Turk J Math
27 (2003) , 425 – 434.

Uniqueness of Primary Decompositions

P. F. Smith

Abstract

Uniqueness properties of primary decompositions in modules over non-commutative rings are presented.

Key Words: Primary, decomposition, normal decomposition, prime ideal, Left Noetherian ring, PI ring.

1. Introduction

Throughout, R is a ring (not necessarily commutative) with identity and all modules are unital left modules. For any submodules N, L of an R-module M, we define $(N : L) = \{r \in R : rL \subseteq N\}$. Note that $(N : L)$ is an ideal of R. Moreover $(N : L) = R$ if and only if $L \subseteq N$. Given a prime ideal P of R, a proper submodule K of an R-module M is called *P-primary* provided

(i) $(K : N) \subseteq P$ for every submodule N of M such that $N \nsubseteq K$; and

(ii) $P^n \subseteq (K : M)$ for some positive integer n.

Note that if K is P-primary, then $P^n \subseteq (K : M) \subseteq P$ for some positive integer n. A submodule L of an R-module M is called *primary* if L is P-primary for some prime ideal P of R. A submodule H of M has a *primary decomposition* if H is the intersection of a finite collection of primary submodules of M. Note that if H has a primary decomposition then H is a proper submodule of M.

In [1], Krull gave necessary and sufficient conditions for a proper ideal I of a commutative ring R to have a primary decomposition. It is a standard fact that, if R is a commutative Noetherian ring and M is a finitely generated R-module then every proper submodule of M has a primary decomposition (see, for example, [5, Theorem 3.10] or [2. Exercise 9.31]). In [3], Fisher gives necessary and sufficient conditions for a proper

AMS Mathematics Subject Classification: Primary 16.D70, Secondary 16D25, 16N60 submodule of an R-module M to have a primary decomposition in case R is a (not necessarily commutative) ring with the property that nil ideals are nilpotent. For a recent new treatment of the existence of primary decompositions see [4], and for the related geometrical aspects see [5, Section 3.8].

In this note we are concerned not with the existence but with uniqueness properties of primary decompositions. Eisenbud [5, Section 3.7] gives a simple example to show that a submodule may have many (even an infinite number of) different primary decompositions. In [6], Gilmer characterizes commutative rings in which each ideal is uniquely the intersection of a finite number of primary ideals.

This paper generalizes results of [11] where the dicussion is concerned with submodules which are finite intersections of prime submodules, a special case of what follows here.

Let N be a submodule of an R-module M such that N has a primary decomposition. Then N will be said to have a *normal decomposition* if there exist a positive integer n, distinct prime ideals $P_i (1 \leq i \leq n)$ of R and P_i-primary submodules $K_i (1 \leq i \leq n)$ of M such that $N = K_1 \cap \cdots \cap K_n$ and $N \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$ for all $1 \leq i \leq n$.

Lemma 1 *Let R be any ring, let P be a prime ideal of R, let n be a positive integer and let K_i be a P-primary submodule of M for each $1 \leq i \leq n$. Then $\cap_{i=1}^{n} K_i$ is also a P-primary submodule of M.*

Proof. Straightforward. □

Corollary 2 *Let R be any ring and let N be a submodule of an R-module M such that N has a primary decomposition. Then N has a normal decomposition.*

Proof. By Lemma 1. □

The proof of the next result is a straightforward adaptation of [7, p.15 Theorem 2].

Theorem 3 *Let R be any ring, let N be a submodule of an R-module M such that N has a primary decomposition and let $N = K_1 \cap \cdots \cap K_n$ and $N = L_1 \cap \cdots \cap L_k$ be normal decompositions of N where K_i is P_i-primary for some prime ideal P_i $(1 \leq i \leq n)$ and L_j is Q_j-primary for some prime ideal $Q_j (1 \leq j \leq k)$ of R. Then $n = k$ and $\{P_i : 1 \leq i \leq n\} = \{Q_j : 1 \leq j \leq k)$.*

Proof. Consider the prime ideals $P_1, \cdots, P_n, Q_1, \cdots, Q_k$. Without loss of generality, we can suppose that $P_n \nsubseteq P_i (1 \leq i \leq n-1)$ and P_n is not strictly contained in $Q_j (1 \leq j \leq k)$. There exists a positive integer t such that $P_n^t M \subseteq K_n$ and hence

$$P_n^t (K_1 \cap \cdots \cap K_{n-1}) \subseteq N = L_1 \cap \cdots \cap L_k.$$

If $K_1 \cap \cdots \cap K_{n-1} \subseteq L_j (1 \leq j \leq k)$, then $N = K_1 \cap \cdots \cap K_{n-1}$, a contradiction. Without loss of generality we can suppose that $K_1 \cap \cdots \cap K_{n-1} \nsubseteq L_k$. Then $P_n^t \subseteq Q_k$ and hence $P_n \subseteq Q_k$. By the choice of P_n, we conclude that $P_n = Q_k$.

Next note that

$$P_n^t(K_1 \cap \cdots \cap K_{n-1}) \subseteq N \subseteq L_1 \cap \cdots \cap L_{k-1},$$

and $P_n \nsubseteq Q_i (1 \leq i \leq k-1)$, so that

$$K_1 \cap \cdots \cap K_{n-1} \subseteq L_1 \cap \cdots \cap L_{k-1}.$$

Similarly, $L_1 \cap \cdots \cap L_{k-1} \subseteq K_1 \cap \cdots \cap K_{n-1}$. Hence $K_1 \cap \cdots \cap K_{n-1} = L_1 \cap \cdots \cap L_{k-1}$ and the result follows by induction. □

In view of Theorem 3, for any submodule N of an R-module M we call prime ideals P_i $(1 \leq i \leq n)$ of R the *associated prime ideals* of N provided there exists a normal decomposition $N = K_1 \cap \cdots \cap K_n$, where K_i is a P_i-primary submodule of M for each $1 \leq i \leq n$.

If A is a proper ideal of a ring R then a prime ideal P of R is a *minimal prime ideal* of A if $A \subseteq P$ and P/A is a minimal prime ideal of the ring R/A.

Lemma 4 *Let R be any ring and let N be a submodule of an R-module M such that N has a primary decomposition. Then every minimal prime ideal of the ideal $(N : M)$ is an associated prime ideal of N.*

Proof. Let $N = K_1 \cap \cdots \cap K_n$ be a normal decomposition of N where K_i is a P_i-primary submodule for some prime ideal P_i for each $1 \leq i \leq n$. There exists a positive integer k such that $P_i^k \subseteq (K_i : M)$ for all $1 \leq i \leq n$. Then $(P_1 \cdots P_n)^k \subseteq (N : M) \subseteq P_1 \cap \cdots \cap P_n$. Let P be any minimal prime ideal of $(N : M)$. Then $(P_1 \cdots P_n)^k \subseteq P$ and hence $P_i \subseteq P$ for some $1 \leq i \leq n$. It follows that $P = P_i$. □

By adapting the proof of Theorem 3, we have the following "uniqueness result".

Theorem 5 *Let R be any ring, let N be a submodule of an R-module M such that N has a primary decomposition and let $P_i (1 \leq i \leq n)$ be the associated prime ideals of N, for some positive integer n, such that $P_j \nsubseteq P_i$ for all $1 \leq i < j \leq n$. Let $N = K_1 \cap \cdots \cap K_n$ and $N = L_1 \cap \cdots \cap L_n$ be normal decompositions of N in terms of P_i-primary submodules K_i and L_i $(1 \leq i \leq n)$. Then $K_1 \cap \cdots \cap K_i = L_1 \cap \cdots \cap L_i$ for all $1 \leq i \leq n$.*

Next, we give a characterization of the associated prime ideals of a submodule with primary decomposition.

Theorem 6 *Let R be any ring and let N be a submodule of an R-module M such that N has a primary decomposition. Then the following statements are equivalent for a prime ideal P of R.*

(i) P is an associated prime ideal of N.

(ii) $P = (N : L)$ for some submodule L of M with $L \nsubseteq N$.

(iii) $P = \{r \in R : rRm \subseteq N$ for some element $m \in M \backslash N\}$.

Proof. (i) $\Rightarrow$ (ii) Let $N = K_1 \cap \cdots \cap K_n$ be a normal decomposition of N where K_i is a P_i-primary submodule of M for some prime ideal P_i of R for each $1 \leq i \leq n$. Let $1 \leq i \leq n$ and let $H_i = K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$. There exists a positive integer $k(i)$ such that $P_i^{k(i)} M \subseteq K_i$ and hence $P_i^{k(i)} H_i \subseteq N$. Since $H_i \nsubseteq N$ there exists an integer $1 \leq t(i) \leq k(i)$ such that $P_i^{t(i)} H_i \subseteq N$ but $P_i^{t(i)-1} H_i \nsubseteq N$. Let $L_i = P_i^{t(i)-1} H_i$. Then L_i is a submodule of M such that $L_i \nsubseteq N$ and $P_i L_i \subseteq N$.

Let $A = (N : L_i)$ and note that $P_i \subseteq A$. On the other hand, $AL_i \subseteq N \subseteq K_i$. If $L_i \subseteq K_i$ then $L_i \subseteq N$, a contradiction. Thus $A \subseteq P_i$. It follows that $P_i = (N : L_i)$.

(ii) $\Longleftrightarrow$ (iii) Clear.

(ii) $\Longrightarrow$ (i) Suppose that $P = (N : L)$ for some submodule $L \nsubseteq N$. There exists $1 \leq i \leq n$ such that $L \nsubseteq K_i$. Without loss of generality, there exists $1 \leq m \leq n$ such that $L \nsubseteq K_i$ $(1 \leq i \leq m)$ and $L \subseteq K_i$ $(m+1 \leq i \leq n)$. Clearly $PL \subseteq N \subseteq K_1 \cap \cdots \cap K_m$ implies that $P \subseteq P_1 \cap \cdots \cap P_m$. On the other hand, there exists a positive integer s such that $(P_1 \cap \cdots \cap P_m)^s M \subseteq K_1 \cap \cdots \cap K_m$ and hence $(P_1 \cap \cdots \cap P_m)^s L \subseteq N$. Thus $(P_1 \cap \cdots \cap P_m)^s \subseteq P$ and we have $P_1 \cap \cdots \cap P_m \subseteq P$, so that $P = P_1 \cap \cdots \cap P_m$. This implies that $P = P_i$ for some $1 \leq i \leq m$. □

If X is a non-empty subset of a ring R then $\ell(X)$ will denote the left annihilator of X, i.e. $\ell(X) = \{r \in R : rX = 0\}$. By a *prime left annihilator* of R we mean a prime ideal P of R such that $P = \ell(X)$ for some non-empty subset X of R, equivalently $P = \ell(A)$, where A is the ideal RXR of R.

Corollary 7 *Let R be any ring, let N be a submodule of an R-module M such that N has a primary decomposition and let P be a prime ideal of R such that $(N : M) \subseteq P$ and*

$P/(N : M)$ is a prime left annihilator of the ring $R/(N : M)$. Then P is an associated prime ideal of N.

Proof. There exists an ideal A of R such that $P = \{r \in R : rA \subseteq (N : M)\}$. Clearly this implies that $A \nsubseteq (N : M)$, i.e. $AM \nsubseteq N$ and $P = (N : AM)$. By Theorem 6, P is an associated prime ideal of N. □

Note that the converse of Corollary 7 is false in general, as the following simple example shows.

Example 8 *Let $\mathbb{Z}$ be the ring of rational integers, let p be any prime in $\mathbb{Z}$, let M be the free $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ and let N be the proper submodule $0 \oplus \mathbb{Z}p$ of M. Then $N = K_1 \cap K_2$ is a normal decomposition where K_1 is the 0-primary submodule $0 \oplus \mathbb{Z}$ and K_2 is the $(\mathbb{Z}p)$-primary submodule $\mathbb{Z}p \oplus \mathbb{Z}p$. Thus the associated prime ideals of N are 0 and $\mathbb{Z}p$. Note that $(N : M) = 0$ and $\mathbb{Z}p$ is not a prime (left) annihilator of $\mathbb{Z}$.*

This brings us to another "uniqueness result".

Theorem 9 *Let R be any ring, let N be a submodule of an R-module M such that N has a primary decomposition and let $P_i (1 \leq i \leq n)$ be the associated prime ideals of N such that $P_i \nsubseteq P_1$ for all $2 \leq i \leq n$. Let $N = K_1 \cap \cdots \cap K_n$ be any normal decomposition of N where K_i is a P_i-primary submodule of M for each $1 \leq i \leq n$. Then $K_1 = \{m \in M : Am \subseteq N$ for some ideal A of R with $A \nsubseteq P_1\}$.*

Proof. Let $m \in M$ satisfy $Am \subseteq N$ for some ideal $A \nsubseteq P_1$. Then $Am \subseteq K_1$ and hence $m \in K_1$. On the other hand, there exists a positive integer k such that $P_i^k M \subseteq K_i$ $(2 \leq i \leq n)$. It follows that if $B = \Pi_{i=2}^n P_i^k$ then B is an ideal of R, $B \nsubseteq P_1$ and $BK_1 \subseteq N$. □

Corollary 10 *Let R be any ring, let N be a submodule of an R-module M such that N has a primary decomposition and let P_i $(1 \leq i \leq n)$ be the associated prime ideals of N such that $P_1, \cdots, P_t$ are minimal in $\{P_i : 1 \leq i \leq n\}$, for some $1 \leq t \leq n$. Let $N = K_1 \cap \cdots \cap K_n$ be any normal decomposition of N where K_i is a P_i-primary submodule of M for each $1 \leq i \leq n$. Then $K_1 \cap \cdots \cap K_t = \{m \in M : Am \subseteq N$ for some ideal $A \nsubseteq P_1 \cup \cdots \cup P_t\}$.*

Proof. Suppose first that $Am \subseteq N$ for some ideal $A \nsubseteq P_1 \cup \cdots \cup P_t$. For each $1 \leq i \leq t$, $A \nsubseteq P_i$ and $Am \subseteq N$ so that $m \in K_i$ by Theorem 9. Thus $m \in K_1 \cap \cdots \cap K_t$.

Conversely, let $x \in K_1 \cap \cdots \cap K_t$. By Theorem 9, for each $1 \le i \le t$ there exists an ideal $B_i \nsubseteq P_i$ such that $B_i x \subseteq N$. Let $B = \sum_{i=1}^{t} B_i\,(\Pi_{j \ne i} P_j)$. Then B is an ideal of R such that $B \nsubseteq P_1 \cup \cdots \cup P_t$ and $Bx \subseteq N$. □

Next we give an algorithm for finding the associated prime ideals of a submodule having a primary decomposition. Such a process for the commutative case is given in [8]. The algorithm we give works for modules over an arbitrary left or right Noetherian ring and depends on being able to find the minimal prime ideals of certain proper ideals of the ring. We shall call a ring R *suitable* if every proper ideal of R has only a finite number of minimal prime ideals. Any ring which satisfies the ascending chain condition on semiprime ideals is suitable by [9, Proposition 33]. Clearly left or right Noetherian rings satisfy the ascending chain condition on semiprime ideals and so too do rings with left or right Krull dimension by [10, Proposition 7.3], so that all such rings are suitable.

Lemma 11 *Let R be any ring, let N be a submodule of an R-module M such that N has a primary decomposition and let $N = K_1 \cap \cdots \cap K_n$ be a normal decomposition of N in terms of primary submodules $K_i (1 \le i \le n)$. Then $(N : K_1 \cap \cdots \cap K_i) = (K_{i+1} \cap \cdots \cap K_n : M)$ for all $1 \le i \le n-1$.*

Proof. Let $r \in (K_{i+1} \cap \cdots \cap K_n : M)$. Then $r(K_1 \cap \cdots \cap K_i) \subseteq K_1 \cap \cdots \cap K_n = M$. Conversely, let $s \in (N : K_1 \cap \cdots \cap K_i)$. For each $i+1 \le j \le n$, $s(K_1 \cap \cdots \cap K_i) \subseteq N \subseteq K_j$ and $K_1 \cap \cdots \cap K_i \nsubseteq K_j$, so that $sM \subseteq K_j$. Hence $sM \subseteq K_{i+1} \cap \cdots \cap K_n$. □

Theorem 12 *Let R be a suitable ring and let N be a submodule of a left R-module M such that N has a primary decomposition, let $P_1, \cdots, P_{k(1)}$ be the minimal prime ideals of the ideal $(N : M)$ of R, let $N_1 = \{m \in M : Am \subseteq N$ for some ideal $A \nsubseteq P_1 \cup \cdots \cup P_{k(1)}\}$, let $P_{k(1)+1}, \cdots, P_{k(2)}$ be the minimal prime ideals of $(N : N_1)$, let $N_2 = \{m \in N_1 : Am \subseteq N$ for some ideal $A \nsubseteq P_{k(1)+1} \cup \cdots \cup P_{k(2)}\}$, let $P_{k(2)+1}, \cdots, P_{k(3)}$ be the minimal prime ideals of $(N : N_2)$, let $N_3 = \{m \in N_2 : Am \subseteq N$ for some ideal $A \nsubseteq P_{k(2)+1} \cup \cdots \cup P_{k(3)}\}$, and so on. Then there exists a positive integer t such that $P_1, \cdots, P_{k(t)}$ are the associated prime ideals of N.*

Proof. Let $N = K_1 \cap \cdots \cap K_n$ be a normal decomposition of N in terms of Q_i-primary submodules K_i for some prime ideal Q_i $(1 \le i \le n)$. Without loss of generality, Lemma 4 gives $Q_i = P_i$ for all $1 \le i \le k(1)$. Suppose that $k(1) < n$. By Corollary 8, $N_1 = K_1 \cap \cdots \cap K_{k(1)}$ and Lemma 9 gives $(N : N_1) = (L_1 : M)$, where $L_1 = K_{k(1)+1} \cap \cdots \cap K_n$.

Again using Lemma 4 we can suppose without loss of generality that $Q_i = P_i$ for all $k(1)+1 \leq i \leq k(2)$. Suppose that $k(2) < n$. Let $N_2 = K_1 \cap \cdots \cap K_{k(2)}$. Then, by Corollary 10,

$$\begin{aligned} K_1 \cap \cdots \cap K_{k(2)} &= N_1 \cap \left(K_{k(1)+1} \cap \cdots \cap K_{k(2)}\right) \\ &= N_1 \cap \{m \in M : Am \subseteq K_{k(1)+1} \cap \cdots \cap K_{k(2)} \text{ for some ideal} \\ &\qquad A \nsubseteq P_{k(1)+1} \cup \cdots \cup P_{k(2)}\} \\ &= N_2. \end{aligned}$$

Again applying Lemma 11 we have $(N : M_2) = (L_2 : M)$, where $L_2 = K_{k(2)+1} \cap \cdots \cap K_n$. Clearly this process must stop since $1 \leq k(1) < k(2) < \cdots \leq n$. □

We can illustrate the process described in Theorem 12 by the following simple example. Again $\mathbb{Z}$ is the ring of rational integers. Let $M = \mathbb{Z}^{(6)}$, let p, q be distinct primes in $\mathbb{Z}$ and let N be the submodule $\mathbb{Z} \oplus \mathbb{Z}p \oplus \mathbb{Z}p^2 \oplus \mathbb{Z}q^3 \oplus \mathbb{Z}q^4 \oplus 0$ of M. Then $(N : M) = 0$ and hence (in the notation of Theorem 12), $k(1) = 1$, $P_1 = 0$ and $N_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus 0$. Next $(N : N_1) = \mathbb{Z}p^2 \cap \mathbb{Z}q^4$, so that $k(2) = 3$, $P_2 = \mathbb{Z}p$ and $P_3 = \mathbb{Z}q$. It can easily be checked that $N_2 = N$. Thus $(N : N_2) = R$ and the process stops, giving the associated prime ideals of N as 0, $\mathbb{Z}p$ and $\mathbb{Z}q$.

Let P be a prime ideal of a ring R. A proper ideal A of R will be called *left P-primary* if

(i) whenever B, C are ideals of R such that $BC \subseteq A$ then $B \subseteq P$ or $C \subseteq A$, and

(ii) $P^n \subseteq A \subseteq P$ for some positive integer n.

Next an ideal will be called *left primary* if it is left P-primary for some prime ideal P of R. An ideal I of R has a *left primary decomposition* if I is the intersection of a finite collection of left primary ideals. Note that an ideal I of R has a left primary decomposition if and only if the submodule I of the left R-module R has a primary decomposition.

Proposition 13 *Let R be any ring and let N be a submodule of an R-module M such that N has a primary decomposition. Then the ideal $(N : M)$ has a left primary decomposition. Moreover, every associated prime ideal of $(N : M)$ is an associated prime ideal of N.*

Proof. Let $N = K_1 \cap \cdots \cap K_n$, where n is a positive integer and K_i is a P_i-primary submodule for some prime ideal P_i for each $1 \leq i \leq n$. Then $(N : M) = (K_1 : M) \cap \cdots \cap (K_n : M)$. Let $1 \leq i \leq n$. There exists a positive integer k such that $P_i^k \subseteq (K_i : M)$. Let

A, B be ideals of R such that $AB \subseteq (K_i : M)$. Then $ABM \subseteq K_i$ and either $A \subseteq P_i$ or $BM \subseteq K_i$, i.e. $B \subseteq (K_i : M)$. It follows that $(K_i : M)$ is left P_i-primary. The result follows. □

A submodule K of an R-module M is called *prime* if $K \neq M$ and $(K : L) = (K : M)$ for every submodule L of M such that $L \not\subseteq K$. In case K is a prime submodule of M it can easily be checked that the ideal $P = (K : M)$ is a prime ideal of R and in this case we call K a *P-prime submodule* of M.

Lemma 14 *Let P be a prime ideal of R. Then the following statements are equivalent for a submodule K of an R-module M.*

(i) K is P-prime.

(ii) K is P-primary and $P \subseteq (K : M)$.

Proof. Straightforward. □

A submodule N of a module M has a *prime decomposition* if N is the intersection of a finite collection of prime submodules of M. Let N be a submodule of an R-module M such that N has a prime decomposition. Then N will be said to have a *normal prime decomposition* if there exist a positive integer n, distinct prime ideals $P_i (1 \leq i \leq n)$ of R and P_i-prime submodules $K_i (1 \leq i \leq n)$ of M such that $N = K_1 \cap \cdots \cap K_n$ and $N \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$ for all $1 \leq i \leq n$. By Corollary 2, any submodule having a prime decomposition has a normal prime decomposition. In certain situations it is possible to write down explicitly a normal prime decomposition for a submodule once its associated prime ideals are known, as we show next.

Theorem 15 *Let R be any ring, let N be a submodule of an R-module M such that N has a prime decomposition and let $P_i (1 \leq i \leq n)$ be the associated prime ideals of N. Suppose further that for each $1 \leq i \leq n$ and for each ideal $A \not\subseteq P_i$ there exists an ideal $B \not\subseteq P$ and a finitely generated left ideal C of R such that $B \subseteq C \subseteq A$. For each $1 \leq i \leq n$ let $H_i = \{m \in M : Dm \subseteq N + P_i M$ for some ideal $D \not\subseteq P_i\}$. Then H_i is a P_i-prime submodule of M for each $1 \leq i \leq n$ and $N = H_1 \cap \cdots \cap H_n$ is a normal prime decomposition of N.*

Proof. Let $N = K_1 \cap \cdots \cap K_n$ be a normal prime decomposition where K_i is a P_i-prime submodule of M for each $1 \leq i \leq n$. Let $1 \leq j \leq n$. It is easy to check that H_j is a submodule of M such that $P_j \subseteq (H_j : M)$ and $N \subseteq H_j$. Let $m \in H_j$. There exists an ideal $E \nsubseteq P_j$ such that $Em \subseteq N + P_jM \subseteq K_j$ so that $m \in K_j$. It follows that $H_j \subseteq K_j$. In particular, H_j is a proper submodule of M.

Now we prove that H_j is a P_j-prime submodule of M. Let $x \in M$ and let F be an ideal of R such that $F \nsubseteq P_j$ and $Fx \subseteq H_j$. By hypothesis, there exist a finitely generated left ideal G and an ideal $G' \nsubseteq P_j$ such that $G' \subseteq G \subseteq F$. Suppose that $G = Rg_1 + \cdots + Rg_k$ for some positive integer k and elements $g_i \in G(1 \leq i \leq k)$. For each $1 \leq s \leq k$ there exists an ideal $I_s \nsubseteq P_j$ such that $I_s g_s x \subseteq N + P_jM$. Let $I = I_1 \cdots I_k$. Then I is an ideal of R such that $I \nsubseteq P_j$ and $IG'x \subseteq IGx = \sum_{s=1}^{k} Ig_s x \subseteq N + P_jM$. Because $IG' \nsubseteq P_j$, the element $x \in H_j$. This proves that H_j is a P_j-prime submodule of M.

To summarise, H_j is a P_j-prime submodule of M such that $N \subseteq H_j \subseteq K_j$ for all $1 \leq j \leq n$. Clearly $N = H_1 \cap \cdots \cap H_n$ and this is a normal prime decomposition of N.□

Note that the condition on associated prime ideals in Theorem 15 is satisfied if R is a left Noetherian ring or a PI-ring. Note further that the proof of Theorem 15 gives the following result.

Corollary 16 *let N be a submodule of an R-module M and let P be a prime ideal of R such that for each ideal $A \nsubseteq P$ there exists an ideal $B \nsubseteq P$ and a finitely generated left ideal C such that $B \subseteq C \subseteq A$. For each positive integer n let $H_n = \{m \in M : Dm \subseteq N + P^nM$ for some ideal $D \nsubseteq P\}$. Then H_n is a P-primary submodule containing N for each positive integer n.*

References

[1] W. Krull, Uber einen Hauptsatz der allgemeinen Idealtheorie, Sitzungsber. Heidelberg. Akad. Wiss. 2 (1929), 11–16.

[2] R.Y. Sharp, Steps in Commutative Algebra, London Math. Soc. Student Texts 19 (Cambridge Univ. Press, Cambridge 1990).

[3] J.W. Fisher, The primary decomposition theory for modules, Pacific J. Math. 35 (1970), 359-367.

[4] B.L. Osofsky, Noether Laskar primary decomposition revisited, Amer. Math. Monthly 101 (1994), 759-768.

[5] D. Eisenbud, Commutative Algebra with a view toward Algebraic Geometry (Springer-Verlag New York 1995).

[6] R.W. Gilmer, Rings in which the unique primary decomposition theorem holds, Proc. Amer. Math. Soc. 14 (1963), 777-781.

[7] D.G. Northcott, Ideal Theory (Cambridge University Press, Cambridge 1965).

[8] D. Eisenbud, C. Huneke and W. Vasconcelos, Direct methods for primary decomposition, Invent. Math. 110 (1992), 207-235.

[9] M. Nagata, On the theory of radicals in a ring, J. Math. Soc. Japan 3 (1951), 330-344.

[10] R. Gordon and J.C. Robson, Krull dimension, Memoirs Amer. Math. Soc. 133 (1973).

[11] R.L. McCasland, J.B.A. Schmidt, P.F. Smith and E.M. Stark, Uniform dimension and radical modules, University of Glasgow preprint 99/19 (1999).

P. F. SMITH
Department of Mathematics
University of Glasgow
Glasgow, G12 8QW, Scotland, UK

Received 31.10.2002