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On Invariant Submanifolds of Riemannian Warped Product Manifold

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Abstract

In this paper, we generalize the geometry of the invariant submanifolds of Riemannian product manifold to the geometry of the invariant submanifolds of Riemannian warped product manifold. We investigate some properties of an invariant submanifolds of a Riemannian warped product manifold. We show that every invariant submanifold of the Riemannian warped product manifold is a Riemannian warped product manifold. Also, we give a theorem on the pseudo-umbilical invariant submanifold. Further, we obtain that integral manifolds on an invariant submanifold are curvature-invariant submanifolds. Finally, we give a necessary condititon on a totally umbilical invariant submanifold to be totally geodesic.

Key Words: Riemannian Warped Product Manifold, Vertical and Horizontal Distributions, Pseudo-Umbilical Submanifold, Curvature-Invariant Submanifold.

1. Introduction

The geometry of a submanifold $(\overline{M}, \overline{g})$ of a locally product Riemannian manifold $(M_1 \times M_2, g_1 \otimes g_2)$ was widely studied by many geometers. In particular, K. Matsumoto has proved that $(\overline{M}, \overline{g})$ is a locally product Riemannian manifold of Riemannian manifolds $(\overline{M}_a, \overline{g}_a)$ and $(\overline{M}_b, \overline{g}_b)$, if it is an invariant submanifold of a Riemannian product manifold $(M_1 \times M_2, g_1 \otimes g_2)$(see [5]). Later, Xu. Senlin, and Ni. Yilong, ([6]) have updated X-Matsumotos and proved that $\overline{M}_a \subset M_1$ and $\overline{M}_b \subset M_2$. Moreover, they have proved that $(\overline{M}_a, \overline{g}_a)$ and $(\overline{M}_b, \overline{g}_b)$ are pseudo-umbilical submanifolds of (M_1, g_1) and (M_2, g_2), respectively, if $(\overline{M}, \overline{g})$ is a pseudo-umbilical submanifold of $(M, g) = (M_1 \times M_2, g_1 \otimes g_2)$. They have also demonstrated that $\overline{M}$ is isometric to the production of its two totally

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geodesic submanifolds $(\overline{M}_a, \overline{g}_a)$ and $(\overline{M}_b, \overline{g}_b)$ which are submanifolds of (M_1, g_1) and (M_2, g_2), respectively (see [6]).

Furthermore, semi-invariant submanifolds of locally product Riemannian manifolds were studied by A. Bejancu (see [2]).

Riemannian and Warped product structures are widely used in geometry to construct new examples of Riemannian manifolds with interesting curvature properties. Warped product metric tensor, as a generalization of Riemannian product metric tensor, have also been useful in the study of several aspect of submanifold theory.

An invariant submanifold of a semi-Riemannian product manifold has been considered by several authors; but , an invariant submanifold of the other product manifold (such as warped product, or twisted product) has not been widely considered so far.

In this work, we have studied the geometry of a submanifold $(\overline{M}, \overline{g})$, a warped product Riemannian manifold of a Riemannian manifold (M_1, g_1) and a Riemannian manifold (M_2, g_2), if it is an invariant submanifold of a Riemannian warped product manifold $(M_1 \times_f M_2, g_1 \otimes f^2 g_2)$. We have also proved that $(\overline{M}, \overline{g})$ is a pseudo-umbilical submanifold of $(M, g) = (M_1 \times_f M_2, g_1 \otimes f^2 g_2)$ if and only if $(\overline{M}_a, \overline{g}_a)$ and $(\overline{M}_b, \overline{g}_b)$ are pseudo-umbilical submanifolds of (M_1, g_1) and (M_2, g_2), respectively, where $(\overline{M}, \overline{g})$ is the Riemannian warped product manifold of the Riemannian manifolds $(\overline{M}_a, \overline{g}_a)$ and $(\overline{M}_b, \overline{g}_b)$. Moreover, we have shown that $(\overline{M}_a, \overline{g}_a)$ and $(\overline{M}_b, \overline{g}_b)$ are the curvature-invariant submanifolds of (M_1, g_1) and (M_2, g_2), respectivley, if $(\overline{M}, \overline{g})$ is the curvature-invariant submanifold of (M, g), and we give a theorem on a totally umbilical invariant submanifold to be totally geodesic.

2. Preliminaries

In this section, we give some notations and terminology used througthout this paper. We recall some necessary facts and formulas from the theory of submanifolds. For an arbitrary submanifold $\overline{M}$ of a Riemannian manifold M, Gauss and Weingarten formulas are given by

$$\nabla_X Y = \overline{\nabla}_X Y + h(X, Y)$$

and

$$\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

respectively, where ∇ and $\overline{\nabla}$ are Levi-Civita connections on the Riemannian manifolds

M and its submanifold $\overline{M}$, respectively; X, Y are vector fields tangent to $\overline{M}$; ξ is a vector field normal to $\overline{M}$; $h : \Gamma(T\overline{M}) \times \Gamma(T\overline{M}) \longrightarrow \Gamma(T\overline{M}^{\perp})$ is the second fundamental form of $\overline{M}$, $\nabla^{\perp}$ is the normal connection in the normal vector bundle $\Gamma(T\overline{M}^{\perp})$; and A_{ξ} is the shape operator of the second quadratic form for a normal vector ξ. From the above formulas it follows that

$$g(A_{\xi}X, Y) = g(h(X, Y), \xi),$$

where the symbol g denotes the Riemannian metric of M.

We denote the Riemannian curvature tensors of the Levi-Civita connections ∇ and $\overline{\nabla}$ on M and $\overline{M}$ by R and $\overline{R}$, respectively. The Gauss, Codazzi, and Ricci equations are given by

$$\begin{aligned} g(R(X, Y)Z, W) &= g(\overline{R}(X, Y)Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \\ (R(X, Y)Z)^{\perp} &= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \\ g(R(X, Y)\xi, \eta) &= g(R^{\perp}(X, Y)\xi, \eta) - g([A_{\xi}, A_{\eta}]X, Y), \end{aligned}$$

respectively, where the vector fields X, Y, Z, W are tangent to $\overline{M}$, the vector fields ξ and η are orthogonal to $\overline{M}$, $(R(X, Y)Z)^{\perp}$ denotes the normal Component of $R(X, Y)Z$ and the derivative ∇h is defined by

$$(\nabla_X h)(Y, Z) = (\nabla_X^{\perp} h)(Y, Z) - h(\overline{\nabla}_X Y, Z) - h(\overline{\nabla}_X Z, Y).$$

$\overline{M}$ is called a curvature-invariant submanifold if it has

$$(R(X, Y)Z)^{\perp} = 0,$$

which is equivalent to

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)$$

for all $X, Y, Z \in \Gamma(T\overline{M})$.

If the ambient space M is a space of constant sectional curvature c, the equations of Gauss, Codazzi and Ricci reduce to

$$\begin{aligned} K(X, Y, Z, W) &= c\{\overline{g}(X, W)\overline{g}(Y, Z) - \overline{g}(X, Z)\overline{g}(Y, W)\} \\ &+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned}$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)$$

and

$$K^{\perp}(X,Y,\xi,\eta)=g([A_{\xi},A_{\eta}]X,Y),$$

respectively, where K denotes the Riemannian-Christoffel curvature tensor of M[4].

Definition 2.1 *For a submanifold $\overline{M}\subseteq M$ the mean-curvature vector field H is defined by the formula*

$$H=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i),$$

where $\{e_i\}$ is a local orthonormal basis in $\Gamma(T\overline{M})$. If a submanifold $\overline{M}\subseteq M$ having one of the conditions

$$h=0,g(h(X,Y),H)=\lambda g(X,Y),H=0,\lambda\in C^{\infty}(M,R),$$

then it is called totally geodesic, pseudo-umbilical and minimal, respectively for all x, $y\in\Gamma(T\overline{M})$ [3].

We recall that the length the mean curvature vector field of $\overline{M}$ is constant if $\overline{M}$ is a totally umbilical submanifold of a Riemannian manifold M[3].

Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds with dimension n_1 and n_2, respectively, and $f\rangle 0$ be a smooth function on M_1. The Riemannian warped product manifold $M=M_1\times_f M_2$ is the product manifold M furnished with metric tensor $g=\pi^{*}g_1+(fo\pi)^2\sigma^{*}g_2$, where $\pi_{*}:\Gamma(T(M_1\times_f M_2))\longrightarrow\Gamma(TM_1)$ and $\sigma_{*}:\Gamma(T(M_1\times_f M_2))\longrightarrow\Gamma(TM_2)$ are the projection mappings. The warped product manifold $M_1\times_f M_2$ is characterized by M_1 is totally geodesic and M_2 is totally umbilical submanifolds of $M_1\times_f M_2$. We denote the Levi-Civita connection of the warped product metric tensor of g by ∇. Then we give the following propositions for later use.

Proposition 2.2 (O'Neill, [7]) *Let $(M_1\times_f M_2,g)$ be a warped Riemannian product manifold with the warping function $f\rangle 0$ on M_1. Then we have*
a) $\nabla_{X_1}Y_1\in\Gamma(TM_1)$
b) $\nabla_{X_1}Y_2=\nabla_{Y_2}X_1=\frac{X_1(f)}{f}Y_2$
c) nor$(\nabla_{X_2}Y_2)=-fg_2(X_2,Y_2)\mathrm{grad}f$
d) $\tan(\nabla_{X_2}Y_2)=\nabla^2_{X_2}Y_2\in\Gamma(TM_2)$,

for all $X_i, Y_i \in \Gamma(TM_i)$, for $i = 1, 2$, respectively, where ∇^2 is the Levi-Civita connection of Riemannian metric tensor g_2.

Proposition 2.3 (O'Neill, [7]) *Let $(M_1 \times_f M_2, g)$ be a warped Riemannian product manifold with the warping function $f\rangle 0$ and Riemannian curvature tensor R. Then we have*
a) $R(X_1, Y_1)Z_1 = R_1(X_1, Y_1)Z_1 \in \Gamma(TM_1)$.
b) $R(X_2, X_1)Y_1 = \frac{1}{f}H^f(X_1, Y_1)X_2$.
c) $R(X_1, Y_1)X_2 = R(X_2, Y_2)X_1 = 0$
d) $R(X_2, Y_2)Z_2 = R_2(X_2, Y_2)Z_2 - g_1(\text{grad}f, \text{grad}f)\{g_2(X_2, Z_2)Y_2 - g_2(Y_2, Z_2)X_2\}$.
e) $R(X_1, Y_2)Z_2 = fg_2(Y_2, Z_2)\nabla_{X_1}\text{grad}f$,

for all $X_i, Y_i, Z_i \in \Gamma(TM_i)$ for $i = 1, 2$, respectively, where R_1 and R_2 denote the Riemannian curvature tensor of M_1 and M_2, respectively, and H^f is the Hessian form of warping function f.

3. Invariant Submanifold of a Riemannian Warped Product Manifold

Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold with (M_1, g_1) and (M_2, g_2). We denote by π_* and σ_* the projection mappings of $\Gamma(T(M_1 \times_f M_2))$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Then we have

$$\pi_*^2 = \pi_*, \sigma_*^2 = \sigma_*, \pi_* \times \sigma_* = \sigma_* \times \pi_* = 0, \pi_* + \sigma_* = I,$$

where I is the identity transformation of $\Gamma(T(M_1 \times_f M_2))$. If we put $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$. It follows that

$$g(FX, Y) = g(X, FY),$$

which is equivalent to

$$g(FX, FY) = g(X, Y),$$

for all $X, Y \in \Gamma(T(M_1 \times_f M_2))$.

Now, let $\overline{M}$ be a submanifold of $M_1 \times_f M_2$ and B the differential of the imbedding i of $\overline{M}$ into $M_1 \times_f M_2$, i.e., $B = i_*$. Let X be a tangent vector field of $\overline{M}$. Then we can write FBX in the following way:

$$FBX = (FBX)^T + (FBX)^\perp = BSX + \xi,$$

where $(FBX)^T = BSX \in \Gamma(T\overline{M})$, $(FBX)^\perp = \xi \in \Gamma(T\overline{M})^\perp$ and $S : \Gamma(T\overline{M}) \longrightarrow \Gamma(T\overline{M})$ is a linear transformation. $\overline{M}$ is said to be an invariant submanifold of $M_1 \times_f M_2$, if $FBX = BSX$ always holds. In the rest of this paper we assume that the submanifold $\overline{M}$ is invariant. In this case, we have $S^2 = I$.

Let $\overline{g}$ be an induced Riemannian metric tensor on $\overline{M}$ by the Riemannian warped metric tensor g, that is, $\overline{g} = i^* g$. Then

$$\begin{aligned}\overline{g}(X,Y) &= i^*g(X,Y) = g(BX,BY) = g(FBX,FBY) = g(BSX,BSY)\\ &= \overline{g}(SX,SY)\end{aligned}$$

for all $X, Y \in \Gamma(T\overline{M})$. Thus S defines an almost Riemannian product structure on $\overline{M}$, that is, $T\overline{M}$ has the vertical and horizontal distributions which are defined by

$$T_1 = \{X \in \Gamma(T\overline{M}) | SX = X\},$$

and

$$T_2 = \{X \in \Gamma(T\overline{M}) | SX = -X\},$$

respectively. Since $S^2 = I$, we know that $\Gamma(T\overline{M}) = T_1 \oplus T_2$. We denote the integral manifolds of the distributions T_1 and T_2 by $\overline{M}_a$ and $\overline{M}_b$, respectively.

Example 3.1 *Let $M = IR^3 \times_f IR^3$ be Riemannian warped product manifold with Riemannian warped metric tensor $\langle , \rangle = \langle , \rangle_1 + f^2 \langle , \rangle_2$, where $\langle , \rangle_i$ denote the standard metric tensors of IR^3 for $i = 1, 2$ and $f : IR^3 \longrightarrow IR^+$ is a smooth function. We consider a submanifold*

$$\overline{M} = \{(x_1, x_2, x_3, x_4, x_5, x_6) | x_3 = \frac{1}{\sqrt{2}}(x_2 + \sin x_1), x_5 = \cos x_4\}$$

of M. By direct calculations we get
$\Gamma(T\overline{M}) = \{U_1 = \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}} \cos x_1 \frac{\partial}{\partial x_3}, U_2 = \frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3}, U_3 = \frac{\partial}{\partial x_4} - \sin x_4 \frac{\partial}{\partial x_5},$
$U_4 = \frac{\partial}{\partial 6}\}$. *We can easily see that $\overline{M}$ is an invariant submanifold of M. It follows that the vertical and horizontal distributions are spanned by $T_1 = Sp\{U_1, U_2\}$ and $T_2 = Sp\{U_3, U_4\}$, respectively.*

Now we can give the following theorem.

Theorem 3.2 *Every invariant submanifold $\overline{M}$ of a Riemannian warped product manifold $(M_1 \times_f M_2, g)$ is a mixed-geodesic submanifold of $(M_1 \times_f M_2, g)$.*

Proof. We denote the integral manifolds of the vertical and horizontal distributions of $\overline{M}$ by $\overline{M}_a$ and $\overline{M}_b$, respectively. If h is the second fundamental form of $\overline{M}$ in $M_1 \times_f M_2$, then we have to show that $h(X_1, X_2) = 0$ for all $X_1 \in \Gamma(T\overline{M}_a)$ and $X_2 \in \Gamma(T\overline{M}_b)$. Using the Gauss formula, we derive

$$\nabla_{X_1} X_2 = \overline{\nabla}_{X_1} X_2 + h(X_1, X_2) = \frac{X_1(f)}{f} X_2.$$

Restrict the above equation to $\Gamma(T\overline{M})$ and $\Gamma(T\overline{M}^{\perp})$, we have $h(X_1, X_2) = 0$, where $\overline{\nabla}$ is the Levi-Civita Connection on $\overline{M}$. This completes the proof of the theorem. □

Theorem 3.3 *Let $(M_1 \times_f M_2, g)$ be Riemannian warped product manifold with the warping function f and $\overline{M}$ be an invariant submanifold of a Riemannian warped product manifold $M_1 \times_f M_2$. We denote the integral manifolds of the vertical and horizontal distributions of $\overline{M}$ by $\overline{M}_a$ and $\overline{M}_b$, respectively. Then $\overline{M}_a$ and $\overline{M}_b$ are totally geodesic and totally umbilical submanifolds of $\overline{M}$, respectively. Moreover, $\overline{M}_a$ and $\overline{M}_b$ are submanifolds of M_1 and M_2, respectively.*

Proof. Let h_a and h_b be the second fundamental forms of $\overline{M}_a$ and $\overline{M}_b$ in $\overline{M}$, respectively. Then

$$\overline{\nabla}_{X_1} Y_1 = \nabla^a_{X_1} Y_1 + h_a(X_1, Y_1),$$

for all $X_1, Y_1 \in \Gamma(T\overline{M}_a)$, where ∇^a is the Levi-Civita connection on $\overline{M}_a$. Hence for all $Z_2 \in \Gamma(T\overline{M}_b)$ we get

$$\begin{aligned} g(h_a(X_1, Y_1), Z_2) &= g(\overline{\nabla}_{X_1} Y_1, Z_2) = -g(Y_1, \overline{\nabla}_{X_1} Z_2) \\ &= -g(Y_1, \frac{X_1(f)}{f} Z_2) = \frac{X_1(f)}{f} g(Y_1, Z_2) = 0. \end{aligned}$$

It follows that $h_a(X_1, Y_1) = 0$, that is, $\overline{M}_a$ is totally geodesic submanifold of $\overline{M}$. In the same way, for all $X_2, Y_2 \in \Gamma(T\overline{M}_b)$ we have

$$\overline{\nabla}_{X_2} Y_2 = \nabla^b_{X_2} Y_2 + h_b(X_2, Y_2),$$

where ∇^b is the Levi-Civita connection on $\overline{M}_b$. For all $Z_1 \in \Gamma(T\overline{M}_a)$

$$\begin{aligned} g(h_b(X_2, Y_2), Z_1) &= g(\overline{\nabla}_{X_2} Y_2, Z_1) = -g(\overline{\nabla}_{X_2} Z_1, Y_2) \\ &= -g(\frac{Z_1(f)}{f} X_2, Y_2) = -\frac{1}{f} g(g(Z_1, \mathrm{grad} f) X_2, Y_2) \\ &= -f g_1(X_1, \mathrm{grad} f) g_2(X_2, Y_2) \\ &= -f g_1(g_2(X_2, Y_2)\mathrm{grad} f, Z_1), \end{aligned} \tag{3.1}$$

that is,

$$h_b(X_2, Y_2) = -f g_2(X_2, Y_2)\mathrm{grad} f,$$

which implies that $\overline{M}_b$ is the totally umbilical submanifold of $\overline{M}$ and $\mathrm{grad} f \in \Gamma(T\overline{M}_a)$.

Now we define the distributions by

$$D_\pi = \{X \in \Gamma(T(M_1 \times_f M_2)) | \pi_* X = X\}$$

and

$$D_\sigma = \{X \in \Gamma(T(M_1 \times_f M_2)) | \sigma_* X = X\}.$$

Then we obtain

$$\begin{aligned} \pi_* B X_1 &= \frac{1}{2}(I + F) B X_1 = \frac{1}{2}(B X_1 + F B X_1) = \frac{1}{2}(B X_1 + B S X_1) \\ &= \frac{1}{2}(B X_1 + B X_1) = B X_1, \end{aligned}$$

and

$$\begin{aligned} \sigma_* B X_1 &= \frac{1}{2}(I - F) B X_1 = \frac{1}{2}(B X_1 - F B X_1) = \frac{1}{2}(B X_1 - B S X_1) \\ &= \frac{1}{2}(B X_1 - B X_1) = 0, \end{aligned}$$

for all $X_1 \in \Gamma(T\overline{M}_a)$. In the same way, we get $\pi_* B X_2 = 0$ and $\sigma_* B X_2 = B X_2$, for all $X_2 \in \Gamma(T\overline{M}_b)$. Because the integral manifolds of D_π and D_σ are manifolds M_1 and M_2, respectively, we can easily see that $\overline{M}_a$ and $\overline{M}_b$ are submanifolds of M_1 and M_2, respectively. □

Since $\overline{M}$ is a warped product manifold and Gauss formula we obtain

$$\begin{aligned}\nabla_{X_1}Y_1 + \nabla_{X_2}Y_2 &= \overline{\nabla}_{X_1}Y_1 + \overline{\nabla}_{X_2}Y_2 \\ &+ h(X_1, Y_1) + h(X_2, Y_2).\end{aligned}$$

Using the Proposition. 2.2(c), we get

$$\begin{aligned}\nabla_{X_1}Y_1 + \nabla^2_{X_2}Y_2 - fg_2(X_2, Y_2)\mathrm{grad}f &= \nabla^a_{X_1}Y_1 + \nabla^b_{X_2}Y_2 + h_b(X_2, Y_2) \\ &+ h(X_1, Y_1) + h(X_2, Y_2)\end{aligned}$$

From (3.1) we have

$$\begin{aligned}\overline{\nabla}_{X_1}Y_1 - \nabla^a_{X_1}Y_1 + \nabla^2_{X_2}Y_2 - \nabla^b_{X_2}Y_2 &= fg_2(X_2, Y_2)\mathrm{grad}f - fg_2(X_2, Y_2)\mathrm{grad}f \\ &+ h(X_1, Y_1) + h(X_2, Y_2) \\ h_1(X_1, Y_1) + h_2(X_2, Y_2) &= h(X_1, Y_1) + h(X_2, Y_2).\end{aligned}$$

Since $h_1(X_1, Y_1) = h(X_1, Y_1) \in \Gamma(T\overline{M}_a{}^{\perp})$ we get $h_2(X_2, Y_2) = h(X_2, Y_2)$. It follows that h_1 and h_2 are the second fundamental forms of $\overline{M}_a$ and $\overline{M}_b$ in M_1 and M_2, respectively. So we have

$$h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2) \tag{3.2}$$

for all $X_1, Y_1 \in \Gamma(TM_a)$ and $X_2, Y_2 \in \Gamma(TM_b)$.
The following corollary is quite easy.

Corollary 3.4 *Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold and $\overline{M}$ be an invariant submanifold of $(M_1 \times_f M_2, g)$. We denote the vertical and horizontal distributions of $\overline{M}$ by T_1 and T_2, respectively. Then the distributions T_1 and T_2 are always involutive, but they are not parallel.*

Theorem 3.5 *Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold with the warping function f and $\overline{M}$ be an invariant submanifold of Riemannian warped product manifold $M_1 \times_f M_2$. We denote the integral manifolds of the vertical and horizontal distributions of $\overline{M}$ by $\overline{M}_a$ and $\overline{M}_b$, respectively. Then $\overline{M}_a$ and $\overline{M}_b$ are curvature-invariant submanifolds of $\overline{M}$.*

Proof. The curvature-invariant submanifold of $\overline{M}_a$ in $\overline{M}$ is trivial because it is totally geodesic submanifold of $\overline{M}$. We denote the Riemannian curvature tensor of $\overline{M}$ and $\overline{M}_b$ by $\overline{R}$ and $\overline{R}_b$, respectively. Then from Proposition. 2.3 and Theorem 3.3 we have

$$\begin{aligned}\overline{R}(X_2, Y_2)Z_2 = & \quad \overline{R}_b(X_2, Y_2)Z_2 \\ - & \quad \overline{g}(\mathrm{grad}f, \mathrm{grad}f)\{\overline{g}(X_2, Z_2)Y_2 - \overline{g}(Y_2, Z_2)X_2\}. \end{aligned} \tag{3.3}$$

Moreover, it is well known that

$$\begin{aligned}\overline{R}(X_2, Y_2)Z_2 &= \overline{R}_b(X_2, Y_2)Z_2 - A_{h_b(Y_2,Z_2)}X_2 + A_{h_b(X_2,Z_2)}Y_2 \\ &+ (\overline{\nabla}_{X_2}h_b)(Y_2, Z_2) - (\overline{\nabla}_{Y_2}h_b)(X_2, Z_2)\end{aligned} \tag{3.4}$$

for all $X_2, Y_2, Z_2 \in \Gamma(T\overline{M}_b)$. Thus from the equations (3.3) and (3.4) we derive

$$A_{h_b(Y_2,Z_2)}X_2 - A_{h_b(X_2,Z_2)}Y_2 = \overline{g}(\mathrm{grad}f, \mathrm{grad}f)\{\overline{g}(X_2, Z_2)Y_2 - \overline{g}(Y_2, Z_2)X_2\},$$

and

$$(\overline{\nabla}_{X_2}h_b)(Y_2, Z_2) - (\overline{\nabla}_{Y_2}h_b)(X_2, Z_2) = 0,$$

which implies that $\overline{M}_b$ is a curvature-invariant submanifold of $\overline{M}$. □

Now we choose a local field of adapted basis $\{e_1, ..., e_a, e_{a+1}, ..., e_{n_1}, e^1, ..., e^b$ $, e^{b+1}, ..., e^{n_2}\}$ with respect to g so that when restricted locally to orthonormal basis over $\Gamma(T\overline{M})$, $\{e_1, ..., e_a\}$ are tangent vectors to $\Gamma(T\overline{M}_a)$ with respect to g_1, $\{e^1, ..., e^b\}$ are tangent vectors to $\Gamma(T\overline{M}_b)$ with respect to g_2 and $\{e_{a+1}, ..., e_{n_1}, e^{b+1}, ..., e^{n_2}\}$ are normal vectors to $\Gamma(T\overline{M})$. Let H be the mean curvature vector field of $\overline{M}$ in $M_1 \times_f M_2$. Then we consider equation (3.2) by a direct calculation we obtain

$$\begin{aligned} mH &= \sum_{i=a+1}^{n_1} \mathrm{tr}h_1 e_i + \sum_{j=b+1}^{n_2} \mathrm{tr}h_2 e^j \\ &= aH_1 + bH_2, m = a + b, \end{aligned}$$

where H_1 and H_2 denote the mean curvature vector fields of $\overline{M}_a$ and $\overline{M}_b$ in M_1 and M_2, respectively.

The following lemma is quite easy.

Lemma 3.6 *H_1 and H_2 are constants if and only if H is constant*

Theorem 3.7 *Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold with the warping function f and $\overline{M}$ be an invariant submanifold of Riemannian warped product manifold $M_1 \times_f M_2$. We denote the integral manifolds of the vertical and horizontal distributions of $\overline{M}$ by $\overline{M}_a$ and $\overline{M}_b$, respectively. Then $\overline{M}$ is a pseudo-umbilical submanifold of $M_1 \times_f M_2$ if and only if $\overline{M}_a$ and $\overline{M}_b$ are pseudo-umbilical submanifolds of M_1 and M_2, respectively. Moreover, $a\|H_1\|^2 = bf^2\|H_2\|^2$.*

Proof. We suppose that $\overline{M}$ is a pseudo-umbilical submanifold of $M_1 \times_f M_2$. Then there exists a smooth function $\lambda \in C^\infty(M, IR)$ such that

$$g(h(X,Y),H) = \lambda \overline{g}(X,Y) \tag{3.5}$$

for all $X, Y \in \Gamma(T\overline{M})$. If we take $e_1, ..., e_a$ for $X = Y$ in the equation (3.5), then we have

$$\begin{aligned} g(\sum_{i=1}^{a} h(e_i, e_i), H) &= \lambda \sum_{i=1}^{a} \overline{g}(e_i, e_i) \\ g(aH_1, H) &= \lambda a \\ g(aH_1, \frac{a}{m}H_1 + \frac{b}{m}H_2) &= \lambda a \\ \frac{a}{m} g_1(H_1, H_1) &= \lambda. \end{aligned}$$

Similarly, taking $e^1, ..., e^b$ for $X = Y$ in the equation (3.5) we get

$$\lambda = g(H_2, H) = \frac{b}{m} g(H_2, H_2) = f^2 \frac{b}{m} g_2(H_2, H_2).$$

Furthermore, we have

$$\begin{aligned} g(H,H) &= \frac{a^2}{m^2} g_1(H_1, H_1) + f^2 \frac{b^2}{m^2} g_2(H_2, H_2) \\ &= \frac{a^2}{m^2} g_1(H_1, H_1) + \frac{ab}{m^2} g_1(H_1, H_1) \\ &= \frac{a}{m} g_1(H_1, H_1), \end{aligned} \tag{3.6}$$

and similarly, we obtain

$$g(H,H) = f^2 \frac{b}{m} g_2(H_2, H_2). \tag{3.7}$$

Hence, taking X_1, Y_1 for X, Y in equation (3.5), respectively, we have

$$\begin{aligned} g(h_1(X_1, Y_1), H) &= \frac{a}{m} g_1(H_1, H_1)\overline{g}(X_1, Y_1) \\ g_1(h_1(X_1, Y_1), \frac{a}{m} H_1) &= \frac{a}{m} g_1(H_1, H_1)\overline{g}(X_1, Y_1) \\ g_1(h_1(X_1, Y_1), H_1) &= g_1(H_1, H_1)\overline{g}(X_1, Y_1). \end{aligned} \tag{3.8}$$

In the same way, if we take X_2, Y_2 for X, Y in the equation (3.5), respectively, then we obtain

$$g_2(h_2(X_2, Y_2), H_2) = g_2(H_2, H_2)\overline{g}(X_2, Y_2). \tag{3.9}$$

The equations (3.8) and (3.9) imply that $\overline{M}_a$ and $\overline{M}_b$ are pseudo-umbilical submanifolds of M_1 and M_2, respectively. We note that $g_1(H_1, H_1)$ and $g_2(H_2, H_2)$ are the smooth functions on M_1 and M_2, respectively. Moreover, we know that they are also constants.

Conversely, we suppose that $\overline{M}_a$ and $\overline{M}_b$ are pseudo-umbilical submanifolds of M_1 and M_2, respectively. Then we have

$$g_1(h_1(X_1, Y_1), H_1) = g_1(H_1, H_1)\overline{g}(X_1, Y_1) \tag{3.10}$$

for all $X_1, Y_1 \in \Gamma(T\overline{M}_a)$ and

$$g_2(h_2(X_2, Y_2), H_2) = g_2(H_2, H_2)\overline{g}(X_2, Y_2) \tag{3.11}$$

for all $X_2, Y_2 \in \Gamma(T\overline{M}_b)$. Then using the projections

$$\pi_* : \Gamma(T(M_1 \times_f M_2)) \longrightarrow \Gamma(TM_1),$$

and

$$\sigma_* : \Gamma(T(M_1 \times_f M_2)) \longrightarrow \Gamma(TM_2),$$

$H = \frac{a}{m} H_1 + \frac{b}{m} H_2$ and $h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2)$, we obtain
$\pi_* H = \frac{a}{m} H_1$, $\sigma_* H = \frac{b}{m} H_2$. Thus we derive

$$\frac{m}{a} g_1(h_1(X_1, Y_1), \pi_* H) = \frac{m^2}{a^2} g_1(\pi_* H, \pi_* H)\overline{g}(X_1, Y_1)$$

and

$$\frac{m}{b} g_2(h_2(X_2, Y_2), \sigma_* H) = \frac{m^2}{b^2} g_2(\sigma_* H, \sigma_* H)\overline{g}(X_2, Y_2).$$

Hence we have

$$g_1(\pi_* h(X,Y), \pi_* H) = \frac{m}{a} g_1(\pi_* H, \pi_* H)\overline{g}(X_1, Y_1) \tag{3.12}$$

and

$$f^2 g_2(\sigma_* h(X,Y), \sigma_* H) = f^2 \frac{m}{b} g_2(\sigma_* H, \sigma_* H)\overline{g}(X_2, Y_2). \tag{3.13}$$

If we add the equations (3.12), (3.13) and using $g(H,H) = \frac{a}{m} g_1(H_1, H_1) = \frac{b}{m} f^2 g_2(H_2, H_2)$, we obtain

$$\begin{aligned}
g(h(X,Y), H) &= \frac{m}{a} g_1(\frac{a}{m} H_1, \frac{a}{m} H_1)\overline{g}(X_1, Y_1) \\
&+ f^2 \frac{m}{b} g_2(\frac{b}{m} H_2, \frac{b}{m} H_2)\overline{g}(X_2, Y_2) \\
&= \frac{a}{m} g_1(H_1, H_1)\overline{g}(X_1, Y_1) + f^2 \frac{b}{m} g_2(H_2, H_2)\overline{g}(X_2, Y_2) \\
&= g(H,H)\{\overline{g}(X_1, Y_1) + \overline{g}(X_2, Y_2)\} \\
&= g(H,H)\overline{g}(X,Y),
\end{aligned}$$

which implies that $\overline{M}$ is a pseudo-umbilical submanifold of $M_1 \times_f M_2$. This completes the proof of the theorem. $\square$

Theorem 3.8 *Let $(M_1 \times_f M_2, g)$ be the Riemannian warped product manifold with the warping function f and $\overline{M}$ be an invariant submanifold of $M_1 \times_f M_2$. We denote the integral manifolds of the vertical and horizontal distributions of $\overline{M}$ by $\overline{M}_a$ and $\overline{M}_b$, respectively. If $\overline{M}$ is a curvature-invariant submanifold of $M_1 \times_f M_2$, then $\overline{M}_a$ and $\overline{M}_b$ are curvature-invariant submanifolds of M_1 and M_2, respectively.*

Proof. We denote the Riemannian curvature tensor fields of Riemannian manifolds $M_1 \times_f M_2$, M_1 and M_2 by R, R_1 and R_2, respectively. Then using the Proposition. 2.3, by direct calculations we get

$$
\begin{aligned}
R(X,Y)Z &= R_1(X_1,Y_1)Z_1 + R_2(X_2,Y_2)Z_2 - g_1(\mathrm{grad}f,\mathrm{grad}f)\{g_2(X_2,Z_2)Y_2 \\
&\quad - g_2(Y_2,Z_2)X_2\} - \frac{1}{f}H^f(X_1,Z_1)Y_2 + \frac{1}{f}H^f(Y_1,Z_1)X_2 \\
&\quad + \frac{1}{f}g(Y_2,Z_2)\nabla_{X_1}\mathrm{grad}f - \frac{1}{f}g(X_2,Z_2)\nabla_{Y_1}\mathrm{grad}f \\
&= R_1(X_1,Y_1)Z_1 + R_2(X_2,Y_2)Z_2 - g_1(\mathrm{grad}f,\mathrm{grad}f)\{g_2(X_2,Z_2)Y_2 \\
&\quad - g_2(Y_2,Z_2)X_2\} - \frac{1}{f}H^f(Z_1,X_1)Y_2 + \frac{1}{f}H^f(Y_1,Z_1)X_2 \\
&\quad + fg_2(Y_2,Z_2)\nabla^a_{X_1}\mathrm{grad}f + fg_2(Y_2,Z_2)h_1(X_1,\mathrm{grad}f) \\
&\quad - fg_2(X_2,Z_2)\nabla^a_{Y_1}\mathrm{grad}f - fg_2(X_2,Z_2)h_1(Y_1,\mathrm{grad}f), \qquad (3.14)
\end{aligned}
$$

where $X_1, Y_1, Z_1 \in \Gamma(T\overline{M}_a)$ and $X_2, Y_2, Z_2 \in \Gamma(T\overline{M}_b)$. From the Gauss equation we have

$$
R(X,Y)Z = R^2(X,Y)Z + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) + A_{h(X,Z)}Y - A_{h(Y,Z)}X
$$

for all $X, Y, Z \in \Gamma(T\overline{M})$, where R^2 and A denote the Riemannian curvature tensor and the shape operator of $\overline{M}$, respectively. Thus from the Codazzi Equation, we obtain

$$
\begin{aligned}
(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) &= (\nabla_{X_1}h_1)(Y_1,Z_1) - (\nabla_{Y_1}h_1)(X_1,Z_1) \\
&\quad + (\nabla^2_{X_2}h_2)(Y_2,Z_2) - (\nabla^2_{Y_2}h_2)(X_2,Z_2) \\
&\quad + fg_2(Y_2,Z_2)h_1(X_1,\mathrm{grad}f) \\
&\quad - fg_2(X_2,Z_2)h_1(Y_1,\mathrm{grad}f), \qquad (3.15)
\end{aligned}
$$

where ∇^2 and ∇ are the Levi-Civita connections on M_2 and $M_1 \times_f M_2$, respectively.

If $\overline{M}$ is a curvature-invariant submanifold of the Riemannian warped product manifold $M_1 \times_f M_2$, from the equation (3.15) we have

$$
(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0,
$$

which implies that

$$
\begin{aligned}
(\nabla_{X_1}h_1)(Y_1,Z_1) - (\nabla_{Y_1}h_1)(X_1,Z_1) &+ fg_2(Y_2,Z_2)h_1(X_1,\mathrm{grad}f) \\
&- fg_2(X_2,Z_2)h_1(Y_1,\mathrm{grad}f) = 0, \qquad (3.16)
\end{aligned}
$$

and

$$(\nabla^2_{X_2} h_2)(Y_2, Z_2) - (\nabla^2_{Y_2} h_2)(X_2, Z_2) = 0. \tag{3.17}$$

So the equation (3.17) implies that $\overline{M}_b$ is a curvature-invariant submanifold of M_2.

If we take FZ for Z in the equation (3.16), then we have

$$(\nabla_{X_1} h_1)(Y_1, Z_1) - (\nabla_{Y_1} h_1)(X_1, Z_1) - fg_2(Y_2, Z_2)h_1(X_1, \mathrm{grad} f) \\ + fg_2(X_2, Z_2)h_1(Y_1, \mathrm{grad} f) = 0. \tag{3.18}$$

So from the equations (3.16) and (3.18) we get

$$(\nabla_{X_1} h_1)(Y_1, Z_1) - (\nabla_{Y_1} h_1)(X_1, Z_1) = 0,$$

which implies that $\overline{M}_a$ is a curvature-invariant submanifold of M_1. This completes the proof of the theorem. □

Theorem 3.9 *Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold and $\overline{M}$ be an invariant submanifold of $(M_1 \times_f M_2, g)$. If $M_1 \times_f M_2$ has constant sectional curvature and $\overline{M}$ is the totally umbilical submanifold of $M_1 \times_f M_2$, then $\overline{M}$ is a totally geodesic submanifold of $M_1 \times_f M_2$.*

Proof. Since M_1 and M_2 are totally geodesic and totally umbilical submanifold of $M_1 \times_f M_2$, respectively, if $M_1 \times_f M_2$ has constant sectional curvature c then M_1 and M_2 have also constant sectional curvatures c and $c+ \parallel \mathrm{grad} f \parallel^2$, respectively. We have

$$h(X, Y) = \overline{g}(X, Y)H \tag{3.19}$$

for all $X, Y \in \Gamma(T\overline{M})$ because $\overline{M}$ is a totally umbilical submanifold of $M_1 \times_f M_2$. In this case, $\overline{M}$ has also constant sectional curvature $c+ \parallel H \parallel^2$. Moreover, $\overline{M}_a$ and $\overline{M}_b$ have constant sectional curvatures

$$c+ \parallel H \parallel^2, c+ \parallel H \parallel^2 + \parallel \mathrm{grad} f \parallel^2,$$

respectively, according to Theorem. 3.3.

We take $X = X_1, Y = Y_1 \in \Gamma(T\overline{M}_a)$ in equation (3.19) and using the projection mapping π_*, we get

$$h_1(X_1, Y_1) = \overline{g}(X_1, Y_1)\frac{a}{m}H_1. \tag{3.20}$$

In the same way, we take $X = X_2, Y = Y_2 \in \Gamma(T\overline{M}_b)$ in equation (3.19) and using the projection mapping σ_* we have

$$h_2(X_2, Y_2) = \overline{g}(X_2, Y_2)\frac{b}{m}H_2. \tag{3.21}$$

Thus we derive that $\overline{M}_a$ and $\overline{M}_b$ have also constant sectional curvatures

$$c+\frac{a^2}{m^2} \parallel H_1 \parallel^2, c + \frac{b^2}{m^2} \parallel H_2 \parallel^2 + \parallel \mathrm{grad} f \parallel^2,$$

respectively, that is,

$$\frac{a^2}{m^2} \parallel H_1 \parallel^2 =\parallel H \parallel^2, \parallel H \parallel^2 + \parallel \mathrm{grad} f \parallel^2 = \frac{b^2}{m^2} \parallel H_2 \parallel^2 .$$

It follow that $H = 0$. $\overline{M}$ is a totally geodesic submanifold of $M_1 \times_f M_2$ because $\overline{M}$ is the totally umbilical submanifold of $M_1 \times_f M_2$.

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