m-1)2^{n-(\lceil a \rceil + 1)} - m, & \text{if } n \text{ is even.} \end{cases}$$

Then

$$I_w(B_n \times P_m) = min\{f(a), 2^n b + \lfloor \frac{m-b}{b+1} \rfloor (2^{n+1} - 1) - 2^n\}.$$

Proof. The proof follows directly from Theorem 3.4. □

4. Conclusion

In this paper we introduce a new stability measure. There are many examples of graphs which suggest that I_w is a suitable measure of stability in that it is able to distinguish between graphs that intuitively should have different levels of stability.

For example consider the trees $T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{19}, T_{21}$ and T_{24} in Appendix. First we give the values integrity, edge-integrity, pure-edge-integrity and weak-integrity of these trees in a table as follows:

Table 5. The values of I, I', I_p and I_w for some trees in Appendix.

i	$I(T_i)$	$I'(T_i)$	$I_p(T_i)$	$I_w(T_i)$
14	4	5	4	3
15	4	5	4	3
16	4	5	4	3
17	3	5	4	2
18	3	5	4	2
19	3	5	4	2
21	3	5	4	2
24	3	5	4	2

Since the edge-integrity of these trees are the same, it does not distinguish these trees. Similiarly, the pure-edge-integrity does not also distinguish these trees. Now let $A = \{T_{14}, T_{15}, T_{16}\}$ and B=$\{T_{17}, T_{18}, T_{19}, T_{21}, T_{24}\}$. Then *the weak integrity distinguishes the sets of trees A and B*. Similiarly, the integrity distinguishes the sets of trees A and B.

The comparison of weak-integrity to the integrity, edge-integrity and pure-edge-integrity of the trees indicates that the *weak-integrity* can be a useful measure of graph stability.

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Alpay KIRLANGIÇ
Department of Mathematics,
Ege University-35100
Bornova-İzmir/TURKEY
e-mail: alpkir@sci.ege.edu.tr

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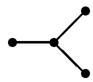
Appendix

All the trees with $p \leq 7$ [5]

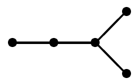
T_1

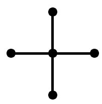
T_2

T_3

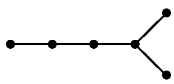
T_4 

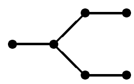
T_5

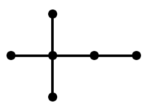
T_6 

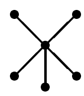
T_7 

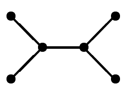
T_8

T_9 

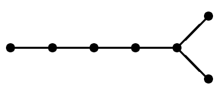
T_{10} 

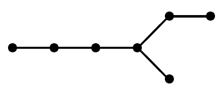
T_{11} 

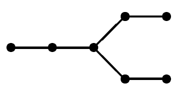
T_{12} 

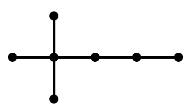
T_{13} 

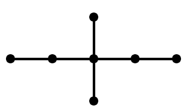
T_{14}

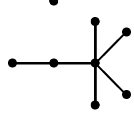
T_{15} 

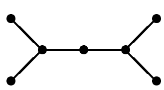
T_{16} 

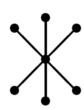
T_{17} 

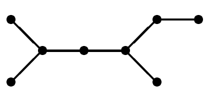
T_{18} 

T_{19} 

T_{20} 

T_{21} 

T_{22} 

T_{23} 

T_{24}