Turk J Math
27 (2003) , 565 – 579.

Rough Oscillatory Singular Integral Operators-II

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Abstract

In this paper, we study certain classes of oscillatory singular integral operators with kernels in $L\log L(\mathbf{S}^{n-1})$ which is known to be the most desirable size condition for the L^p boundedness to hold. We prove that such operators are bounded on L^p. Our results extend and improve previously known results. Variations of our approach in this paper can be applied to handle more general oscillatory singular integral operators. This concludes by indicating a variety of results that can be obtained.

Key Words: Oscillatory singular integral operators, Rough kernels, L^p estimates, Hardy Littlewood maximal function.

1. Introduction and Statement of Results

Let $n \geq 2$ and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\mathbf{N}^0$ denote the set of all nonnegative integers. Suppose that $\Omega \in L^1\left(\mathbf{S}^{n-1}\right)$ is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(x)\ d\sigma(x) = 0. \tag{1.1}$$

For a real valued polynomial mapping $\mathcal{P}$ on $\mathbf{R}^n \times \mathbf{R}^n$, consider the oscillatory singular integral operator

$$\mathbf{T}_{\mathcal{P},\Omega} f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\mathcal{P}(x,y)} \left|x-y\right|^{-n} \Omega(x-y) f(y) dy. \tag{1.2}$$

2000 *Mathematics Subject Classification:* Primary 42B20; Secondary 42B15, 42B25.

When $\mathcal{P} = 0$, $\mathbf{T}_{\mathcal{P},\Omega}$ is the classical Calderón-Zygmund singular integral operator which is known to be bounded on L^p for all $1 < p < \infty$, provided that Ω is in the Hardy space $H^1(\mathbf{S}^{n-1})$. When $\mathbf{T}_{\mathcal{P},\Omega}$ is of convolution type, i.e., $\mathcal{P}(x,y) = P(x-y)$ for some real valued polynomial mapping P on $\mathbf{R}^n$, the L^p boundedness properties of $\mathbf{T}_{\mathcal{P},\Omega}$ are well understood (for more information, see [1], [6], [10]). In [9], Ricci-Stein proved that $\mathbf{T}_{\mathcal{P},\Omega}$ is bounded on L^p for all $1 < p < \infty$, provided that Ω is in $\mathcal{C}^1(\mathbf{S}^{n-1})$. Later, Lu-Zhang [8] showed that the L^p boundedness of $\mathbf{T}_{\mathcal{P},\Omega}$ still holds if the condition $\Omega \in \mathcal{C}^1(\mathbf{S}^{n-1})$ is replaced by the weaker condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Subsequently, the condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ was very much relaxed by Jiang and Lu in ([7]). In fact, they proved that $\mathbf{T}_{\mathcal{P},\Omega}$ is bounded on L^p, $1 < p < \infty$ provided that $\Omega \in L\log L(\mathbf{S}^{n-1})$, where $L\log L(\mathbf{S}^{n-1})$ is the space of all $L^1\left(\mathbf{S}^{n-1}\right)$ functions Ω that satisfies

$$\int_{\mathbf{S}^{n-1}} \left|\Omega(y^{'})\right| \log^{+}(\left|\Omega(y^{'})\right|) d\sigma(y^{'}) < \infty.$$

It is worth pointing out that $L\log L(\mathbf{S}^{n-1})$ properly contains the space $L^q(\mathbf{S}^{n-1})$ (for any $q > 1$). Moreover, it is known that the condition $\Omega \in L\log L(\mathbf{S}^{n-1})$ is the most desirable size condition for the L^p boundedness of $\mathbf{T}_{0,\Omega}$ to hold ([4]). In fact, Calderón-Zygmund ([4]) showed that the L^p boundedness of $\mathbf{T}_{0,\Omega}$ for any $1 < p < \infty$ may fail if the condition $\Omega \in L\log L(\mathbf{S}^{n-1})$ is replaced by $\Omega \in L(\log L)^{1-\varepsilon}(\mathbf{S}^{n-1})$ for some $\varepsilon > 0$.

In this paper, we study a more general class of oscillatory singular integral operators. More specifically, we investigate the L^p boundedness of the class of operators $\mathbf{T}_{\Phi,\Omega}$ for phase functions Φ of the form

$$\Phi(x,y) = \sum_{j=0}^{l} P_j\ (x)\phi_j(y-x),$$

where, $\phi_j : \mathbf{R}^n \to \mathbf{R}$ is a homogenous function which is real analytic on $\mathbf{S}^{n-1}$ and P_j is a real valued polynomial on $\mathbf{R}^n$. It is clear that the class of such functions Φ contains properly the class of all real valued polynomial mappings $\mathcal{P}$ on $\mathbf{R}^n \times \mathbf{R}^n$ investigated by Jiang and Lu ([7]). This naturaly leads to the following question:

Question. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1) and that $\Omega \in L\log L(\mathbf{S}^{n-1})$. Is the corresponding operator $\mathbf{T}_{\Phi,\Omega}$ bounded on $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$?*

In this paper, we shall answer this question in the affirmative under some certain conditions. In fact, we shall present a systematic approach for dealing with oscillatory singular integral operators when their kernels Ω belong to $L\log L(\mathbf{S}^{n-1})$. The key idea of our approach is determining the dependence of the L^p estimates of the operators under consideration on the size of Ω. In order to apply this idea, we pave the way by a sequence of lemmas in Section 2. It is worth pointing out that, a great deal more can be obtained by applying variations of this approach to more general oscillatory singular integral operators.

Our main results in this paper are the following:

Theorem 1.1. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1), and $\Omega \in L^\infty(\mathbf{S}^{n-1})$ with $\|\Omega\|_{L^1} \leq 1$ and $\|\Omega\|_{L^\infty} \leq 2^A$ for some $A > 1$. Suppose also that $\{d_j, m_j : 0 \leq j \leq l\} \subset \mathbf{N}^0$* and *that $\Phi(x,y) = \sum_{j=0}^{l} P_j\ (x)\phi_j(y-x)$, where $\phi_j : \mathbf{R}^n \to \mathbf{R}$ is a homogenous function of degree m_j which is real analytic on $\mathbf{S}^{n-1}$, and $P_j(x)$ is a real valued polynomial on $\mathbf{R}^n$ with degree d_j. If ϕ_j is a constant function whenever $m_j = 0$, then*

$$\|\mathbf{T}_{\Phi,\Omega}(f)\|_p \leq C_p A \|f\|_p \tag{1.3}$$

for all $1 < p < \infty$ with constant C_p independent of A and the coefficients of the polynomials $\{P_j : 0 \leq j \leq l\}$.

As a consequence of Theorem 1.1 and certain decomposition of the function Ω, we obtain the following result:

Theorem 1.2. *Suppose that $\Omega \in L\log L(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1). Suppose also that $\{d_j, m_j : 0 \leq j \leq l\} \subset \mathbf{N}^0$ and that $\Phi(x,y) = \sum_{j=0}^{l} P_j\ (x)\phi_j(y-x)$, where , $\phi_j : \mathbf{R}^n \to \mathbf{R}$ is a homogenous function of degree m_j which is real analytic on $\mathbf{S}^{n-1}$, and $P_j(x)$ is a real valued polynomial on $\mathbf{R}^n$ with degree d_j. If ϕ_j is a constant function whenever $m_j = 0$, then*

$$\|\mathbf{T}_{\Phi,\Omega}(f)\|_p \leq C_p \|f\|_p \tag{1.4}$$

for all $1 < p < \infty$ with constant C_p independent of the coefficients of the polynomials $\{P_j : 0 \leq j \leq l\}$.

Clearly, Theorem 1.2 is a proper extension of the result in ([7]).

Throughout this paper the letter C will denote a constant that may vary at each occurrence, but it is independent of the essential variables.

Finally, the authors would like to thank the refree for his/her valuable remarks.

2. Some Lemmas

Lemma 2.1 ([5]). For $j \in \{1,2\}$ let U_j be a domain in $\mathbf{R}^{n_j}$ and K_j a compact subset of U_j. Let $h(\cdot,\cdot)$ be real analytic function on $U_1 \times U_2$ such that $h(\cdot, z)$ is a nonzero function for every $z \in U_2$. Then there exists a positive constant $\delta = \delta(h, K_1, K_2)$ such that

$$\sup_{z\in K_2} \int_{K_1} |h(w,z|^{-\delta}\, dw < \infty.$$

Lemma 2.2 (van der Corput [12]). *Suppose ϕ and ψ are real-valued and smooth in (a, b), and that $\left|\phi^{(k)}(t)\right| \geq 1$ for all $t \in (a,\, b)$. Then the inequality*

$$\left|\int_a^b e^{-i\lambda\phi(t)}\psi(t)dt\right| \leq C_k \left|\lambda\right|^{-\frac{1}{k}} [|\psi(b)| + \int_a^b \left|\psi^{'}(t)\right| dt],$$

holds when:

(i) $k \geq 2$, or

(ii) $k = 1$ and $\phi^{'}$ is monotonic.

The bound C_k is independent of a, b, ϕ, and λ.

For a real valued function Φ on $\mathbf{R}^n \times \mathbf{R}^n$ and a homogeneous function Ω of degree zero on $\mathbf{R}^n$, define the operator

$$\mathbf{T}^0_{\Phi,\Omega} f(x) = \int_{|x-y|<1} e^{i\Phi(x,y)} \left|x-y\right|^{-n} \Omega(x-y) f(y)dy. \tag{2.1}$$

Then we have the following result:

Lemma 2.3. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ and $\Omega \in L^1(\mathbf{S}^{n-1})$. Suppose also that Φ and ϕ are real valued functions on $\mathbf{R}^n \times \mathbf{R}^n$ with the following properties:*

(**i**) *For $h \in \mathbf{R}^n$ there exists a real valued function Ψ_h on $\mathbf{R}^n \times \mathbf{R}^n$ such* *that* $\Phi(x,y)=\phi(x-h,y-h)+\Psi_h(x,y)$;

(**ii**) *There exist* $\alpha\geq 0$ *and* $\beta>0$ *such that for all* z, $x,y\in\mathbf{R}^n$, *we have*

$$|\phi(x-z,y-z)|\leq B\,|x-z|^{\alpha}\,|x-y|^{\beta}\,;$$

where B *is a constant independent of* z, x, *and* y.

(**iii**) $\left\|\mathbf{T}^0_{\Psi_h,\Omega}f\right\|_p\leq C_pA\,\|f\|_p$ *with constants* C_p *and* A *independent of* h.

Then

$$\left\|\mathbf{T}^0_{\Phi,\Omega}f\right\|_p\leq (B\,\|\Omega\|_{L^1}+A)C_p\,\|f\|_p\,, \tag{2.2}$$

where C_p *is independent of* A *and* Ω.

Proof. Given $h\in\mathbf{R}^n$. Then

$$\mathbf{T}^0_{\Phi,\Omega}=\mathbf{T}^0_{\Psi_h,\Omega}+(\mathbf{T}^0_{\Phi,\Omega}-\mathbf{T}^0_{\Psi_h,\Omega}). \tag{2.3}$$

Now, by conditions (i) and (ii), whenever $|x-h|<\frac{1}{4}$, we have

$$\begin{aligned}
&\left|\mathbf{T}^0_{\Phi,\Omega}f(x)-\mathbf{T}^0_{\Psi_h,\Omega}f(x)\right|\\
\leq\;& \int_{|x-y|<1}\left|e^{i(\phi(x-h,y-h)+\Psi_h(x,y))}-e^{i\Psi_h(x,y)}\right|\,|x-y|^{-n}\,|\Omega(x-y)|\,|f(y)|\,dy\\
\leq\;& \int_{|x-y|<1}|\phi(x-h,y-h)|\,|x-y|^{-n}\,|\Omega(x-y)|\,|f(y)|\,dy\\
\leq\;& B\int_{|x-y|<1}|x-h|^{\alpha}\,|x-y|^{-n+\beta}\,|\Omega(x-y)|\,|f(y)|\,dy\\
\leq\;& B\int_{|x-y|<1}|x-y|^{-n+\beta}\,|\Omega(x-y)|\,|f(y)|\,dy.
\end{aligned}$$

Therefore, by Minkowski's inequality, we obtain

$$\begin{aligned}
&\int_{|x-h|<\frac{1}{4}}\left|\mathbf{T}^0_{\Phi,\Omega}f(x)-\mathbf{T}^0_{\Psi_h,\Omega}f(x)\right|^p dx\\
\leq\;& B(\int_{|z|<1}|z|^{-n+\beta}\,|\Omega(z)|\,dz)^p\int_{|y-h|<\frac{5}{4}}|f(y)|^p\,dy\\
\leq\;& \frac{B}{\beta}\,\|\Omega\|^p_{L^1}\int_{|y-h|<\frac{5}{4}}|f(y)|^p\,dy.
\end{aligned} \tag{2.4}$$

Thus, by (2.3), (2.4), and condition (iii), we have

$$\int_{|x-h|<\frac{1}{4}} \left|\mathbf{T}^0_{\Phi,\Omega} f(x)\right|^p dx \leq C_p^p A^p \|f\|_p^p + \frac{B}{\beta} \|\Omega\|_{L^1}^p \int_{|y-h|<\frac{5}{4}} |f(y)|^p \, dy. \tag{2.5}$$

Hence, since $h \in \mathbf{R}^n$ is arbitrary, (2.5) implies (2.2). This completes the proof. □

Lemma 2.4. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1). Suppose also that $\Omega \in L^1(\mathbf{S}^{n-1}) \cap L^\infty(\mathbf{S}^{n-1})$ with $\|\Omega\|_{L^1} \leq 1$ and $\|\Omega\|_{L^\infty} \leq 2^A$ for some $A > 1$. Then the singular integral operator*

$$\mathbf{T}_{I,\Omega} f(x) = \text{p.v.} \int_{\mathbf{R}^n} |x-y|^{-n} \, \Omega(x-y) f(y) dy \tag{2.6}$$

satisfies

$$\|\mathbf{T}_{I,\Omega} f\|_p \leq C_p A \|f\|_p \tag{2.7}$$

for all $1 < p < \infty$. The constant C_p is independent of A.

Proof. The proof of this lemma is based on an argument developed in ([1]). Let $\{\sigma_j : j \in \mathbf{Z}\}$ be a sequence of measures defined in the Fourier transform side by

$$\hat{\sigma}_j(\xi) = \int_{2^j \leq |y| < 2^{j+1}} e^{-i\xi\cdot y} \, |y|^{-n} \, \Omega(y) dy, \; j \in \mathbf{Z}.$$

Then

$$\mathbf{T}_{I,\Omega} f(x) = \sum_{j\in\mathbf{Z}} \sigma_j * f(x). \tag{2.8}$$

Now, by a standard argument (see [1]), we can show that

$$|\hat{\sigma}_j(\xi)| \leq C \max\{\left|2^j \xi\right|, 2^A \left|2^j \xi\right|^{-\epsilon}\} \tag{2.9}$$

for some $\epsilon > 0$ with constant C independent of A, j and ξ. Therefore, by (2.9) and the trivial bound $|\hat{\sigma}_j(\xi)| \leq 1$, we get

$$|\hat{\sigma}_j(\xi)| \leq C \left|2^j \xi\right|^{\pm \frac{\epsilon}{A}}. \tag{2.10}$$

Let M be the maximal function

$$Mf(x) = \sup_{j\in\mathbf{Z}} \int_{2^j \leq |y| < 2^{j+1}} |y|^{-n} |\Omega(y)| |f(x-y)| dy.$$

Then by a theorem on page 477 in ([12]) and the assumption that $\|\Omega\|_{L^1} \leq 1$, we have

$$\|Mf\|_p \leq C\,\|f\|_p \tag{2.11}$$

for all $1 < p \leq \infty$ with constant C independent of A. Hence (2.7) follows by (2.8), (2.10), (2.11), and Theorem 2.1 in ([1]). This completes the proof.

By a careful inspection of the proof of Lemma 1 in ([8]), we have the following version of Lemma 1 in ([8]):

Lemma 2.5. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ and $\Omega \in L^1(\mathbf{S}^{n-1})$. If*

$$\mathbf{T}f(x) = \text{p.v.}\int_{\mathbf{R}^n} K(x,y)f(y)dy \tag{2.12}$$

is a (L^p, L^p) type operator with $1 < p < \infty$, and $K(x,y)$ satisfies

$$|K(x,y)| \leq \frac{|\Omega[(x-y)']|}{|x-y|^n}, \tag{2.13}$$

then the operators

$$\mathbf{T}_\varepsilon f(x) = \int_{|x-y|<\varepsilon} K(x,y)f(y)dy$$

are (L^p, L^p) type operators, and $\|\mathbf{T}_\varepsilon\| \leq C(\|\mathbf{T}\| + \|\Omega\|_{L^1})$, where C is independent of $\mathbf{T}$ and ε.

It should be pointed out that Lemma 2.5 was proved in ([8], Lemma 1) under the assumption that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq \infty$. Moreover, the dependence of $\|\mathbf{T}_\varepsilon\|$ on the function Ω was not explicitly stated, partly because such information was not needed for the treatment used there. It is worth noticing that all hypotheses imposed in Lemma 2.5 above can be satisfied under the condition $\Omega \in L^1(\mathbf{S}^{n-1})$. This can be easily seen by taking $\Omega \in L^1(\mathbf{S}^{n-1})$ to be an odd homogeneous function of degree zero on $\mathbf{R}^n$(see [4]).

Finally, we end this section by recalling the following lemma in ([2]) which will be a key step in proving Theorem 1.3.

Lemma 2.6 ([2]). *Suppose that* $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ *that satisfies (1.1). Then there exist a subset* $\mathbf{D}$ *of* $\mathbf{N}$*, a sequence* $\{\lambda_m : m \in \mathbf{N}\}$ *of non negative real numbers, and a sequence of functions* $\{\Omega_m : m \in \mathbf{D} \cup \{0\}\}$ *in* $L^1(\mathbf{S}^{n-1})$ *such that*

(i) $\int_{\mathbf{S}^{n-1}} \Omega_m d\sigma = 0$, *for* $m \in \mathbf{D} \cup \{0\}$;

(ii) $\|\Omega_m\|_\infty \leq 2^{4(m+2)}$, *and* $\|\Omega_m\|_{L^1(\mathbf{S}^{n-1})} \leq 2$, *for* $m \in \mathbf{D} \cup \{0\}$;

(iii) $\sum_{m \in \mathbf{D}} (m+2)\lambda_m < \infty$;

(iv) $\Omega = \sum_{m \in \mathbf{D} \cup \{0\}} \lambda_m \Omega_m$.

3. Proofs of Main Results

We start this section by presenting a proof of Theorem 1.1.

Proof(of Theorem 1.1). We shall use induction on

$$d(\Phi) = \inf \max_{0 \leq j \leq m} \{d_j + m_j\},$$

where the infimum is taken over all representations of Φ of the form $\Phi(x, y) = \sum_{j=0}^{m} P_j(x)\phi_j(y - x)$ with d_j is the degree of P_j and m_j is the degree of homogeneity of ϕ_j.

It is clear that if $d(\Phi) = 0$, then $|\mathbf{T}_{\Phi,\Omega}(f)(x)| = |\mathbf{T}_{I,\Omega} f(x)|$, where $\mathbf{T}_{I,\Omega}$ is the operator in (2.6). Therefore, (1.3) holds by Lemma 2.4 and Lemma 2.5.

Now assume that (1.3) holds for all Φ with $d(\Phi) \leq d$ and given $\Phi(x, y) = \sum_{j=0}^{l} P_j(x)\phi_j(y - x)$ with $d(\Phi) = d + 1$. Let $j_1, j_2, \ldots, j_k$ be all $0 \leq j \leq l$ with $d_j + m_j = d + 1$. For $1 \leq s \leq k$, let $h_s(x) = \sum_{|\alpha_{j_s}| = d_{j_s}} a_{\alpha_{j_s}} x^{\alpha_{j_s}}$ and $H(x, y) = \sum_{s=1}^{k} h_s(x)\phi_{j_s}(y - x)$. It is straightforward to see that H can be written as

$$H(x, y) = \sum_{m=1}^{M} \lambda_m \phi_m(x, y), \tag{3.1}$$

for some integer $M > 0$, constants $\{\lambda_m : 1 \leq m \leq M\}$ with

$$\sum_{m=1}^{M} |\lambda_m| = \sum_{s=1}^{k} \sum_{|\alpha_{j_s}| = d_{j_s}} \left| a_{\alpha_{j_s}} \right|, \tag{3.2}$$

and functions ϕ_m, $1 \leq m \leq M$ of the form $x^{\alpha}\theta(y-x)$ for some multi-index α and a homogenous function θ of degree $d+1-|\alpha|$ which is real analytic on $\mathbf{S}^{n-1}$. Then

$$\Phi(x,y) = \sum_{m=1}^{M} \lambda_m \phi_m(x,y) + \sum_{0\leq j\leq l, d_j+m_j\leq d} P_j\ (x)\phi_j(y-x). \tag{3.3}$$

Now set

$$\begin{aligned}\delta &= \left(\sum\nolimits_{m=1}^{M} |\lambda_m|\right)^{\frac{1}{d+1}};\\ \Phi_{\delta}(x,y) &= \sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)}\phi_m(x,y) + \sum_{0\leq j\leq l, d_j+m_j\leq d} P_j\ (\delta^{-1}x)\phi_j(\delta^{-1}(y-x));\end{aligned}$$

and

$$f_{\delta}(x) = f(\delta^{-1}x).$$

Thus, it is easy to see that the following hold:

$$\Phi(x,y) = \Phi_{\delta}(\delta x, \delta y), \tag{3.4}$$

$$\sum\nolimits_{m=1}^{M} \left|\lambda_m \delta^{-(d+1)}\right| = 1 \tag{3.5}$$

$$\left\|\mathbf{T}_{\Phi,\Omega} f\right\|_p = \delta^{-\frac{n}{p}} \left\|\mathbf{T}_{\Phi_{\delta},\Omega} f_{\delta}\right\|_p. \tag{3.6}$$

Therefore, by (3.6) and the fact that $\delta^{-\frac{n}{p}}\ \left\|f_{\delta}\right\|_p = \left\|f\right\|_p$, it suffices to show that

$$\left\|\mathbf{T}_{\Phi_{\delta},\Omega} f\right\|_p \leq CA\left\|f\right\|_p \tag{3.7}$$

for all $1 < p < \infty$, where C is a constant independent of δ and the coefficients of the polynomials P_j. To this end, by writing $\mathbf{T}_{\Phi_{\delta},\Omega}$ as

$$\mathbf{T}_{\Phi_{\delta},\Omega} f(x) = \mathbf{T}^{0}_{\Phi_{\delta},\Omega} f(x) + \mathbf{T}^{\infty}_{\Phi_{\delta},\Omega} f(x), \tag{3.8}$$

where

$$\mathbf{T}^{\infty}_{\Phi_{\delta},\Omega} f(x) = \int_{|x-y|\geq 1} e^{i\Phi_{\delta}(x,y)} \left|x-y\right|^{-n} \Omega(x-y) f(y)dy,$$

it suffices to show that

$$\left\|\mathbf{T}^{0}_{\Phi_\delta,\Omega}f\right\|_p \leq AC_p \left\|f\right\|_p \tag{3.9}$$

and

$$\left\|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega}f\right\|_p \leq AC_p \left\|f\right\|_p \tag{3.10}$$

for all $1 < p < \infty$, where C is a constant independent of δ and the coefficients of the polynomials P_j.

We start by proving (3.9). For $h \in \mathbf{R}^n$, let

$$\begin{aligned}\Psi_{h,\delta}(x,y) \quad &= \quad \sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)}\{\phi_m(x,y) - \phi_m(x-h,y-h)\} \\ &\quad + \sum_{0\leq j\leq l, d_j+m_j\leq d} P_j\ (\delta^{-1}x)\phi_j(\delta^{-1}(y-x)).\end{aligned} \tag{3.11}$$

Since Ψ_h satisfies the induction assumption, Lemma 2.5 and the fact that $\|\Omega\|_{L^1} \leq 1$ imply that

$$\left\|\mathbf{T}^{0}_{\Psi_{h,\delta},\Omega}f\right\|_p \leq AC_p \left\|f\right\|_p \tag{3.12}$$

for all $1 < p < \infty$, where C is a constant independent of δ and the coefficients of the polynomials P_j and hence of h. Moreover, by the choice of ϕ_m and (3.11), straightforward calculations imply that $\Psi_{h,\delta}$ satisfies the assumptions (i)-(ii) of Lemma 2.3 with Φ replaced by Φ_δ and $\phi(x,y) = \sum_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(x,y)$. Hence (3.9) follows by Lemma 2.3. This proves (3.9).

Now, we prove (3.10). For $j \in \mathbf{Z}$, let

$$\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)(x) = \text{p.v.} \int_{2^{j-1}<|x-y|\leq 2^j} e^{i\Phi_\delta(x,y)} \left|x-y\right|^{-n} \Omega(x-y) f(y)dy.$$

Then

$$\mathbf{T}^{\infty}_{\Phi_\delta,\Omega}(f)(x) = \sum_{j=1}^{\infty} \mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)(x). \tag{3.13}$$

By similar argument as in ([8]), for fixed $y' \in \mathbf{S}^{n-1}$, let Y be the hyperplane through the origin orthogonal to y'. Then for $x \in \mathbf{R}^n$, there exist $s \in \mathbf{R}$ and $z \in Y$ such that $x = z + sy'$. Therefore,

$$\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)(x) = \int_{\mathbf{S}^{n-1}} \Omega(y') N_{\delta,j,y',z}(f(z+\cdot y')(t) dt d\sigma(y'), \tag{3.14}$$

where $N_{\delta,j,y',z}$ is the operator defined on $L^2(\mathbf{R})$ by

$$N_{\delta,j,y',z}(g)(s) = \int_{2^{j-1}\le s-t<2^j} e^{i\Phi_\delta(z+sy',z+ty')}(s-t)^{-1} g(t) dt.$$

Now, it is easy to see that the operator $(N_{\delta,j,y',z})^* N_{\delta,j,y',z}$ has the kernel

$$M_{\delta,j}(u,v) = \int_{\frac{1}{2}<r\le 1, 2^{j-1}<2^j r+v-u\le 2^j} e^{iE_{\delta,j}(y',z,u,v,r)} r^{-1}(2^j r+v-u)^{-1} dr,$$

where

$$E_{\delta,j}(y',z,u,v,r) = \Phi_\delta(2^j ry'+z+vy', z+vy') - \Phi_\delta(2^j ry'+z+vy', z+uy').$$

Now by the choice of ϕ_m, simple manipulations imply that

$$\begin{aligned} E_{\delta,j}(y',z,u,v,r) \quad &= \quad \sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \{[2^{(d+1)j} r^{(d+1)} - (2^j r+v-u)^{(d+1)}]\phi_m(y',0) \\ &\quad + [2^{dj} r^d - (2^j r+v-u)^d] \sum\nolimits_{j=1}^{n} z_j \phi_{m,j}(y',0) \\ &\quad + n[2^{dj} r^d v - (2^j r+v-u)^d u]\phi_m(y',0)\} + R(r,j,z,y',u,v,\delta), \end{aligned}$$

where $x'_j \phi_{m,j}(x',0) = \phi_m(x',0)$ and R is a function with $\frac{d^d}{dr^d}\left(R(r,j,z,y',u,v,\delta)\right) = 0$. Therefore,

$$\frac{d^d}{dr^d}(E_{\delta,j}(y',z,u,v,r)) = C(n,d)(v-u)2^{dj} \sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0) \tag{3.15}$$

with constant $C(n,d)$ that depends only on n and d.

By (3.15) and Lemma 2.2, we get

$$|M_{\delta,j}(u,v)| \le C\,|v-u|^{-\frac{1}{d}}\, 2^{-j} \left| \sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0) \right|^{-\frac{1}{d}}. \tag{3.16}$$

Now by (3.16) and the estimate $|M_{\delta,j}(u,v)| \le 2^{-j} C \chi_{[0,2^{j-1}]}(|v-u|)$, we have

$$|M_{\delta,j}(u,v)| \le C\,|v-u|^{-\frac{\varepsilon}{d}}\,2^{-j}\left|\sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0)\right|^{-\frac{\varepsilon}{d}} \chi_{[0,2^{j-1}]}(|v-u|) \quad (3.17)$$

for all $\varepsilon \in (0,1]$. Thus if we choose $\varepsilon < d$, we have

$$\int_{|v-u|<2^{j-1}} |M_{\delta,j}(u,v)|\,dv \le C 2^{-\frac{\varepsilon}{d}j}\left|\sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0)\right|^{-\frac{\varepsilon}{d}}; \quad (3.18)$$

which implies that

$$\|(N_{\delta,j,y',z})^* N_{\delta,j,y',z}\|_{L^\infty \to L^\infty} \le C 2^{-\frac{\varepsilon}{d}j}\left|\sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0)\right|^{-\frac{\varepsilon}{d}}. \quad (3.19)$$

Similarly, we obtain

$$\|(N_{\delta,j,y',z})^* N_{\delta,j,y',z}\|_{L^1 \to L^1} \le C 2^{-\frac{\varepsilon}{d}j}\left|\sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0)\right|^{-\frac{\varepsilon}{d}}.$$

Hence, we have

$$\|N_{\delta,j,y',z}\|_{L^2 \to L^2} \le C 2^{-\frac{\varepsilon}{d}j}\left|\sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0)\right|^{-\frac{\varepsilon}{d}}. \quad (3.20)$$

Now, since

$$|N_{\delta,j,y',z} g(s)| \le C HL(g)(s),$$

where HL is the Hardy Littlewood maximal function which is bounded on L^p for all $1 < p < \infty$, we have

$$\|N_{\delta,j,y',z}\|_{L^p \to L^p} \le C \quad (3.21)$$

for all $1 < p < \infty$.

By interpolation between (3.20) and (3.21), we have

$$\|N_{\delta,j,y',z}\|_{L^p \to L^p} \le C 2^{-\frac{\theta\varepsilon}{d}j}\left|\sum\nolimits_{m=1}^{M} \lambda_m \delta^{-(d+1)} \phi_m(y',0)\right|^{-\frac{\theta\varepsilon}{d}} \quad (3.22)$$

for $0<\theta<1$. Therefore, we immediately obtain

$$\left\|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)\right\|_{L^p} \leq C2^{-\frac{\theta\varepsilon}{d}j}2^{A}\left(\int_{\mathbf{S}^{n-1}}\left|\sum\nolimits_{m=1}^{M}\lambda_m\delta^{-(d+1)}\phi_m(y',0)\right|^{-\frac{\theta\varepsilon}{d}}d\sigma(y')\right)\|f\|_{L^p}. \tag{3.23}$$

Thus if we choose ε very small, then (3.5), Lemma 2.1, and (3.23) imply that

$$\left\|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)\right\|_{L^p} \leq C2^{-\frac{\theta\varepsilon}{d}j}2^{A}\|f\|_{L^p}. \tag{3.24}$$

On the other hand, it is easy to see that

$$\left|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)(x)\right| \leq \int_{2^{j-1}<|y|\leq 2^j}|y|^{-n}\,\Omega(y)\,|f(x-y)|\,dy. \tag{3.25}$$

Thus, by (3.25) and the fact that $\|\Omega\|_1\leq 1$, we have

$$\left\|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)\right\|_{L^p} \leq \|f\|_p. \tag{3.26}$$

Therefore, by interpolation between (3.24) and (3.26), we get

$$\left\|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega,j}(f)\right\|_{L^p} \leq C2^{-\frac{\theta\varepsilon}{dA}j}\|f\|_p \tag{3.27}$$

for all $1<p<\infty$. Hence by (3.13) and (3.27), we have

$$\begin{aligned}\left\|\mathbf{T}^{\infty}_{\Phi_\delta,\Omega}(f)\right\|_{L^p} &\leq C\{\sum_{j=1}^{\infty}2^{-\frac{\theta\varepsilon}{dA}j}\}\|f\|_p\\ &\leq CA\|f\|_p \end{aligned}\tag{3.28}$$

for all $1<p<\infty$. This completes the proof. □

Now we give the proof of Theorem 1.2.

Proof(of Theorem 1.2). Suppose $\Omega\in L(\log^+L)(\mathbf{S}^{n-1})$ that satisfies (1.1). Then by Lemma 2.6 there exist a subset $\mathbf{D}$ of $\mathbf{N}$, a sequence $\{\lambda_m : m\in\mathbf{N}\}$ of non negative real numbers, and a sequence of functions $\{\Omega_m : m\in\mathbf{D}\cup\{0\}\}$ in $L^1(\mathbf{S}^{n-1})$ that satisfy

$$\int_{\mathbf{S}^{n-1}}\Omega_m d\sigma=0,\ \text{for } m\in\mathbf{D}\cup\{0\}; \tag{3.29}$$

$$\|\Omega_m\|_\infty \le 2^{4(m+2)} \text{ and } \|\Omega_m\|_{L^1(\mathbf{S}^{n-1})} \le 2, \text{ for } m \in \mathbf{D}\cup\{0\}\,; \tag{3.30}$$

$$\sum_{m\in\mathbf{D}\cup\{0\}} (m+2)\lambda_m < \infty; \tag{3.31}$$

$$\Omega = \sum_{m\in\mathbf{D}\cup\{0\}} \lambda_m\Omega_m. \tag{3.32}$$

For $m \in \mathbf{D}\cup\{0\}$, let $\mathbf{T}_{\Phi,\Omega,m}(f)$ be the operator defined by (1.2) with Ω replaced by Ω_m and $\mathcal{P}$ by Φ. Then by (3.32), $\mathbf{T}_{\Phi,\Omega}$ is decomposed as

$$\mathbf{T}_{\Phi,\Omega}(f)(x) = \sum_{m\in\mathbf{D}\cup\{0\}} \lambda_m \mathbf{T}_{\Phi,m}(f)(x). \tag{3.33}$$

Now by (3.29)-(3.30) and Theorem 1.2 with $A = 4(m+2)$, we have

$$\|\mathbf{T}_{\Phi,\Omega,m}(f)\|_{L^p} \le (m+2)C\,\|f\|_p \tag{3.34}$$

for all $1 < p < \infty$. Hence by (3.31) and (3.34), the proof is complete. □

Finally, the authors would like to point out that the class of operators discussed in this paper has been introduced by the same authors in ([3]) under the assumption that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$, from a different point of view.

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Received 12.12.2002