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On the Extended Hecke Groups $\overline{H}(\lambda_q)$

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Abstract

Hecke groups $H(\lambda_q)$ have been studied extensively for many aspects in the literature, [5], [8]. The Hecke group $H(\lambda_3)$, the modular group $PSL(2,\mathbb{Z})$, has especially been of great interest in many fields of mathematics, for example number theory, automorphic function theory and group theory. In this paper we consider the extended Hecke groups $\overline{H}(\lambda_q)$ which are defined analogously with the extended modular group. We find the conjugacy classes of torsion elements in $\overline{H}(\lambda_q)$. Using this we give some results about the normal subgroups and Fuchsian subgroups of $\overline{H}(\lambda_q)$.

Key Words: Extended Hecke group, conjugacy class, extended modular group.

1. Introduction

Hecke introduced an infinite class of discrete groups $H(\lambda_q)$ of linear fractional transformations preserving the upper-half plane [2]. The Hecke group $H(\lambda)$ is the group generated by

$$x(z) = -\frac{1}{z} \quad \text{and} \quad u(z) = z + \lambda,$$

where $\lambda = \lambda_q = 2\cos\pi/q$, $q \geq 3$ integer, or $\lambda \geq 2$. We consider the former case. These Hecke groups $H(\lambda_q)$ are Fuchsian groups of the first kind. Let

$$y = xu = \frac{-1}{z + \lambda_q}.$$

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Then $H(\lambda_q)$ has a presentation

$$H(\lambda_q) =< x, y : x^2 = y^q = 1 > .$$

For $q = 3$, the resulting Hecke group $H(\lambda_3)$ is the modular group $PSL(2, \mathbb{Z})$. In a sense Hecke groups are generalizations of the modular group. Similar to the extended modular group case, the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, is the group obtained by adding the reflection

$$r(z) = 1 \ / \ \overline{z}$$

to the generators of $H(\lambda_q)$, and the Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. The extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) =< x, y, r : x^2 = y^q = r^2 = (xr)^2 = (yr)^2 = 1 > . \qquad (1.1.1)$$

Again, for $q = 3$, we obtain the extended modular group studied in [3], [4].

In this paper, firstly we find the conjugacy classes of finite order elements of the extended Hecke groups $\overline{H}(\lambda_q)$. In studying normal subgroups (especially Fuchsian ones), it is important to find the conjugacy classes of torsion elements. Also we show that in $\overline{H}(\lambda_p)$, $p \geq 3$ prime number, every normal subgroup with torsion has finite index (in fact the index is $2, 4$ or $2p$).

2. Conjugacy Classes

It is well known that the Hecke group $H(\lambda_q)$ is a free product of a cyclic group of order 2 and a cyclic group of order q, i.e. $H(\lambda_q) \cong \mathbb{Z}_2 * \mathbb{Z}_q$, [1]. The extended Hecke group $\overline{H}(\lambda_q)$ is also known to be a free product with amalgamation as $\overline{H}(\lambda_q) \cong D_2 *_{\mathbb{Z}_2} D_q$, [7], [4].

One of the consequences of the amalgam decomposition is a classification of the conjugacy classes of torsion elements in $\overline{H}(\lambda_q)$.

The following lemmas are trivial due to the presentation of $\overline{H}(\lambda_q)$ given in (1.1.1).

Lemma 2.1 *In $\overline{H}(\lambda_q)$, we have $y^s r = r y^{q-s}$, $1 \leq s \leq q-1$.*

Lemma 2.2 *In $\overline{H}(\lambda_q)$, we have* *(i)* $y^n r, 1 \leq n \leq q-1$, *is conjugate to* r *by* $y^t r$ *where* $t = \frac{q.k+n}{2}$ *for some* $k \in \mathbb{Z}$ *satisfying the condition* $t \in \mathbb{Z}$ *(note that* t *is not unique).*

(ii) y^s, $1 \leq s \leq \frac{q-1}{2}$, *is conjugate to* y^{q-s} *by* r.

Now we can give the following theorem for the extended Hecke group $\overline{H}(\lambda_p)$ where $p \geq 3$ prime number. Note that, by the definition, $\overline{H}(\lambda_q)$ contains reflections.

Theorem 2.3 *Let* $p \geq 3$ *be prime number. There are exactly* $\frac{p+5}{2}$ *conjugacy classes of torsion elements in* $\overline{H}(\lambda_p)$*, three for those of order* 2 *and* $\frac{p-1}{2}$ *for those of order* p*. In particular, any elliptic transformation of order 2 is conjugate to* $x : z \to -1/z$ *and any reflection is conjugate to one of* $r : z \to 1/\overline{z}$ *or* $xr : z \to -\overline{z}$ *while any elliptic transformation of order* p *is conjugate to one of* $y : z \to -1/(z + \lambda_p)$, y^2 *,... or* $y^{\frac{p-1}{2}}$.

Proof. We consider the following presentation of $\overline{H}(\lambda_p)$ given in (1.1.1):

$$\overline{H}(\lambda_p) =< x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1 > .$$

The decomposition of $\overline{H}(\lambda_p) \cong D_2 *_{\mathbb{Z}_2} D_p$ arose by letting

$$G_1 = \ < x, r : x^2 = r^2 = (xr)^2 = 1 > \ \cong D_2$$

and

$$G_2 = \ < y, r : y^p = r^2 = (yr)^2 = 1 > \ \cong D_p.$$

From a theorem of Kurosh, see [6], we know that any element of finite order in a generalized free product $A *_H B$ is conjugate to an element of finite order in one of the factors. Now let g be any element of finite order in $\overline{H}(\lambda_p)$. Then g must be conjugate to an element of finite order in one of the factors G_1 and G_2. Therefore to find the conjugacy classes of finite order elements in $\overline{H}(\lambda_p)$, we must find the conjugacy classes in these factors.

G_1 has one conjugacy class of elliptic elements of order 2 with representative x, two conjugacy classes of reflections with representatives r and xr.

Similarly, in G_2, there are p classes of reflections with representatives r, $yr, \ldots, y^{p-1}r$ and $p-1$ classes of elliptic elements of order p with representatives y, $y^2, \ldots, y^{p-1}$. By Lemma 2.2 (i), $y^n r$ is conjugate to r, $1 \leq n \leq p-1$, then we have one conjugacy class of reflections with representative r. Also by Lemma 2.2 (ii), y^s is conjugate to y^{p-s} and so we have $\frac{p-1}{2}$ conjugacy classes of elliptic elements of order p with representatives $y, y^2, ..., y^{\frac{p-1}{2}}$.

Therefore, in $\overline{H}(\lambda_p)$ as a whole there are: one class of elliptic elements of order 2 with representative x; two classes of reflections with representatives r, xr; and, $\frac{p-1}{2}$ classes of elliptic elements of order p with representatives y, $y^2, \ldots, y^{\frac{p-1}{2}}$. □

The situation is more complex for any odd q which is not prime.

For odd and composite values of q, there are $\frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order q with representatives $y, y^{r_1}, ..., y^{r_{\varphi(q)/2}}$ where $(r_i, q) = 1$, $1 \leq i \leq \frac{\varphi(q)}{2}$ and $\varphi(q)$ is the Euler function. There are: one conjugacy class of elliptic elements of order 2 with representative x; two conjugacy classes of reflections with representatives r, xr; and, in total $\frac{q-1}{2} - \frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order t_i where $t_i \mid q$.

For even q, firstly let us start with $q = 4$ and $q = 6$.

Theorem 2.4 *(i) In $\overline{H}(\sqrt{2})$, there are two conjugacy classes of elliptic elements of order 2 with representatives x, y^2, two conjugacy classes of reflections with representatives r, xr and one class of elliptic elements of order 4 with representative y.*

(ii) In $\overline{H}(\sqrt{3})$, there are two conjugacy classes of elliptic elements of order 2 with representatives x, y^3, two conjugacy classes of reflections with representatives r, xr and one class of elliptic elements of order 3 with representative y^2 and one class of elliptic elements of order 6 with representative y.

For any even $q > 6$, we can only say that there are $\frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order q with representatives $y, y^{r_1}, ...y^{r_{\varphi(q)/2}}$ where $(r_i, q) = 1$. Also, there are two conjugacy classes of elliptic elements of order 2 with representatives $x, y^{\frac{q}{2}}$ and two conjugacy classes of reflections with representatives r, xr. Furthermore, there are totally $\frac{q}{2} - \frac{\varphi(q)}{2} - 1$ conjugacy classes of elliptic elements of order t_i where $t_i \mid q$ and $t_i \neq 2$.

Here we mention briefly conjugacy classes of torsion elements in Hecke groups $H(\lambda_q)$. Note that most of these results can be obtained by use of the notion of fundamental region of Hecke groups. Similar to the extended Hecke group case, for any even q, we can only say that there are $\varphi(q)$ conjugacy classes of elliptic elements of order q with representative $y, y^{r_1}, ..., y^{r_{\varphi(q)}}$ where $(r_i, q) = 1$, two classes of elliptic elements of order 2 with representatives $x, y^{\frac{q}{2}}$. Furthermore, there are in total $q - \varphi(q) - 2$ conjugacy classes of elliptic elements of order t_i where $t_i \mid q$, $t_i \neq 2$. If q is prime, then there are only q conjugacy classes of torsion elements in $H(\lambda_q)$, one for those of order 2 and $q - 1$ for those of order q. In particular, any elliptic transformation of order 2 is conjugate to $x : z \to -1/z$ and any elliptic transformation of order q is conjugate to one of $y : z \to -1/(z + \lambda_p)$, y^2,...,y^{q-1}. For odd and composite values of q, there are $\varphi(q)$ conjugacy classes of elliptic elements of order q with representatives $y, y^{r_1}, ..., y^{r_{\varphi(q)}}$ where $(r_i, q) = 1$, $1 \leq i \leq \varphi(q)$. There are one conjugacy classes of elliptic elements of order 2 with representative x and totally $q - 1 - \varphi(q)$ conjugacy classes of elliptic elements of order t_i where $t_i \mid q$.

Using Theorem 2.3 we get the following.

Theorem 2.5 *If G is a normal subgroup of $\overline{H}(\lambda_p)$, p prime, and G has torsion, then the index $\left|\overline{H}(\lambda_p) : G\right|$ is finite.*

Proof. Suppose $g \in \overline{H}(\lambda_p)$ and g has finite order. Since $G \lhd \overline{H}(\lambda_p)$, if $g \in G$ then $N(g) \subseteq G$ implies $\left|\overline{H}(\lambda_p) : G\right| \mid \left|\overline{H}(\lambda_p) : N(g)\right|$ where $N(g)$ is the normal closure of g in $\overline{H}(\lambda_p)$.

Since $\left|\overline{H}(\lambda_p) : N(g)\right| = \left|\overline{H}(\lambda_p) : N(g^*)\right|$ where g^* is any conjugate of g, we complete the argument by showing that $\left|\overline{H}(\lambda_p) : N(h)\right|$ is finite. Here h is any of the class representatives of torsion elements listed in Theorem 2.3. Now $h = x, r, xr, y, y^2, ..., y^{\frac{p-1}{2}}$. The quotient group $\overline{H}(\lambda_p) \ / \ N(h)$ is the group obtained by adding the relation $h = 1$ to the relations of $\overline{H}(\lambda_p)$, [6].

(1) Suppose $h = x$. Then

$$\overline{H}(\lambda_p)/N(x) \cong < x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, \ x = 1 >$$

$$\cong < y, r : y^p = r^2 = (yr)^2 = 1 > \cong D_p.$$

Therefore $\left|\overline{H}(\lambda_p) : N(x)\right| = 2p$.

(2) Suppose $h = r$. Then we find

$$\overline{H}(\lambda_p)/N(r) \cong < x, y : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, \ r = 1 >$$

$$\cong < x : x^2 = 1 > \cong C_2$$

since $y^2 = y^p = 1$. Therefore $\left|\overline{H}(\lambda_p) : N(r)\right| = 2$.

(3) Let $h = xr$. Then

$$\overline{H}(\lambda_p)/N(xr) \cong < x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1,\ xr = 1 > .$$

As $xr = 1$ we see that $x = r$ since $r^2 = 1$. Therefore we get

$$\overline{H}(\lambda_p)/N(xr) \cong < x, y : x^2 = y^p = (xy)^2 = 1 > \cong D_p,$$

so $\left|\overline{H}(\lambda_p) : N(xr)\right| = 2p$.

(4) Let $h = y$. Then

$$\overline{H}(\lambda_p)/N(y) \cong < x, r : x^2 = r^2 = (xr)^2 = 1 > \cong D_2,$$

so $\left|\overline{H}(\lambda_p) : N(y)\right| = 4$.

(5) Similarly, if $h = y^a$, $2 \leq a \leq \frac{p-1}{2}$, then $(a, p) = 1$ and so we have

$$\overline{H}(\lambda_p)/N(y^a) \cong < x, r : x^2 = r^2 = (xr)^2 = 1 > \cong D_2.$$

Hence $\left|\overline{H}(\lambda_p) : N(y^a)\right| = 4$.

Thus in all cases, the index is finite. □

We can restate this as in the following corollary.

Corollary 2.6 *If $G \lhd \overline{H}(\lambda_p)$ and G has an elliptic element or reflection then $\left|\overline{H}(\lambda_p) : G\right|$ divides $4p$ (divides $2, 4,$ or $2p$, depending on elliptic element or reflection).*

Since the index of Hecke group $H(\lambda_q)$ in $\overline{H}(\lambda_q)$ is 2 and since if G is normal in $H(\lambda_q)$, then is also normal in $\overline{H}(\lambda_q)$, we have

Corollary 2.7 *If $G \lhd H(\lambda_p)$, $p \geq 3$ prime number, and G has an elliptic element then the index $|H(\lambda_p) : G|$ divides $2p$ (divides 2 or p, depending on elliptic element).*

3. Fuchsian Subgroups

It is well-known that $H(\lambda_q)$ is discontinuous in the upper half-plane and Fuchsian with the real line as a fixed circle.

If C is a circle, we let $P(C)$ be the Fuchsian stabilizer of C in $\overline{H}(\lambda_q)$, i.e. the subgroup of $\overline{H}(\lambda_q)$ which maps both C and the interior of C on itself and $P_N(C)$ the normal closure of $P(C)$ in $\overline{H}(\lambda_q)$.

Now we can give the following corollary for the extended Hecke group $\overline{H}(\lambda_p)$ for prime number $p \geq 3$.

Corollary 3.1 *If $P(C)$ contains any elliptic element in $\overline{H}(\lambda_p)$ then the index $\left|\overline{H}(\lambda_p) : P_N(C)\right|$ is finite. In particular, $\left|\overline{H}(\lambda_p) : P_N(C)\right|$ divides $2, 4, 2p$.*

Let $L(C)$ be the general stabilizer of the circle C in $\overline{H}(\lambda_p)$, i.e. the subgroup of $\overline{H}(\lambda_p)$ which maps C on itself and let $L_N(C)$ be its normal closure. Then we have

Theorem 3.2 *If the circle C is fixed by either an elliptic or parabolic element or a reflection in $\overline{H}(\lambda_p)$ then $\left|\overline{H}(\lambda_p) : L_N(C)\right|$ is finite (in fact $\left|\overline{H}(\lambda_p) : L_N(C)\right| \mid 4p$).*

Proof. If the circle C is fixed by an elliptic element or reflection, then the result follows from Theorem 2.5. Now let C is fixed by a parabolic map t. Any parabolic element in $\overline{H}(\lambda_p)$ is conjugate to a translation $t' : z \to z + \alpha$. So, if $vtv^{-1} = t'$, then $v(C)$ is a fixed circle of t'. The real line is fixed by t'. Further, the real line is fixed by $x : z \to -1/z$. Then $v^{-1}xv$ fixes C, so $L(C)$ contains an elliptic map. From Theorem 2.5, $\left|\overline{H}(\lambda_p) : L_N(C)\right|$ is finite and in fact it divides $4p$. □

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