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Pushouts of Profinite Crossed Modules and cat^1-profinite groups*

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Abstract

In this paper, we presented a brief review of crossed modules [7], cat^1-groups [6], pullback crossed modules [4], pullback cat^1-group [1], profinite crossed modules [5], cat^1-profinite groups [5], pullback profinite crossed modules [5], pullback cat^1-profinite groups [3]. We defined the pushout cat^1-profinite groups and gave the left adjoint constructions.

Key Words: Crossed modules, Cat^1-groups, Profinite groups, Pushout, Pullback, Adjoint.

1. Introduction

Crossed module was introduced by J. H. C. Whitehead in [7]. In [6], Loday reformulated the notion of a crossed module as a cat^1-groups and showed that the category **XMod** is equivalent to the category **Cat1**.

In section 2, we recall the basic properties of crossed modules and their morphisms and cat^1-groups and their morphisms. Section 3 includes the definition of pullback crossed modules, which is defined by Brown and Higgins in [4], and the definition of pullback cat^1-groups is due to Alp in [1]. We introduced profinite crossed modules and cat^1-profinite groups which are defined by Korkes and Porter in [5] in Section 4. Section 5 includes pullback profinite crossed modules which is defined by Korkes and Porter in

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[5] and pullback cat^1-profinite groups which is defined by Alp in [3]. In Section 6, we gave the definitions of pushout profinite crossed modules which is defined by Korkes and Porter in [5] and pushout cat^1-profinite groups. In section 7, we presented left adjoint construction of pushout cat^1-profinite group and combined the pictures of pullback cat^1-profinite groups and pushout cat^1-profinite group together.

2. Crossed Modules and Cat1-Groups

In this section we recall the descriptions of two equivalent categories which are **XMod**, the category of crossed modules and their morphisms; **Cat1**, the category of cat^1-groups and their morphisms.

A crossed module $\mathcal{X} = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary* of $\mathcal{X}$, together with an action $\alpha : R \to \text{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\begin{array}{rrcl} \textbf{XMod 1:} & \partial(s^r) & = & r^{-1}(\partial s)r \\ \textbf{XMod 2:} & s^{\partial s'} & = & s'^{-1}ss'. \end{array}$$

The standard examples of crossed modules can be found in [1].

A morphism between two crossed modules $\mathcal{X}_1$ and $\mathcal{X}_2$ is a pair (σ, ρ), where $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$\partial_2\sigma = \rho\partial_1, \quad \sigma(s^r) = (\sigma s)^{\rho r}.$$

When $\mathcal{X}_2 = \mathcal{X}_1$ and σ, ρ are automorphisms then (σ, ρ) is an automorphism of $\mathcal{X}_1$. The group of automorphisms is denoted by $\text{Aut}(\mathcal{X}_1)$.

The notion of a crossed modules is reformulated as a cat^1-group by Loday in [6]. For computational purposes we find it convenient to define a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ as a group G with two surjections $t, h : G \to R$ and an embedding $e : R \to G$ satisfying:

$$\begin{array}{rl} \textbf{Cat 1:} & te = he = \text{id}_R, \\ \textbf{Cat 2:} & [\ker t, \ker h] = \{1_G\}. \end{array}$$

A morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of cat1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$h_2\gamma = \rho h_1, \quad t_2\gamma = \rho t_1, \quad e_2\rho = \gamma e_1$$

The crossed module $\mathcal{X}$ associated to $\mathcal{C}$ has $S = \ker t$ and $\partial = h \mid_S$. The cat^1-group associated to $\mathcal{X}$ has $G = R \ltimes S$, using the action from $\mathcal{X}$, and

$$t(r, s) = r,\ h(r, s) = r(\partial s),\ er = (r, 1).$$

3. Pullback Crossed modules and Pullback Cat^1-groups

Let $\mathcal{X} = (\partial : S \to R)$ be a crossed R-module and $\iota : Q \to R$ be a morphism of groups. Then $\iota^*\mathcal{X} = (\partial^\bullet : \iota^*S \to Q)$ is the pullback of $\mathcal{X}$ by ι, where $\iota^*S = \{(q, s) \in Q \times S \times Q \mid \iota q = \partial s\}$ and $\partial^\bullet(q, s) = q$. The action of Q on $\iota^{**}S$ is given by

$$(q_1, s)^q \quad = \quad (q^{-1}q_1q, s^{\iota q}). \tag{3.1}$$

The verification of the crossed module axioms is given in [4].

A pullback cat^1-group is defined by Alp in [1] as follows.

$$\begin{tikzcd}
\iota^{**}G \arrow[r, "h^{**}"] \arrow[d, "t^{**}"] \arrow[dr, "\pi"] & Q \arrow[l, bend right, "e^{**}"'] \arrow[dr, "\iota"] & \\
Q \arrow[u, bend left, "e^{**}"] \arrow[dr, "\iota"] & G \arrow[r, "h"] \arrow[d, "t"] & R \arrow[l, bend right, "e"'] \\
 & R \arrow[u, bend left, "e"] &
\end{tikzcd}$$

Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat^1-group and let $\iota : Q \to R$ be a group homomorphism. Define $\iota^{**}\mathcal{C} = (e^{**}; t^{**}, h^{**} : \iota^{**}G \to Q)$ to be the pullback of G where

$$\iota^{**}G = \{(q_1, g, q_2) \in Q \times G \times Q \mid \iota q_1 = tg, \iota q_2 = hg\},$$

$t^{**}(q_1, g, q_2) = q_1$, $h^{**}(q_1, g, q_2) = q_2$ and $e^{**}(q) = (q, e\iota q, q)$. Multiplication in $\iota^{**}G$ is componentwise. The pair (π, ι) is a morphism of cat^1-groups where $\pi : \iota^{**}G \to G,\ (q_1, g, q_2) \mapsto g$.

A verification of the cat^1-group axioms is given in [1].

4. Profinite crossed modules and cat^1-profinite groups

A profinite crossed module [5] $\mathcal{PX} = (\partial : S \to R)$ is a crossed module in which S and R are profinite groups, S acts continuously on R and ∂ is a continuous group homomorphism.

Examples of profinite crossed module were given in [5].

Proposition 4.1 Let $\partial : A \to G$ and $\delta : B \to G$ be two profinite crossed modules and let $(\phi, Id) : (\partial : A \to G) \to (\delta : B \to G)$ be a morphism of profinite crossed modules. Then by defining a continuous $B-$ action on A by ${}^{b}a = {}^{\delta(b)} a$ we have $\phi : A \to B$ is a profinite crossed module [5].

Proof. We can show two crossed modules axioms as follows:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\partial"'] & B \arrow[d, "\delta"] \\ G \arrow[r, "\mathrm{id}"'] & G \end{tikzcd}$$

where $\partial = \delta\phi$ and $\phi({}^{g}a) = {}^{g}\phi(a)$. We can verify the axioms of crossed modules as follows:

XMod1:

$$\begin{aligned} \phi({}^{b}a) &= \phi({}^{\delta b}a) \\ &= {}^{\delta b}(\phi a) \\ &= b\phi(a)b^{-1} \end{aligned}$$

XMod2:

$$\begin{aligned} {}^{\phi a_2}a_1 &= {}^{\delta(\phi a_2)}a_1 \\ &= {}^{\delta\phi(a_2)}(a_1) \\ &= {}^{\partial a_2}a_1 \\ &= a_2a_1a_2^{-1} \end{aligned}$$

□

If $\mathcal{PX} = (\partial : S \to R)$ and $\mathcal{PX}' = (\partial' : S' \to R')$ are profinite crossed modules and $(\mu, \eta) : (\partial : S \to R) \to (\partial' : S' \to R')$ is a morphism between them in which the pair (μ, η) are both continuous then the pair (μ, η) is called a morphism of profinite crossed modules [5].

A cat^1-profinite group [5] is a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ in which G is a profinite group and t and h are continuous endomorphisms of G.

A morphism of cat^1-profinite groups is a morphism $\phi : \mathcal{C} = (e; t, h : G \to R) \to \mathcal{C}' = (e'; t', h' : G' \to R')$ of the underlying cat^1-groups such that ϕ is a continuous morphism of profinite groups.

5. Pullbacks of Profinite Crossed modules and cat^1-profinite groups

Let $\mathcal{PX} = (\partial : S \to R)$ be a profinite crossed module and $\iota : Q \to R$ be a continuous homomorphism of profinite groups. Then $\iota^{**}\mathcal{X} = (\partial^{**} : \iota^{**}S \to Q)$ is the pullback of $\mathcal{PX}$ by ι. So that $\iota^{**}, S \subset Q \times S$ is a closed subgroup given by

$$\iota^{**}S = \{(q, s) \in Q \times S \mid \iota q = \partial s\}$$

and Q acts continuously on the right of $\iota^{**}S$ by

$$(q_1, s)^q = (q^{-1}q_1 q, s^{\iota q}),$$

since $\partial^{**}(q_1, s) = q_1$. The verification of crossed module axioms can be found in [5].

A pullback cat^1-profinite group is defined by Alp in [3] as follows. Let $\mathcal{PC} = (e; t, h : G \to R)$ be a cat^1-profinite group and $\iota : Q \to R$ be a continuous homomorphism. Then $e^{**}; t^{**}, h^{**} : \iota^{**}G \to Q$ is a pullback of G where $\iota^{**}G \subset Q \times G \times Q$ and

$$\iota^{**}G = \{(q_1, g, q_2) \in Q \times G \times Q \mid \iota q_1 = tg,\ \iota q_2 = hg\}.$$

Now we can define tail, head and emmbedding as follows:

$$\begin{aligned} t^{**}(q_1, g, q_2) &= q_1 \\ h^{**}(q_1, g, q_2) &= q_2 \\ e^{**}(q) &= (q, e\iota q, q). \end{aligned}$$

The verification of cat^1-group axioms can be found in [3].

6. Pushouts of Profinite Crossed Modules and Cat^1-profinite groups

Let $PX = (\partial : S \to R)$ be a profinite crossed module over R and let $\phi : R \to H$ be a continuous homomorphism of profinite groups. Consider profinite group $\phi_*(S)$ topologically generated by the profinite space $S \times H$ with relations

1. $(s_1, h)(s_2, h) = (s_1 s_2, h)$
2. $({}^r s, h) = (s, h\phi(r))$
3. $(s_1, h_1)(s_2, h_2)(s_1, h_1)^{-1} = (s_2, h_1(\phi\partial s_1)h_1^{-1}h_2)$

for all $h, h_1, h_2 \in H, s, s_1, s_2 \in S$ and $r \in R$.

Define a continuous homomorphism $\delta : \phi_*(S) \to H$ by extending $\delta(s, h) = h(\phi\partial s)h^{-1}$ to the whole of $\phi_*(S)$ and define a continuous $H-$ action on the left of $\phi_*(S)$ by ${}^h(s, h_1) = (s, hh_1)$ for $h, h_1 \in H, s \in S$ and a continuous homomorphism $\psi : S \to \phi_*(S)$ by $\psi(s) = (s, 1)$ [5].

Proposition 6.1 [5] With the notation above, $\delta : \phi_*(S) \to H$ is a profinite crossed module over H.

Proof. The statement about continuity are fairly trivial and the axioms of crossed module were checked in [5] as follows.

XMod1:

$$\begin{aligned} \delta({}^h(s, h_1)) &= \delta(s, hh_1) \\ &= hh_1(\phi\partial s)(hh_1)^{-1} \\ &= h(\delta(s, h_1))h^{-1} \end{aligned}$$

XMod2:

$$\begin{aligned} {}^{\phi a_2}a_1 &= {}^{\delta(\phi a_2)}a_1 \\ &= {}^{\delta\phi(a_2)}(a_1) \\ &= {}^{\partial a_2}a_1 \\ &= a_2 a_1 a_2^{-1} \end{aligned}$$

□

A pushout cat^1- profinite group is defined as

$$\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "t"'] \arrow[dr, "\psi"] & R \arrow[l, bend right, "e"'] \arrow[dr, "\phi"] & \\
R \arrow[u, bend left, "e"] \arrow[dr, "\phi"'] & \phi_{**}G \arrow[r, "h_{**}"] \arrow[d, "t_{**}"'] & H \arrow[l, bend right, "e_{**}"'] \\
& H \arrow[u, bend left] &
\end{tikzcd}$$

Let $C = (e, t, h : \ G \to R)$ be a cat^1- group and let $\phi : R \to H$ be a continuos group homomorphism Define $e_{**},\ t_{**},\ h_{**} : \phi_{**}G \to H$ to the pushout of G where

$$\phi_{**} = \{(h_1, s, h_2) \in H \times S \times H\}$$

$$\begin{aligned}
t_{**}(h_1, s, h_2) &= h_1 \\
h_{**}(h_1, s, h_2) &= h_2 \\
e_{**}(h) &= (h, e\phi^{-1}h, h).
\end{aligned}$$

The pair (ψ, ϕ) is a pair morphism of cat^1-profinite groups where
$\psi : G \to \phi_{**}G,\ s \mapsto (h_1, s, h_2)$. We can show that t_{**} and h_{**} are homomorphisms.

$$\begin{aligned}
t_{**}\{(h_1, s_1, h_2)(h_3, s_2, h_4)\} &= t_{**}(h_1h_3, s_1s_2, h_2h_4) \\
&= h_1h_3 \\
&= t_{**}(h_1, s_1, h_2)t_{**}(h_3, s_2, h_4).
\end{aligned}$$

Now we can give the verification of cat^1-group axioms as follows:
CAT1:

$$\begin{aligned}
h_{**}e_{**}(h) &= h_{**}(h_1, e\phi^{-1}h, h) = h \\
t_{**}e_{**}(h) &= t_{**}(h_1, e\phi^{-1}h, h) = h.
\end{aligned}$$

So, $t_{**}e_{**} = h_{**}e_{**} = \mathrm{id_H}$ then CAT1 is satisfied.

CAT2:

Suppose $a = (h_1', s_1, h_1) \in \ker t_{**}$, $b = (h_2, s_2, h_2') \in \ker h_{**}$. Then $h_1' = h_2' = 1$ so, by the definition of ϕ_{**}, we have $s_1 \in \ker t$, $s_2 \in \ker h$. Then $[a, b] = (1_H, [s_1, s_2], 1_H) = (1_H, 1_S, 1_H)$.

7. Construction of the left adjoint

Proposition 7.1 The category of cat^1-profinite groups is cocomplete.

Proposition 7.2 The functor $\phi_{**} : Cat^1ProGrp/U \to Cat^1ProGrp/R$ has a left adjoint $\phi_{**} : Cat^1ProGrp/R \to Cat^1ProGrp/U$.

The proofs of above propositions are clear since left adjoint construction of pullback cat^1-groups was given in [2].

Combining pictures together we get the following diagram.

$$
\begin{tikzcd}
\phi^{**}G \arrow[r, "h^{**}"] \arrow[d, "t^{**}"'] \arrow[dr, "\pi"] & Q \arrow[l, bend right, "e^{**}"'] \arrow[dr, "\iota"] & & \\
Q \arrow[u, bend left, "e^{**}"] \arrow[dr, "\iota"'] & G \arrow[r, "h"] \arrow[d, "t"'] \arrow[dr, "\psi"] & R \arrow[l, bend right, "e"'] \arrow[dr, "\phi"] & \\
 & R \arrow[u, bend left] \arrow[dr, "\phi"'] & \phi_{**}(G) \arrow[r, "h_{**}"] \arrow[d, "t_{**}"'] & H \arrow[l, bend right, "e_{**}"'] \\
 & & H \arrow[u, bend left] &
\end{tikzcd}
$$

$$\begin{aligned} \phi\iota t^{**} &= t_{**}\psi\pi \\ h_{**}\psi\pi &= \phi\iota h^{**}. \end{aligned}$$

Since

$$\begin{aligned} \phi\iota t^{**}(q_1, g, q_2) &= \phi\iota(q_1) \\ &= (\phi t)(g) \\ &= (t^{**}\psi)(g) \\ &= t^{**}(\psi g) \\ &= t^{**}(q_1, g, q_2) \\ &= q_1 \\ t_{**}\psi\pi(q_1, g, q_2) &= t_{**}\psi(g) \\ &= t_{**}(q_1, g, q_2) \\ &= q_1 \end{aligned}$$

diagram is commutative.

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