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On General Fibonacci Sequences in Groups

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Abstract

In this paper, we have constituted 3-step general Fibonacci sequences in a nilpotent group with exponent p (p is a prime number) and nilpotency class 4 and given formulas to find the α term of the sequence.

Key Words: General Fibonacci sequences; nilpotent group; nilpotency class; fundamental period.

1. Introduction

Let s_i denote the 3-step general recurrence defined by $s_i = ls_{i-1} + ms_{i-2} + ns_{i-3}$ for some $l,\ m,\ n \in \mathbb{N}$. We assume that p does not divide n; then we get the definition of a 3-step general standard Fibonacci sequence as $(0,\ 0,\ 1,\ l,\ l^2+m,\ l(l^2+m)+lm+n, ...)$ in Z/pZ. If p were permitted to divide n, then the sequence would ultimately be periodic, but would never return to 0, 0, 1. This sequence or loop must be periodic and we use the letter k to denote the fundamental period of s_i that is the shortest period of that sequence. The fundamental period of a sequence satisfying a linear recurrence is sometimes called the Wall number of that sequence. Obviously k depends on p.

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi gives a fast algorithm which is based on the product of Lucas numbers to compute large Fibonacci numbers [8]. Fibonacci sequences have been an interesting subject in applied mathematics. West has shown by using transfer matrices

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that the number $|S_n(123, 3214)|$ of permutations avoiding the patterns 123 and 3214 is the Fibonacci number F_{2n} [11].

The study of Fibonacci sequences in groups began with the earlier work of Wall [10], where the ordinary Fibonacci sequences in cyclic groups were investigated. Vinson was particularly interested in ranks of apparition in ordinary Fibonacci sequences [9]. In the mid 1980's, Wilcox extended the problem to abelian groups [12]. Prolific co-operation among Campbell, Doostie and Robertson expanded the theory to some finite simple groups [3]. Ryba constructed and analized a pair of sequences of representations of the symmetric groups [7]. Aydın and Smith proved in [2] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime number p exponent. Furthermore, Aydın and Dikici proved in [1] that the length of the 2-step general Fibonacci sequences are equal to the length of the 2-step general Fibonacci recurrences constructed by two generating elements of finite nilpotent groups of nilpotency class 2 and exponent a prime number p. Dikici and Smith proved in [5] that, for the 3-step Fibonacci recurrence and any finite $p-$group of exponent p and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step Fibonacci sequence in the field *GF(p)*. Dikici and Özkan proved in [4] for the 3-step general Fibonacci recurrence and any finite $p-$group of exponent p and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step general Fibonacci sequence in the field *GF(p)*.

One of the latest works in this area is [6] in which if G is a non-trivial finite p-group of exponent p and nilpotency class 4, then the followings hold:

i. $k(G) = k$, except for finite primes;

and

ii. $k(G) = kp$ for all primes. (p is a prime number and $p > 3$).

Definition 1 *Let H and K be normal in G and $K \leqslant H$. If H/K is contained in the centre of G/K, then H/K is called a central factor of G. A group G is called nilpotent if it has a finite series of normal subgroups*

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_r = 1$$

such that G_{i-1}/G_i is a central factor of G for each $i = 1, 2, ..., r$. The smallest possible r is called the nilpotency class of G.

The Main Result

Let G be a nilpotent group with nilpotency class 4 and prime number exponent $p > 3$. G has two generators x and y. A presentation of G is

$$G =< x, y, z, t, u : (y, x) = z, (z, x) = t, (t, x) = u >,$$

where pairs of generators with unspecified commutator are implicitly deemed to commute. The subgroup $< y, z, t, u >$is abelian and u is in the centre of G. Every element of G has a unique representation as

$$x^a y^b z^c t^d u^e,$$

where the exponents are elements of $GF(p)$. Having established this way of writing elements, we can even think of groups of elements as vectors of dimension 5 over $GF(p)$, i.e. as (a,b,c,d,e).

Let $(x^a y^b z^c t^d u^e)$ be an element of G. We claim that

$$(x^a y^b z^c t^d u^e)^n = \Big(x^{na} y^{nb} z^{nc+\binom{n}{2}ba} t^{nd+\binom{n}{2}ca+\binom{n}{2}b\binom{a}{2}+\binom{n}{3}ba^2} u^{ne+\binom{n}{2}da+\binom{n}{2}c\binom{a}{2}+\binom{n}{2}b\binom{a}{3}}.$$
$$u^{\left[\binom{n-1}{2}+\binom{n-2}{2}+\cdots+\binom{2}{2}\right]ca^2+\left[\binom{n-1}{3}+\binom{n-2}{3}+\cdots+\binom{3}{3}\right]ba^3+2\left[\binom{n-1}{2}+\binom{n-2}{2}+\cdots+\binom{2}{2}\right]ba\binom{a}{2}} \Big) \quad (1)$$

We show this by induction on n. Since $\binom{1}{2} = \binom{1}{3} = \binom{2}{3} = 0$ our argument is obvious for $n = 1$. So assume that

$$(x^a y^b z^c t^d u^e)^{n-1} = \Big(x^{(n-1)a} y^{(n-1)b} z^{(n-1)c+\binom{n-1}{2}ba} t^{(n-1)d+\binom{n-1}{2}ca+\binom{n-1}{2}b\binom{a}{2}+\binom{n-1}{3}ba^2}$$
$$u^{(n-1)e+\binom{n-1}{2}da+\binom{n-1}{2}c\binom{a}{2}+\binom{n-1}{2}b\binom{a}{3}}$$
$$u^{\left[\binom{n-2}{2}+\binom{n-3}{2}+\cdots+\binom{2}{2}\right]ca^2+\left[\binom{n-2}{3}+\binom{n-3}{3}+\cdots+\binom{3}{3}\right]ba^3+2\left[2\binom{n-2}{2}+\binom{n-3}{2}+\cdots+\binom{2}{2}\right]ba\binom{a}{2}} \Big)$$

Multiplication of both sides by $x^a y^b z^c t^d u^e$ gives the left hand side of (1). Thus we have

$$(x^a y^b z^c t^d u^e)^n = \Big(x^{(n-1)a} y^{(n-1)b} z^{(n-1)c+\binom{n-1}{2}ba} t^{(n-1)d+\binom{n-1}{2}ca+\binom{n-1}{2}b\binom{a}{2}+\binom{n-1}{3}ba^2}$$

$$u^{(n-1)e+\binom{n-1}{2}da+\binom{n-1}{2}c\binom{a}{2}+\binom{n-1}{2}b\binom{a}{3}}$$

$$u^{[\binom{n-2}{2}+\binom{n-3}{2}+\cdots+\binom{2}{2}]ca^2+[\binom{n-2}{3}+\binom{n-3}{3}+\cdots+\binom{3}{3}]ba^3+2[\binom{n-2}{2}+\binom{n-3}{2}+\cdots+\binom{2}{2}]ba\binom{a}{2}} \Big) (x^a y^b z^c t^d u^e)$$

Notice that u is in the centre of G and $(y,x) = z, (z,x) = t$ and $(t,x) = u$, so that

$$(x^a y^b z^c t^d u^e)^n = x^{na} y^{nb} z^{nc+\binom{n}{2}ba} t^{nd+\binom{n}{2}ca+\binom{n}{2}b\binom{a}{2}+\binom{n}{3}ba^2}$$
$$u^{ne+\binom{n}{2}da+\binom{n}{2}c\binom{a}{2}+\binom{n}{2}b\binom{a}{3}+[\binom{n-1}{2}+\binom{n-2}{2}+\cdots\binom{2}{2}]ca^2}$$
$$u^{[\binom{n-1}{3}+\binom{n-2}{3}+\cdots+\binom{3}{3}]ba^3+2[\binom{n-1}{2}+\binom{n,2}{2}+\cdots\binom{a}{2}]ba\binom{a}{2}}$$

Let upper side of x, y, z, t and u be A, B, C, D and E, respectively. That is,

$$A = na$$
$$B = nb$$

$$C = nc + \binom{n}{2}ba$$

$$D = nd + \binom{n}{2}ca + \binom{n}{2}b\binom{a}{2} + \binom{n}{3}ba^2$$

and

$$\begin{aligned} E \quad = \quad & ne + \binom{n}{2}da + nc\binom{n}{2} + \binom{n}{2}b\binom{n}{3} + [\binom{2}{2} + \binom{3}{2} + ... + \binom{n-2}{2} \\ & + \binom{n-1}{2}]ca^2 + [\binom{3}{3} + \binom{4}{3} + ... + \binom{n-2}{3} + \binom{n-1}{3}]ba^3 + 2[\binom{2}{2} \\ & + \binom{3}{2} + ... + \binom{n-2}{2} + \binom{n-1}{2}]ba\binom{a}{2}. \end{aligned}$$

Since $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n-1}{r}=\binom{n}{r+1}$,

$$E=ne+\binom{n}{2}da+\binom{n}{2}c\binom{a}{2}+\binom{n}{2}b\binom{a}{3}+\binom{n}{3}ca^2+\binom{n}{4}ba^3+2\binom{n}{3}ba\binom{a}{2}.$$

Let $(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})$, $(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})$ and $(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})$be three elements of G. We get the formula for multiplication of powers of these three elements. That is, using (1),

$$(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})^l(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})^m(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})^n=(x^{a_3}y^{b_3}z^{c_3}t^{d_3}u^{e_3})$$

where $(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})^l=x^{A_0}y^{B_0}z^{C_0}t^{D_0}u^{E_0}$,$(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})^m=x^{A_1}y^{B_1}z^{C_1}t^{D_1}u^{E_1}$and $(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})^n=x^{A_2}y^{B_2}z^{C_2}t^{D_2}u^{E_2}$ for simplicity. Furthermore,

$$A_0=la_0$$
$$B_0=lb_0$$

$$C_0=lc_0+\binom{l}{2}b_0a_0$$

$$D_0=ld_0+\binom{l}{2}c_0a_0+\binom{l}{2}b_0\binom{a_0}{2}+\binom{l}{2}b_0a_0^2$$

and

$$E_0=le_0+\binom{l}{2}d_0a_0+\binom{l}{2}c_0\binom{a_0}{2}+\binom{l}{2}b_0\binom{a_0}{3}+\binom{l}{3}c_0a_0^2+\binom{l}{4}4b_0a_0^3+2\binom{l}{3}b_0a_0\binom{a_0}{2}.$$

Similarly,

$$A_1=ma_1$$
$$B_1=mb_1$$

$$C_1=mc_1+\binom{m}{2}b_1a_1$$

$$D_1=md_1+\binom{m}{2}c_1a_1+\binom{m}{2}b_1\binom{a_1}{2}+\binom{m}{3}b_1a_1^2$$

and

$$E_1 = le_1 + \binom{m}{2} d_1 a_1 + \binom{m}{2} c_1 \binom{a_1}{2} + \binom{m}{2} b_1 \binom{a_1}{3} + \binom{m}{3} c_1 a_1^2$$

$$+ \binom{m}{4} 4b_1 a_1^3 + 2\binom{m}{3} b_1 a_1 \binom{a_1}{2}.$$

Finally,

$$A_2 = na_2$$
$$B_2 = nb_2$$

$$C_2 = nc_2 + \binom{n}{2} b_2 a_2$$

$$D_2 = nd_2 + \binom{n}{2} c_2 a_2 + \binom{n}{2} b_2 \binom{a_2}{2} + \binom{n}{3} b_2 a_2^2$$

and

$$E_2 = ne_2 + \binom{n}{2} d_2 a_2 + \binom{n}{2} c_2 \binom{a_2}{2} + \binom{n}{2} b_2 \binom{a_2}{3} + \binom{n}{3} c_2 a_2^2 + \binom{n}{4} b_2 a_2^3 + 2\binom{n}{3} b_2 a_2 \binom{a_2}{2}.$$

Thus,

$$\begin{aligned}
&(x^{A_0} y^{B_0} z^{C_0} t^{D_0} u^{E_0})(x^{A_1} y^{B_1} z^{C_1} t^{D_1} u^{E_1})(x^{A_2} y^{B_2} z^{C_2} t^{D_2} u^{E_2}) = x^{A_3} y^{B_3} z^{C_3} t^{D_3} u^{E_3} \\
&= (x^{A_0} y^{B_0} z^{C_0} x^{A_1} y^{B_1} z^{C_1} t^{D_0+D_1} u^{E_0+E_1+D_0A_1})(x^{A_2} y^{B_2} z^{C_2} t^{D_2} u^{E_2}) \\
&= (x^{A_0} y^{B_0} x^{A_1} y^{B_1} z^{C_0+C_1} t^{D_0+D_1+C_0A_1} u^{E_0+E_1+D_0A_1+C_0\binom{A_1}{2}})(x^{A_2} y^{B_2} z^{C_2} t^{D_2} u^{E_2}) \\
&= (x^{A_0+A_1} y^{B_0+B_1} z^{C_0+C_1+B_0A_1} t^{D_0+D_1+C_0A_1+B_0\binom{A_1}{2}} \\
&u^{E_0+E_1+D_0A_1+C_0\binom{A_1}{2}+b_0\binom{A_1}{3}})(x^{A_2} y^{B_2} z^{C_2} t^{D_2} u^{E_2}) \\
&= x^{A_0+A_1} y^{B_0+B_1} z^{C_0+C_1+B_0A_1} x^{A_2} y^{B_2} z^{C_2} t^{D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2}} \\
&u^{E_0+E_1+E_2+D_0A_1+C_0\binom{A_1}{2}+B_0\binom{A_1}{3}+A_2(D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2})} \\
&= x^{A_0+A_1} y^{B_0+B_1} x^{A_2} y^{B_2} z^{C_0+C_1+C_2+B_0A_1} t^{D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2}+A_2(C_0+C_1+B_0A)} \\
&u^{E_0+E_1+E_2+D_0A_1+C_0\binom{A_1}{2}+B_0\binom{A_1}{3}+A_2(D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2})+(C_0+C_1+B_0A_1)\binom{A_2}{2}} \\
&= x^{A_0+A_1+A_2} y^{B_0+B_1+B_2} z^{C_0+C_1+C_2+B_0A_1+(B_0+B_1)A_2} t^{D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2}} \\
&t^{A_2(C_0+C_1+B_0A_1)+(B_0+B_1)\binom{A_2}{2}} u^{E_0+E_1+E_2+D_0A_1+C_0\binom{A_1}{2}+B_0\binom{A_1}{3}} \\
&u^{A_2(D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2})+(C_0+C_1+B_0A_1)\binom{A_2}{2}+(B_0+B_1)\binom{A_2}{3}}.
\end{aligned}$$

We can write

$$A_3 = A_0 + A_1 + A_2$$
$$B_3 = B_0 + B_1 + B_2$$

$$C_3 = C_0 + C_1 + C_2 + B_0A_1 + (B_0 + B_1)A_2$$

$$D_3 = D_0 + D_1 + D_2 + C_0A_1 + B_0\binom{A_1}{2} + A_2(C_0 + C_1 + B_0A_1) + \binom{A_2}{2}(B_0 + B_1)$$

$$E_3 = E_0 + E_1 + E_2 + D_0A_1 + C_0\binom{A_1}{2} + B_0\binom{A_1}{3} + A_2\left(D_0 + D_1 + C_0A_1 + B_0\binom{A_1}{2}\right)$$

$$+\binom{A_2}{2}(C_0 + C_1 + B_0A_1) + \binom{A_2}{3}(B_0 + B_1)$$

Thus, we can write

$$A_3 = la_0 + ma_1 + na_2$$
$$B_3 = lb_0 + mb_1 + nb_2$$

$$C_3 = lc_0 + \binom{l}{2}b_0a_0 + mc_1 + \binom{m}{2}b_1a_1 + nc_2 + \binom{n}{2}b_2a_2 + lmb_0a_1 + (lb_0 + mb_1)na_2$$

$$\begin{aligned} D_3 \quad = \quad & ld_0 + \binom{l}{2}c_0a_0 + \binom{l}{2}b_0\binom{a_0}{2} + \binom{l}{3}b_0a_0^2 + md_1 + \binom{m}{2}c_1a_1 \\ & + \binom{m}{2}b_1\binom{a_1}{2} + \binom{m}{3}b_1a_1^2 \ + nd_2 + \binom{n}{2}c_2a_2 + \binom{n}{2}b_2\binom{a_2}{2} + \binom{n}{3}b_2a_2^2 \\ & + (lc_0 + \binom{l}{2}b_0a_0)ma_1 + lb_0\binom{ma_1}{2} + na_2(lc_0 + \binom{l}{2}b_0a_0 \\ & + mc_1 + \binom{m}{2}b_1a_1 + lb_0ma_1) + \binom{na_2}{2}(lb_0 + mb_1) \end{aligned}$$

$$\begin{aligned}
E_3 \;=\;& le_0 + \binom{l}{2}d_0a_0 + \binom{l}{2}c_0\binom{a_0}{2} + \binom{l}{3}b_0\binom{a_0}{3} + \binom{l}{3}c_0a_0^2 + \binom{l}{4}b_0a_0^3 \\
&+2\binom{l}{3}b_0a_0\binom{a_0}{2} + me_1 + \binom{m}{2}d_1a_1 + \binom{m}{2}c_1\binom{a_1}{2} + \binom{m}{2}b_1\binom{a_1}{3} \\
&+\binom{m}{3}c_1a_1^2 + \binom{m}{4}b_1a_1^3 + 2\binom{m}{3}b_1a_1\binom{a_1}{2} + ne_2 + \binom{n}{2}d_2a_2 + \binom{n}{2}c_2\binom{a_2}{2} \\
&+\binom{n}{2}b_2\binom{a_2}{3} + \binom{n}{3}c_2a_2^2 + \binom{n}{4}b_2a_2^3 + 2\binom{n}{3}b_2a_2\binom{a_2}{2} \\
&+(ld_0 + \binom{l}{2}c_0a_0 + \binom{l}{2}b_0\binom{a_0}{2} + \binom{l}{3}b_0a_0^2)ma_1 \\
&+(lc_0 + \binom{l}{2}b_0a_0)\binom{ma_1}{2} + lb_0\binom{ma_1}{3} \\
&+na_2(ld_0 + \binom{l}{2}c_0a_0 + \binom{l}{2}b_0\binom{a_0}{2} + \binom{l}{3}b_0a_0^2 + md_1 + \binom{m}{2}c_1a_1 \\
&+\binom{m}{2}b_1\binom{a_1}{2} + \binom{m}{3}b_1a_1^2 + (lc_0 + \binom{l}{2}b_0a_0)ma_1 + lb_0\binom{ma_1}{2}) \\
&+\binom{na_2}{2}(lc_0 + \binom{l}{2}b_0a_0 + mc_1 + \binom{m}{2}b_1a_1 + lb_0ma_1) + \binom{na_2}{3}(lb_0 + mb_1)
\end{aligned}$$

We shall use vector notation to calculate the sequence and define a bi-infinite sequence $(r_i) = (a_i, b_i, c_i, d_i, e_i)$ via the 3-step general recurrence. We must consider two types of initial data for loops in G. We have a loop v of type I with initial data

$$\begin{aligned}
v_0 &= (0,\ 0,\ 0,\ 0,\ 0) \\
v_1 &= (1,\ 0,\ 0,\ 0,\ 0) \\
v_2 &= (0,\ 1,\ 0,\ 0,\ 0).
\end{aligned}$$

and another w of type II with initial data

$$\begin{aligned}
w_0 &= (1,\ 0,\ 0,\ 0,\ 0) \\
w_1 &= (0,\ 1,\ 0,\ 0,\ 0) \\
w_2 &= (0,\ 0,\ 0,\ 0,\ 0)
\end{aligned}$$

The analysis of the type II loop is entirely similar to that of type I. Thus the type I loop begins

$$v_0 = (t_0,\ s_0,\ 0,\ 0,\ 0)$$
$$v_1 = (t_1,\ s_1,\ 0,\ 0,\ 0)$$

and

$$v_2 = (t_2, s_2, 0, 0, 0)$$

so that $(a_i) = (t_i)$ and $(b_i) = (s_i)$.

It can easily be seen that the sequence t_i can be written in terms of s_i as $t_i = s_{i+1} - ns_i$. We shall need two formulas for C_α and D_α to work out the formula for E_α, $\alpha \geq 0$. Now, let the notation $\sum$denote the notation $\sum\limits_{i=0}^{\alpha-1}$. Then by induction,

$$C_\alpha = \binom{l}{2}\sum s_{\alpha-i-1}s_it_i + \binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}t_{i+1} + \binom{n}{2}\sum s_{\alpha-i-1}s_{i+2}t_{i+2}$$

$$+ lm\sum s_{\alpha-i-1}s_it_{i+1} + nl\sum s_{\alpha-i-1}s_it_{i+2} + mn\sum s_{\alpha-i-1}s_{i+1}t_{i+2}$$

and

$$\begin{aligned} D_\alpha \ =\ & \binom{l}{2}\sum s_{\alpha-i-1}c_it_i + \binom{l}{2}\sum s_{\alpha-i-1}s_i\binom{t_i}{2} + \binom{l}{3}\sum s_{\alpha-i-1}s_it_i^2 \\ & + \binom{m}{2}\sum s_{\alpha-i-1}c_{i+1}t_{i+1} + \binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}\binom{t_{i+1}}{2} + \binom{m}{3}\sum s_{\alpha-i-1}s_{i+1}t_{i+1}^2 \\ & + \binom{n}{2}\sum s_{\alpha-i-1}c_{i+2}t_{i+2} + \binom{n}{2}\sum s_{\alpha-i-1}s_{i+2}\binom{t_{i+2}}{2} + \binom{n}{3}\sum s_{\alpha-i-1}s_{i+2}t_{i+2}^2 \\ & + lm\sum s_{\alpha-i-1}c_it_{i+1} + m\binom{l}{2}\sum s_{\alpha-i-1}s_it_it_{i+1} + l\sum s_{\alpha-i-1}s_i\binom{mt_{i+1}}{2} \\ & + nl\sum s_{\alpha-i-1}c_it_{i+2} + n\binom{l}{2}\sum s_{\alpha-i-1}s_it_it_{i+2} + nm\sum s_{\alpha-i-1}c_{i+1}t_{i+2} \\ & + n\binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}t_{i+1}t_{i+2} + nlm\sum s_{\alpha-i-1}s_it_{i+1}t_{i+2} + l\sum s_{\alpha-i-1}s_i\binom{nt_{i+2}}{2} \\ & + m\sum s_{\alpha-i-1}s_{i+1}\binom{nt_{i+2}}{2} \end{aligned}$$

for $\alpha \geq 0$. These enable us, via a similar process, to describe E_α for $\alpha \geq 0$ as

$$
\begin{aligned}
E_\alpha = & \binom{l}{2}\sum s_{\alpha-i-1}d_it_i + \binom{l}{2}\sum s_{\alpha-i-1}c_i\binom{t_i}{2} + \binom{l}{2}\sum s_{\alpha-i-1}s_i\binom{t_i}{3} \\
& +\binom{l}{3}\sum s_{\alpha-i-1}c_it_i^2 + \binom{l}{4}\sum s_{\alpha-i-1}s_it_i^3 + 2\binom{l}{3}\sum s_{\alpha-i-1}s_it_i\binom{t_i}{2} \\
& +\binom{m}{2}\sum s_{\alpha-i-1}d_{i+1}t_{i+1} + \binom{m}{2}\sum s_{\alpha-i-1}c_{i+1}\binom{t_{i+1}}{2} \\
& +\binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}\binom{t_{i+1}}{3} + \binom{m}{3}\sum s_{\alpha-i-1}c_{i+1}t_{i+1}^2 \\
& +\binom{n}{2}\sum s_{\alpha-i-1}d_{i+2}t_{i+2} + \binom{n}{2}\sum s_{\alpha-i-1}c_{i+2}\binom{t_{i+2}}{2} \\
& +\binom{n}{2}\sum s_{\alpha-i-1}s_{i+2}\binom{t_{i+2}}{3} + \binom{n}{3}\sum s_{\alpha-i-1}c_{i+2}t_{i+2}^2 \\
& +\binom{n}{4}\sum s_{\alpha-i-1}s_{i+2}t_{i+2}^3 + m\binom{l}{2}\sum s_{\alpha-i-1}s_it_{i+1}\binom{t_i}{2} \\
& +m\binom{l}{2}\sum s_{k-i-1}s_it_{i+1}t_i^2 + l\sum s_{\alpha-i-1}c_i\binom{mt_{i+1}}{2} \\
& +\binom{l}{2}\sum s_{\alpha-i-1}s_it_i\binom{mt_{i+1}}{2} + l\sum s_{\alpha-i-1}s_i\binom{mt_{i+1}}{3} + nl\sum s_{\alpha-i-1}d_it_{i+2} \\
& +n\binom{l}{2}\sum s_{\alpha-i-1}c_it_it_{i+2} + n\binom{l}{2}\sum s_{\alpha-i-1}s_it_{i+2}\binom{t_i}{2} \\
& +n\binom{l}{3}\sum s_{\alpha-i-1}s_it_{i+2}t_i^2 + nm\sum s_{\alpha-i-1}d_{i+1}t_{i+2} \\
& +n\binom{m}{2}\sum s_{\alpha-i-1}c_{i+1}t_{i+1}t_{i+2} + n\binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}t_{i+2}\binom{t_{i+1}}{2} \\
& +n\binom{m}{3}\sum s_{\alpha-i-1}s_{i+1}t_{i+2}t_{i+1}^2 + nlm\sum s_{\alpha-i-1}c_it_{i+1}t_{i+2} \\
& +nm\binom{l}{2}\sum s_{\alpha-i-1}s_it_it_{i+1}t_{i+2} + nl\sum s_{\alpha-i-1}s_it_{i+2}\binom{mt_{i+1}}{2} \\
& +l\sum s_{\alpha-i-1}c_i\binom{nt_{i+2}}{2} + \frac{l}{2}\sum s_{\alpha-i-1}s_it_i\binom{nt_{i+1}}{2}
\end{aligned}
$$

$$+m\sum s_{\alpha-i-1}c_{i+1}\binom{nt_{i+2}}{2}+\binom{m}{2}\sum s_{\alpha-i-1}s_{i+1}t_{i+1}\binom{nt_{i+2}}{2}$$

$$+lm\sum s_{\alpha-i-1}s_i t_{i+1}\binom{nt_{i+2}}{2}+l\sum s_{\alpha-i-1}s_i\binom{nt_{i+2}}{3}+m\sum s_{\alpha-i-1}s_{i+1}\binom{nt_{i+2}}{3}.$$

In the table, $A(i)=T(i)$ and $B(i)=S(i)$.

Now we use these formulas to calculate the Fibonacci loop $l(x,y,z,\ldots)$. The first few terms are as follows:

(0, 0, 0, 0, 0)

(1, 0, 0, 0, 0)

(0, 1, 0, 0, 0)

(m, n, 0, 0, 0)

$(l+mn,\ m+n^2,\ \binom{n}{2}mn+nm^2,\ \binom{n}{2}n\binom{m}{2}+\binom{n}{3}nm^2+m\binom{n.m}{2},\ \binom{n}{2}nm+\binom{n}{4}nm^3+2\binom{n}{3}nm\binom{m}{2}+\binom{n.m}{3}m).$

The following table shows elements of the sequence for various values of l, m and n. The elements can be obtained by hand and computer program. Moreover, we have more data for various values of l, m and n. Computer codes and their results are available on a request.

Table

$l = 1$	A(3)	1	B(3)	1	C(3)	0	D(3)	0	E(3)	0
	A(4)	2	B(4)	2	C(4)	1	D(4)	0	E(4)	0
$m = 1$	A(5)	3	B(5)	4	C(5)	6	D(5)	4	E(5)	1
	A(6)	6	B(6)	7	C(6)	18	D(6)	23	E(6)	16
$n = 1$	A(7)	11	B(7)	13	C(7)	67	D(7)	16	E(7)	415
	A(8)	20	B(8)	24	C(8)	236	D(8)	1460	E(8)	6434
$l = 1$	A(3)	2	B(3)	2	C(3)	0	D(3)	0	E(3)	0
	A(4)	5	B(4)	6	C(4)	12	D(4)	14	E(4)	8
$m = 2$	A(5)	14	B(5)	17	C(5)	112	D(5)	462	E(5)	1310
	A(6)	40	B(6)	48	C(6)	928	D(6)	11616	E(6)	106474
$n = 2$	A(7)	113	B(7)	136	C(7)	7618	D(7)	281329	E(7)	7773408
	A(8)	320	B(8)	385	C(8)	61390	D(8)	6480888	E(8)	515371456
l=1	A(3)	2	B(3)	4	C(3)	0	D(3)	0	E(3)	0
	A(4)	9	B(4)	18	C(4)	64	D(4)	144	E(4)	208
m=2	A(5)	40	B(5)	81	C(5)	1564	D(5)	19874	E(5)	184202
	A(6)	180	B(6)	364	C(6)	32458	D(6)	1920918	E(6)	84789040
n=4	A(7)	809	B(7)	1636	C(7)	660424	D(7)	177539344	E(7)	1409806336
	A(8)	3636	B(8)	7353	C(8)	13361924	D(8)	-999957504	E(8)	-1920991232
l=2	A(3)	3	B(3)	5	C(3)	0	D(3)	0	E(3)	0
	A(4)	17	B(4)	28	C(4)	195	D(4)	915	E(4)	2990
m=3	A(5)	94	B(5)	157	C(5)	7243	D(5)	222964	E(5)	5090233
	A(6)	527	B(6)	879	C(6)	230513	D(6)	40249532	E(6)	967275520
n=5	A(7)	2951	B(7)	4922	C(7)	7256201	D(7)	-460688640	E(7)	-108789760
	A(8)	16524	B(8)	27561	C(8)	227678144	D(8)	-574029824	E(8)	-2013265920
l=2	A(3)	4	B(3)	1	C(3)	0	D(3)	0	E(3)	0
	A(4)	6	B(4)	5	C(4)	16	D(4)	24	E(4)	16
m=4	A(5)	22	B(5)	11	C(5)	108	D(5)	790	E(5)	4416
	A(6)	54	B(6)	33	C(6)	888	D(6)	14520	E(6)	176863
n=1	A(7)	154	B(7)	87	C(7)	6630	D(7)	33399	E(7)	13348748
	A(8)	414	B(8)	241	C(8)	49800	D(8)	6585600	E(8)	6879074592
l=4	A(3)	1	B(3)	2	C(3)	0	D(3)	0	E(3)	0
	A(4)	6	B(4)	5	C(4)	4	D(4)	1	E(4)	0
m=1	A(5)	13	B(5)	16	C(5)	114	D(5)	557	E(5)	1750
	A(6)	36	B(6)	45	C(6)	838	D(6)	9986	E(6)	84425
n=2	A(7)	109	B(7)	126	C(7)	6458	D(7)	217889	E(7)	5458466
	A(8)	306	B(8)	361	C(8)	55258	D(8)	5660503	E(8)	433183040

Conjecture

One can prove theorems similar to those in [4]. That is, there is a relation between the fundamental period (Wall number) of a 3-step general Fibonacci sequence and the length of a fundamental period of any loop satisfying the recurrence.

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