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Perelman's Monotonicity Formula and Applications

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Abstract

This article relies on [15] that the author wrote with Gang Tian and Xiaodong Wang. In view of Hamilton's important work on the Ricci flow and Perelman's paper on the Ricci flow where he developes the techniques that he will later use in completing Hamilton's program for the geometrization conjecture, there may be more interest in the area. We will also discuss the author's theorem which says that the curvature tensor stays uniformly bounded under the unnormalized Ricci flow in a finite time, if the curvatures are uniformly bounded. We will prove that in the case of a Kähler-Ricci flow with uniformly bounded Ricci curvatures, for each sequence of flows $g(t_i + t)$ for $t_i \to \infty$ there exists a subsequence of metrics converging to a solution to the flow outside a set of codimension 4.

1. Introduction

The study of the Ricci flow began with Hamilton's seminal paper [8]. In this paper he introduced the notion of the Ricci flow, showed its short time existence and applied it to classify closed 3-manifolds with positive Ricci curvature. In his paper [9] Hamilton extended his methods and showed that closed 4-manifolds with positive curvature operator are topologically either S^4 or RP^4. In [10] R.Hamilton proved that for any initial metric on a closed surface (except that of a 2-sphere with variable curvature) the Ricci flow converges to a constant curvature metric. In [3] B. Chow extended the result to the case of any metric on a 2-sphere. The Ricci flow equation is the equation

$$\frac{d}{dt} g_{ij} = -2\mathrm{Ric}_{ij}.$$

Key words and phrases. Ricci flow, smooth convergence.

It is a very powerful equation, because if we start with an arbitrary initial metric on a manifold, sometimes we can expect to get nice metrics in the limit such as Einstein metrics or soliton type solutions. The stationary points of the Ricci flow on the space of metrics on a given manifold are the Ricci-flat metrics, or Einstein metrics in the case of the normalized Ricci flow when the volume of a manifold is fixed. Ricci solitons are important in the study of the Ricci flow, in particular regarding studying the singularities that appear along the flow. Soliton solutions evolve by diffeomorphisms $g(t) = \phi^* g(0)$, so that $\frac{d}{dt} g(t) = \mathcal{L}_V(g)$, where vector field V induces one parameter family of diffeomorphisms $\phi(t)$. Gradient solitons, where $V = \nabla f$ for some function f satisfy the equation $\mathrm{Ric} + D^2 f = 0$. The Ricci flow is invariant under the whole diffeomorphism group of a manifold. The stationary points of the Ricci flow on the moduli space $M/\mathcal{D}$ are the equivalence classes of Ricci solitons.

The organization of the paper is as follows: in section 2 we will discuss the functional $\mathcal{W}$ that was introduced by Perelman in [13]. We will also give the detailed proof of Perelman's noncollapsing theorem and we will mention the author's theorem that gives a sufficient condition for the existence of a solution to the unnormalized Ricci flow for all times. In section 3 we will state Perelman's pseudolocality theorem whose proof can be found in [13] and [15]. In this section we will also prove a subsequential smooth convergence of a Kähler-Ricci flow on a given manifold, outside a set of real codimension 4. We believe it can be proved in a more elementary way, using the parabolic regularity theory, but we will give a prove that uses Perelman's pseudolocality theorem and Theorem 7 proved by J. Cheeger, T. Colding and G. Tian in [5].

2. Functional W and the noncollapsing theorem

On a closed manifold M consider the following functional in a metric g and a smooth function f on M

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dV. \tag{1}$$

If we fix a measure $dm = e^{-f} dV$ we get a functional $\mathcal{F}^m(g)$ as f is determined by g. Its first variation is

$$d\mathcal{F}^m(v) = \int_M -\langle v, \mathrm{trRic} + D^2 f \rangle dm. \tag{2}$$

This leads to the consideration of the gradient flow

$$\frac{d}{dt}g = -2(\mathrm{Ric} + D^2 f),$$

where f evolves according to the backward heat equation

$$\frac{df}{dt} = -R - \Delta f.$$

In other words, the Ricci flow can be viewed as L^2 gradient flow of functional $\mathcal{F}$. A very important property of this functional is its monotonicity along the flow, i.e.

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |\mathrm{Ric} + D^2 f|^2 e^{-f} dV \geq 0.$$

To generalize F consider the functional

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-\frac{n}{2}} dV,$$

where $\tau > 0$ is a scale parameter. It is also monotone along the flow $\frac{d}{dt}g_{ij} = -2R_{ij}$, while f and τ satisfy

$$\frac{d}{dt}f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},$$
$$\dot{\tau} = -1,$$

respectively. Perelman used the monotonicity of these functionals to rule out nontrivial breathers.

Definition 1. A solution $g(t)$ to the Ricci flow is called a breather if for some $t_1 < t_2$ we have $g(t_2) = \alpha\phi^* g(t_1)$ for some constant α and diffeomorphism ϕ. The cases $\alpha = 1, \alpha > 1$ and $\alpha < 1$ correspond to steady, expanding and shrinking breathers, respectively.

Denote by $\mu(g, \tau) = \inf_{\{f | (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} dv = 1\}} \mathcal{W}(g, f, \tau)$. It is not difficult to show that this infimum is achieved for some function f, that $\lim_{\tau\to 0}\mu(g, \tau) = 0$ and that $\mu(g, \tau)$ is negative for small τ.

One of very nice applications of functional $\mathcal{W}$ and the monotonicity formula for $\mathcal{W}$ is the noncollapsing theorem for solutions for a finite time. This noncollapsing theorem together with curvature bounds give a uniform lower bound on the injectivity radii along the flow.

Definition 2. Let $g(t)$ be a solution to the Ricci flow on $[0,T)$. We say $g(t)$ is locally collapsing at T if $\exists t_k \to T$ and $B_k = B(p_k, r_k)$ at t_k such that r_k^2/t_k is bounded, $|\mathrm{Rm}|(g(t_k) \le C r_k^{-2}$ in B_k and $r_K^{-n}\mathrm{Vol}B_k \to 0$.

Theorem 1. *If M is closed and $T < \infty$, then $g(t)$ is not locally collapsing at T.*

Proof. Suppose it is locally collapsing at T, then we have a sequence $t_k \to T$ and B_k as described in the definition. Let $u = e^{-f/2}$. $\mu(g,\tau)$ is the infimum of

$$\mathcal{W}(u) = \int_M (\tau(4|\nabla u|^2 + Ru^2) - u^2 \ln u^2 - nu^2)(4\pi\tau)^{-n/2} dV \tag{3}$$

under the constraint

$$\int_M u^2 (4\pi\tau)^{-n/2} dV = 1. \tag{4}$$

Let $\tau = r_k^2$ and

$$u_k = e^{C_k}\phi(r_k^{-1} d(x,p_k)) \tag{5}$$

at t_k, where ϕ is a smooth function, equal 1 on $[0,1/2]$, decreasing on $[1/2,1]$ and equal 0 on $[1,\infty)$. C_k is a constant so that

$$\begin{aligned}(4\pi)^{n/2} &= e^{2C_k} r_k^{-n} \int_{B(p_k,r_k)} \phi(r_k^{-1} d(x,p_k))^2 dV \\ &\le e^{2C_k} r_k^{-n} \mathrm{Vol}B_k.\end{aligned}$$

Since $r_k^{-n}\mathrm{Vol}B_k \to 0$, $C_k \to +\infty$. We compute

$$\mathcal{W}(u_k) \le (4\pi)^{-n/2} r_k^{-n} e^{2C_k} \int_{B(p_k,r_k)} (4|\phi'|^2 - 2\phi^2 \log\phi) dV + r_k^2 \max_{B_k} R - n - 2C_k. \tag{6}$$

Let $V(r) = \mathrm{Vol}B(p_k, r)$. We have that $\mathrm{Ric} \ge -(n-1)C^2 r_k^{-2}$ in B_k. Let H_k be the simply connected space of constant sectional curvature $-C^2 r_k^{-2}$. Let $\bar{V}(r)$ be the corresponding volume in H_k. $\bar{V} r_k/\bar{V}(r_k/2)$ is bounded above by a uniform constant C'. By Bishop comparison theorem, $V r_k/V(r_k/2) \le \bar{V} r_k/\bar{V}(r_k/2) \le C'$. Hence $V(r_k) - V(r_k/2) \le C'V(r_k/2)$. Therefore

$$\begin{aligned}\int_{B(p_k,r_k)} (4|\phi'|^2 - 2\phi^2 \log\phi) dV \le& \quad C(V(r_k) - V(r_k/2)) \\ \le& \quad CV(r_k/2) \\ \le& \quad C \textstyle\int_{B_k} \phi^2 dV.\end{aligned}$$

Finally we get

$$\mathcal{W}(u_k) \leq C'' - 2C_k. \tag{7}$$

Since $C_k \to +\infty$ and $\mu(g(t_k), r_k^2) \leq \mathcal{W}(g(t_k), u_k, r_k^2)$, we conclude that $\mu(g(t_k), r_k^2) \to -\infty$. By the monotonicity $\mu(g(0), t_k + r_k^2) \leq \mu(g(t_k), r_k^2)$ and hence $\mu(g(0), t_k + r_k^2) \to -\infty$. This is impossible since $t_k + r_k^2$ is bounded. □

We will now state a corollary of noncollapsing theorem.

Corollary 1. *Let $g_{ij}(t)$, $t \in [0, T)$ be a solution to the Ricci flow on a closed manifold M, where $T < \infty$. Assume that for some sequences $t_k \to T$, $p_k \in M$ and some constant C we have $Q_k = |\mathrm{Rm}|(p_k, t_k) \to \infty$ and $|\mathrm{Rm}|(x, t) \leq CQ_k$, whenever $t < t_k$. Then a subsequence of scalings of $g_{ij}(t_k)$ at p_k with factors Q_k converges to a complete ancient solution to the Ricci flow, which is κ-noncollapsed on all scales for some $\kappa > 0$.*

Short time existence of solutions to the Ricci flow has been proved by Hamilton ([8]) and simplified by DeTurck ([6]). A very interesting and important question that we can ask is under which conditions the flow will exist for all times, i.e. when the curvature does not blow up in a finite time. The other very important question is when the curvature of the flow stays uniformly bounded when the flow exists for all times (i.e. when we can guarantee that the curvature will not blow up at $t = \infty$).

From Hamilton's work in [11] we know that in the case of unnormalized flow we have the following result.

Theorem 2 (Hamilton's theorem)**.** *For any smooth initial metric on compact manifold there exists a maximal time T on which there is a unique smooth solution to the Ricci flow for $0 \leq t < T$. Either $T = \infty$ or else the curvature is unbounded as $t \to T$.*

In [16] the author has proved the following result on the existence time for a solution.

Theorem 3. *Let $g(t)$ be a solution to $(g_{ij})_t = -2R_{ij}$ with $|\mathrm{Ric}| \leq C$ uniformly for all times when the solution exist. Then the solution exists for all times $t \in [0, \infty)$.*

The theorem above is an immediate consequence of Theorem 2 and theorem that we will state below.

Theorem 4. *Consider the unnormalized flow $(g_{ij})_t = -2R_{ij}$ on a compact manifold M for $t \in [0,T)$, where $T < \infty$. Assume that $|\mathrm{Ric}(g(t))| \le C$ for all $t \in [0,T)$. Then the curvature tensor can not blow up at T.*

Since the Ricci tensor of our flow stays uniformly bounded in a finite time, all metrics are uniformly equivalent to each other, which prevents the collapsing case to happen when we take a sequence of dilations. Namely, if the statement of Theorem 4 were not true, we would get a sequence of dilatations of our flow converging to an ancient, complete, Ricci-flat, nonflat solution to the Ricci flow. On the other hand, using the fact that the uniform bounds on the Ricci curvature give a good control over the volume forms and distances of metrics under the Ricci flow, we can get that for a limit of the sequence of dilatations the following holds: $\frac{\mathrm{Vol}(B(p,r)}{r^n} = w_n$ for all $r > 0$, where w_n is the volume of a unit ball in the euclidean space and p is a point in the limit manifold. That would mean that our limit solution would have to be Euclidean, which is not the case. The detailed proof of this theorem can be found in [16].

We would like to mention the local version of monotonicity formula for $\mathcal{W}$ that can be found in [13]. The monotonicity of $\mathcal{W}$ along the flow immediatelly follows from Proposition 1 as we will see below.

Proposition 1 (Perelman)**.** *Let $g_{ij}(t)$ be a solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \le t \le T$, and let $u = (4\pi(T-t))^{-\frac{n}{2}}e^{-f}$ satisfy the conjugate heat equation $\Box^* = -u_t - \Delta u + Ru$. Then $v = [(T-t)(2\Delta f - |\nabla f|^2 + R) + f - n]u$ satisfies*

$$\Box^* v = -2(T-t)|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2(T-t)}g_{ij}|^2$$

This immediatelly implies the monotonicity for $\mathcal{W}$, for the following reason. By partial integration we get

$$\int_M v = \int_M [(T-t)(|\nabla f|^2 + R) + f - n]e^{-f}(4\pi(T-t))^{-\frac{n}{2}}dV = W(g(t), f(t), \tau(t))$$

$$\begin{aligned} \frac{d}{dt}W &= \frac{d}{dt}\int_M v dV_t = \int_M v_t DV_t - \int_M -RvdV_t \\ &= \int_M (-\Box^* v - \Delta v)dV_t = -\int_M \Box^* v \ge 0 \end{aligned}$$

by Proposition 1. The advantage of this version of monotonicity formula shows up when one wants to work locally. There is a nice application of it in the proof of pseudolocality theorem in section 10.

3. Pseudolocality theorem and the Kähler-Ricci flow

We will state Perelman's pseudolocality theorem whose proof can be found in [12], [13] and [15]. We will show below how it can be used to prove the following theorem about the Kähler-Ricci flow.

Theorem 5. *Assume we are given the Kähler-Ricci flow* $(g_{i\bar{j}})_t = -R_{i\bar{j}} + g_{i\bar{j}}$ *on a closed, Kähler manifold* M *with* $c_1(M) > 0$. *Assume that the Ricci curvatures are uniformly bounded, i.e.* $|\mathrm{Ric}| \le C$ *for all* t. *Then for every sequence* $t_i \to \infty$ *there exists a subsequence such that* $(M, g(t_i + t)) \to (Y, \bar{g}(t))$, *where* $(Y, \bar{g}(t))$ *is an orbifold and the convergence is smooth outside the set* S *of codimension* 4. *Moreover,* $\bar{g}(t)$ *solves the Kähler-Ricci flow equation off* S.

Theorem 6 (Perelman's pseudolocality theorem)**.** *For every* $\alpha > 0$ *there exists* $\delta > 0$, $\epsilon > 0$ *with the following property. Suppose we have a smooth solution to the Ricci flow* $(g_{ij})_t = -2R_{ij}$ *for* $0 \le t \le (\epsilon r_0)^2$ *and assume that at* $t = 0$ *we have* $R(x) \ge -r_0^{-2}$ *and* $\mathrm{Vol}(\partial\Omega)^n \ge (1-\delta)c_n \mathrm{Vol}(\Omega)^{n-1}$ *for any* x *and* $\Omega \subset B(x_0, r_0)$, *where* c_n *is the euclidean isoperimetric constant. Then we have an estimate* $|\mathrm{Rm}|(x,t) \le \alpha t^{-1} + (\epsilon r_0)^{-2}$, *whenever* $0 < t \le (\epsilon r_0)^{-2}$, $d(x,t) = \mathrm{dist}_t(x, x_0) \le \epsilon r_0$.

Before we start proving Theorem 5 we will prove simple lemmas about the convergence of a sequence of 1 parameter families of metrics changing under any Ricci flow that are consequences of statements that can be found in [7].

Proposition 2 (D. Glickenstein)**.** *Let* $\{(M_i, g_i(t), p_i)\}_{i=1}^{\infty}$, *where* $t \in [0, T]$, *be a sequence of pointed Riemannian manifolds of dimension* n *which is continuous in the* t *variable in the following way: for each* $\delta > 0$ *there exists* $\eta > 0$ *such that if* $t_0, t_1 \in [0, T]$ *satisfies* $|t_0 - t_1| < \eta$ *then*

$$(1+\delta)^{-1} g_i(t_0) \le g_i(t_1) \le (1+\delta) g_i(t_0), \tag{8}$$

for all $i > 0$, and such that $\mathrm{Ric}(g_i(t)) \geq cg_i(t)$, where c does not depend on t or i. Then there is a subsequence $\{(M_i, g_i(t), p_i)\}_{i=1}^{\infty}$ and a 1-parameter family of complete pointed metric spaces $(X(t), d(t), x)$ such that for each $t \in [0, T]$ the subsequence converges to $(X(t), d(t), x)$ in the pointed Gromov-Hausdorff topology.

Since $(M_i, g_i(t))$ and $(M_i, g_i(0))$ are homeomorphic by Lipschitz homeomorphisms, it can be showed that $X(t)$ is homeomorphic to $X(0)$.

Theorem 7 (Cheeger, Colding, Tian)**.** *If $\{M_i, g_i, p_i)\}$ converges to (Y, d, y) in pointed Gromov-Hausdorff topology, if $|\mathrm{Ric}|_{M_i} \leq C$ and if $\mathrm{Vol}(B_1(p_i)) \geq C$ for all i, then the regular part $\mathcal{R}$ of Y is a $C^{1,\alpha}$-Riemannian manifold and at points of $\mathcal{R}$, the convergence is $C^{1,\alpha}$. Moreover the codimension of the set of singular points (which is a closed set in Y) is at most 4.*

We will consider the Kähler-Ricci flow

$$(g_{i\bar{j}})_t = g_{i\bar{j}} - R_{i\bar{j}} = \partial_i \bar{\partial}_j u. \tag{9}$$

In [2] H.D. Cao proved that the solution to this flow exists for all times $t \in [0, \infty)$. Perelman proved that there are uniform bounds on $C^{1,\alpha}$ norms on the potential functions $u(t)$, the scalar curvature $R(t)$ and the diamteres $\mathrm{diam}(M, g(t))$. He also proved that the noncollapsing condition holds along the Kähler-Ricci flow, i.e. that there exists C so that $\mathrm{Vol}B(p, r) \geq Cr^n$ for all $p \in M$, all $r > 0$ and all times $t > 0$, where C is a constant depending on $g(0)$. If t_i is any sequence such that $t_i \to \infty$, Proposition 2 will apply to $(M, g(t_i + t))$ for all i and all t belonging to a time interval of finite length (for $|\mathrm{Ric}|_{g(t)} \leq C$ for all t and the condition (8) in Proposition 2 is satisfied).

We will restrict ourselves to the case of Kähler manifolds of complex dimension 2, but it is easy to generalize the whole argument to an arbitrary dimension, without the essential changes (in higher dimensions we deal with a singular set of dimension $2n - 4$, which is 0, i.e. a set of points in complex dimension 2). In the case of complex dimension 2, for every sequence $t_i \to \infty$ there is a subsequence $\{(M, g(t_i + t))\}$ converging to a compact orbifold $(Y, \bar{g}(t))$ with isolated singularities. This is due to the fact that L^2 norm of the curvature operator (in the Kähler case) can be uniformly bounded in terms of the first and the second Chern class of a manifold and its Kähler class. Combining Proposition 2 and Theorem 7 gives us that $(Y, \bar{g}(t))$ is an 1 parameter family of orbifolds (it is even a Lipschitz family for parameter t belonging to an interval of finite length), such that a regular part of $(Y, \bar{g}(t))$ is $C^{1,\alpha}$ manifold and the convergence $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ takes place in $C^{1,\alpha}$ topology, away from the set of singular points (which is common for all orbifolds $(Y, \bar{g}(t))$).

The main tools in the proof of our theorem will be Perelman's pseudolocality theorem and Theorem $(A.1.5)$ in [4].

Proof. Assume that the curvature blows up. Let $t_i \to \infty$ be such that $Q_i = |\mathrm{Ric}|(p_i, t_i) \geq \max_{M\times[0,t_i]}$ and $Q_i \to \infty$. We have already seen that, since $|\mathrm{Ric}|(g(t)) \leq C$, there exists a subsequence $(M, g(t_i + t))$ converging to orbifolds $(Y, \bar{g}(t))$ in $C^{1,\alpha}$ norm off the set of singular points. Moreover, metrics $\bar{g}(t)$ are $C^{1,\alpha}$ off the singular set. We may assume that $\mathrm{Sing}(Y) = \{p\}$. Our goal is to show that we actually have C^∞ convergence off the singular point, due to the fact we are changing our metrics under the Kähler-Ricci flow.

Adopt the notation of [5]. Let $\mathcal{R}$ denote the regular set in Y. Let $\mathcal{R}_\epsilon = \{y | d_{GH}(B_1(y_\infty), B_1(0)) < \epsilon$ for every tangent cone $(Y_y, y_\infty)\}$ and $\mathcal{R}_{\epsilon,r}$ is the set of all points y such that there exists x such that $(0, x) \in R^4 \times \{x\}$ and for some $u > r$ and every $s \in (0, u]$ $d_{GH}(B_s(y), B_s((0, x))) < \epsilon s$. $\mathcal{R}_\epsilon = \cup_r \mathcal{R}_{\epsilon,r}$.

Choose ϵ_P and δ_P as in Perelman's pseudolocality theorem. Choose $\epsilon' > 0$ such that $\delta_P > \epsilon'$ and $\epsilon' \leq \epsilon_0$, where ϵ_0 is such that $\mathcal{R} = \mathcal{R}_\epsilon$ for all $\epsilon \leq \epsilon_0$ (such ϵ_0 exists by section 7 in [4]).

Pick up any point $q \in Y \backslash \{p\}$. Then $q \in \cap_{\epsilon \leq \epsilon_0} \mathcal{R}_\epsilon$.

Claim 3.1. For any regular point $q \in \mathcal{R}$ there exists i_0, A and $r > 0$ such that for all $B_{g(t_i+t)}(s, q') \subset B_{g(t_i+t)}(r, q)$ we have $\mathrm{Vol}_{g(t_i+t)} B_{g(t_i+t)}(s, q') \geq (1 - \epsilon')s^n$, for all $i \geq i_0$ and all $t \in [-A, A]$.

Proof. For ϵ' find r and δ as in Theorem $(A.1.5$ (i) and (ii) in [4]). For this δ choose δ_1 and r_1 (by Theorem $(A.1.5$ (iii))) such that $x \in (\mathcal{WR})_{8\delta_1,r'}$ implies that

$$y \in \mathcal{R}_{\delta,s} \;\; \forall y \in B_{r'}(x) \;\; \forall s \leq (1 - \delta)r' - \mathrm{dist}(x, y) \;\; , r' \leq r_1. \tag{10}$$

Take any sequence $\delta_i \to 0$ as $i \to \infty$. We can choose a sequence r_i such that $q \in \mathcal{R}_{\delta_i,r_i}$. We claim that $q \in \mathcal{R}_{\delta_1,r}$, for some $r < r_1$. In order to prove that we may assume

$r_i \to 0$ (otherwise if $r_i \geq \kappa$ for all i, $d_{GH}(B_\kappa(q), B_\kappa(0)) \leq \delta_i \to 0$ and therefore we would have $\mathrm{Vol}B_r(q) = \mathrm{Vol}B_r(0)$ for all $r \leq \kappa$, and $q \in \mathcal{R}_{\delta_1,s}$ for some $s < r_1$, by Theorem (A.1.5, (i)) in [4]). Therefore, there exist $\delta'' < \delta_1$ and $r'' < r_1$ such that $q \in (\mathcal{WR})_{\delta'',r''}$. This implies $q \in (\mathcal{WR})_{\delta_1,r''}$, since $\delta'' < \delta_1$. Choose a sequence $q_i \in M$ such that $B_{g(t_i+t)}(r'', q_i) \stackrel{GH}{\to} B_{\bar{g}(t)}(r'', q)$ as $i \to \infty$ for all $t \in [-A, A]$. Choose i_0 such that for $i \geq i_0$ we have that $d_{GH}(B_{g(t_i+t)}B(r'', q_i), B_{\bar{g}(t)}(r'', q) < \delta_1$ for all $t \in [-A, A]$. We can choose such A because our metrics $g(t)$ are equivalent for t belonging to the interval of finite length (for the uniform bound on the Ricci curvatures). By the 'triangle inequality' for the Gromov-Hausdorff distance we get that

$$d_{GH}(B_{g(t_i+t}(r'', q_i), B_{r''}(0)) < 4\delta_1,$$

for all $i \geq i_0$ and $t \in [-A, A]$, i.e. $q_i \in (\mathcal{WR})_{8\delta_1,r''}$. (10) gives us that $q' \in \mathcal{R}_{\delta,s}$ for all $q' \in B_{g(t_i+t)}(r'', q_i)$ and $s \leq (1-\delta)r'' - \mathrm{dist}_{t_i+t}(q_i, q')$ for all $i \geq i_0$ and all $t \in [-A, A]$. By (A.1.5 (ii)) in [4] we get that (since $r'' < r$)

$$\mathrm{Vol}B_{g(t_i+t)}(s, q') \geq (1-\epsilon')\mathrm{Vol}B_s(0), \tag{11}$$

for all $q' \in B_{g(t_i+t)}(r'', q)$ and $s \leq (1-\delta)r'' - \mathrm{dist}_{g(t_i+t)}(q_i, q')$. By reducing r'' we get that there exists r'' such that the estimate (11) holds for all $q' \in B_{g(t_i+t)}(r'', q)$ and all s such that $B_{g(t_i+t)}(s, q') \subset B_{g(t_i+t)}(r'', q)$.

□

Choose r, i_0 and A as in the claim above (for our regular point q that we have fixed earlier). Reduce r'' if necessary, so that $(\epsilon' r'')^2 < A$. Since $1-\epsilon' > 1-\delta_P$, and since for every ball $B_{g(t_i-(\epsilon' r'')^2/2)}(s, q') \subset B_{g(t_i-(\epsilon' r'')^2/2)}(r'', q_i)$ we have that $\mathrm{Vol}_{g(t_i-(\epsilon' r'')^2/2)}B_s(q') \geq (1-\delta_P)s^n$, by Perelman's pseudolocality theorem

$$|\mathrm{Rm}|_(x, t) \leq \frac{1}{(\epsilon' r'')^2} + (\epsilon' r'')^2,$$

for all $x \in B_{g(t)}(q_i, \epsilon' r'')$ and for every $t \in [0, t_i + (\epsilon' r'')^2/2]$. $g_i(t) = g(t_i + t)$ is a sequence of Ricci flows with uniformly bounded curvatures for $t \in [0, (\epsilon' r'')^2/2]$ on balls $B_{g_i(t)}(q_i, \epsilon' r'')$. This together with the volume noncollapsing condition and Hamilton's compactness theorem we get that the convergence of the sequence of our metrics is smooth, and $\bar{g}(t)$ are smooth metrics on $B(q, \epsilon' r'')$, for $t \in [0, (\epsilon' r'')^2/2]$. Repeating all these above to translated time intervals by $(\epsilon' r'')^2/2$ we get that $\bar{g}(t)$ are smooth metrics on $B(q, \epsilon' r'')$ for all times.

Since q was an arbitrary chosen point, the conclusion of our Theorem (5) holds for all points in Y that are not singular.

□

We will state now a simple corollary of Theorem 5.

Corollary 2. *Let $g_{i\bar{j}}(t)$, $t \in [0,\infty)$ be a solution to the Kähler-Ricci flow on a closed manifold M. Assume there exist a sequence $t_k \to \infty$, $p_k \in M$ and some constant C such that $Q_k = |\mathrm{Ric}|(p_k, t_k) \to \infty$ and $|\mathrm{Ric}|(x,t) \le CQ_k$ for all $t < t_k$. Then a subsequence of scalings of $g_{i\bar{j}}(t_k)$ at p_k with factors Q_k converges to an ancient, Ricci-flat solution to the Ricci flow, outside the set of codimension 4.*

Proof. The proof that a subsequence of scalings of $g_{i\bar{j}}(t_k)$ at p_k with factors Q_k converges to an ancient solution to the Ricci flow outside the set of codimension 4 (call it S) is the same as in 5. The fact that it is Ricci flat off the set of codimension 4 follows from Perelman's theorem which says that there exists a constant C so that $|R(g(t))| \le C$ for all $t > 0$ and the evolution equation for the scalar curvature

$$\frac{d}{dt}R = \Delta R + |\mathrm{Ric}|^2 + \frac{1}{2n}R(R-n),$$

which after rescaling becomes

$$\frac{d}{ds}R_i = \Delta_i R_i + |\mathrm{Ric}|_i^2 + \frac{1}{2n}R_i(R_i - \frac{n}{Q_i}),$$

where $s = s(t) = (t - t_i)Q_i$. Since $R_i = \frac{R}{Q_i} \to 0$ as $i \to \infty$, smoothly outside S, we have that the limit solution is Ricci flat outside S.

□

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