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# Flops of Crepant Resolutions

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**Abstract**

Let $G$ be a finite subgroup of $SL(3,\mathbb{C})$ acting with an isolated singularity on $\mathbb{C}^3$. A crepant resolution of $\mathbb{C}^3/G$ comes together with a set of tautological line bundles associated to each irreducible representation of $G$. In this note we give a formula for the triple product of the first Chern class of the tautological bundles in terms of both the geometry of the crepant resolution and the representation theory of $G$. From here we derive the way these triple products change when we perform a flop.

## 1. Introduction

Let $G \subset SL(3,\mathbb{C})$ be a nontrivial finite subgroup. Then $G$ acts on $\mathbb{C}^3$, and the quotient $\mathbb{C}^3/G$ is a Calabi-Yau orbifold. We are interested in studying crepant resolutions $\pi : X \to \mathbb{C}^3/G$ of this orbifold. These are resolutions of singularities of $\mathbb{C}^3/G$ with the property that $X$ is a smooth Calabi-Yau manifold. A result of Roan shows that a crepant resolution exists for any quotient $\mathbb{C}^3/G$; it uses the classification of the finite subgroups of $SL(3,\mathbb{C})$, [Ro]. In general such a crepant resolution is not unique: a flop on $X$ gives another crepant resolution. Recall that if we have a nonsingular complex manifold $X$ of dimension 3 containing a rational curve $l_0$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ we can contract the curve to get a variety with a single node and then resolve it in another way to get a nonsingular manifold $X'$; we say that $X'$ is obtained from $X$ by performing a *flop* along $l_0$.

Even if the orbifold $\mathbb{C}^3/G$ might have more that a crepant resolution, the Betti numbers of these resolutions are the same; these are the stringy orbifold numbers of [DHVW]. In order to explain this we first need to define the "age function" on $G$ (see Joyce's book [Jo] for most of the definitions included).

**Definition 1.** Let $G \subset SL(3,\mathbb{C})$ be a finite subgroup. Each element $g \in G$ has 3 eigenvalues $e^{2\pi i a(g)}, e^{2\pi i b(g)}, e^{2\pi i c(g)}$, where $a(g), b(g), c(g) \in [0,1)$ are uniquely defined up

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to order. We define the *age* of $g$ to be $age(g) := a(g) + b(g) + c(g)$; it is a well-defined integer with $0 \leq \mathrm{age}\,(\mathrm{g}) < 3$. It gives the *age function*, $age : G \to \{0, 1, 2\}$.

Note that since the age function does not change under conjugation, it is an invariant of the conjugacy class of $g$. The relation between the age function and the Betti numbers of a crepant resolution is given by the following theorem of Ito and Reid [IR96].

**Theorem 1.** *Let $X$ be a crepant resolution of $\mathbb{C}^3/G$. Then there exists a one-to-one correspondence between compact prime divisors in $X$, which form a basis for $H^2_c(X, \mathbb{Q})$, and conjugacy classes of elements of $G$ with age 1.*

In 1995, Nakamura introduced a candidate for a crepant resolution of $\mathbb{C}^3/G$

$$\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3) := \{\mathrm{Z} \subset \mathbb{C}^3 \;\; G\text{-invariant subscheme of dimension zero} \,|\, \mathrm{H}^0(\mathcal{O}_{\mathrm{Z}}) = \mathrm{R}\}; \quad (1)$$

it is now known under the name of *Nakamura's G-Hilbert scheme* [Nak]. Here $R$ is the *regular representation* of $G$: if $R_0, R_1, \ldots, R_{r-1}$ are the irreducible representations of $G$ of dimensions $n_i$, and $R_0$ denotes the trivial one, then $R := \oplus_{i=0}^{r-1} \mathbb{C}^{n_i} \otimes R_i$. In 1999, Bridgeland, King and Reid proved that indeed the Hilbert-Chow morphism $\pi : \mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3) \to \mathbb{C}^3/\mathrm{G}$ is a crepant resolution [BKR]. This space comes with a natural set of tautological sheaves associated to each irreducible representation of $G$: we define the sheaf $\mathcal{R}_i$ associated to $R_i$ as

$$\mathcal{R}_i = \pi^* \mathcal{O}_i/_{\mathrm{Tors}}, \quad (2)$$

where $\mathcal{O}_i$ is the coherent sheaf on $\mathbb{C}^3/G$ corresponding to the irreducible representation $R_i$ (see the discussion in 2.2.1). The proof of [BKR] also implies that the sheaves $\mathcal{R}_i$ are locally free (therefore they are holomorphic vector bundles) and form a $\mathbb{Z}$-basis in $K$-theory.

Any other crepant resolution is obtained from $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ by a sequence of flops. In the case when $G \subset SL(3, \mathbb{C})$ is a finite abelian subgroup, Craw and Ishii exhibited all the crepant resolutions as moduli spaces. Moreover their construction allows one to define tautological vector bundles on $X$ – associated to each irreducible representation of $G$– and show that they form a basis in $K$-theory (thus generalizing the known results for $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ to all the crepant resolutions of $\mathbb{C}^3/G$).

In this note we assume that $G \subset SL(3, \mathbb{C})$ is a finite group which acts with an isolated singularity on $\mathbb{C}^3$; this implies that $G$ is an abelian group. Let $X$ be a crepant resolution of $\mathbb{C}^3/G$ and let $\mathcal{R}_0, \ldots, \mathcal{R}_r$ be its corresponding tautological bundles. The manifold $X$ is endowed with a Ricci-flat ALE metric, and therefore allows us to use index theory methods to understand the cohomology of $X$. In this way we obtained the cohomological generalization of the McKay Correspondence [De2]. We describe it now: Let $Q = \mathbb{C}^3$ be the natural 3-dimensional representation of $G$ induced by the embedding $G \subset SL(3, \mathbb{C})$.

We consider the tensor product decompositions:

$$R_i \otimes Q = \sum_{j=0}^{r-1} a_{ij} \, R_j \ , \quad R_i \otimes \Lambda^2 Q = \sum_{j=0}^{r-1} b_{ij} \, R_j \ .$$

Using the positive integers $a_{ij}$ and $b_{ij}$ we define the *generalized Cartan matrix*

$$C := [a_{ij} - b_{ij}]_{i,j=1,\dots r-1} \, . \tag{3}$$

In this set-up we obtain:

**Proposition 1** ([De2]). *Let $G \subset SL(3, \mathbb{C})$ be a finite subgroup which acts with an isolated singularity on $\mathbb{C}^3$, and let $(X, \pi)$ be a crepant resolution of $\mathbb{C}^3/G$ with tautological bundles $\mathcal{R}_0, \dots, \mathcal{R}_{r-1}$. Then, their Chern characters – which form a basis for $H^*(X; \mathbb{Q})$ – satisfy the following multiplicative relations*

$$\left[ \int_X \left( \mathrm{ch}(\mathcal{R}_\mathrm{i}) - \mathrm{rk}(\mathcal{R}_\mathrm{i}) \right) \left( \mathrm{ch}(\mathcal{R}_\mathrm{j}^*) - \mathrm{rk}(\mathcal{R}_\mathrm{j}) \right) \right]_{i,j=1,\dots,r-1} = C^{-1} \, . \tag{4}$$

The above formula is the generalization of Kronheimer and Nakajima's formula [KN] for the case of surface singularities. It gives the Cartan matrix $C$ from the topology of the crepant resolution and therefore it is a geometrical interpretation of the McKay Correspondence.

However, the relations (4) are common for all the crepant resolutions of $\mathbb{C}^3/G$. They do not give any insight about what changes when two crepant resolutions differ by a flop. The purpose of this note is to use the analytical approach developed in [De2] to show how the triple intersection $\int_X c_1(\mathcal{R}_j)^3$ depends on the crepant resolution $X$ of $\mathbb{C}^3/G$ (see Theorem 5). As a consequence, we prove that given $X$ with the tautological line bundles $\mathcal{R}_0, \dots, \mathcal{R}_{r-1}$, and given $X'$ obtained from $X$ via the flop of a $(-1,-1)$ curve $l_0$ with its tautological bundles $\mathcal{R}'_0, \dots, \mathcal{R}'_{r-1}$ given by the proper transforms of the ones on $X$, we have either

$$\int_X c_1(\mathcal{R}'_j)^3 = \int_X c_1(\mathcal{R}_j)^3, \quad \text{or} \quad \int_X c_1(\mathcal{R}'_j)^3 = \int_X c_1(\mathcal{R}_j)^3 - 1. \tag{5}$$

The choice above depends on the irreducible representation $R_j$ and on the crepant resolution $X$ (i.e. it is independent of the choice of tautological line bundles) in a manner which is made explicit in Corollary 4.

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## 2. Crepant Resolutions

We consider $G \subset SL(3, \mathbb{C})$ a finite subgroup of $SL(3, \mathbb{C})$ which acts with an isolated singularity on $\mathbb{C}^3$. It follows that $G$ is an abelian group, and therefore we can understand a crepant resolution $X$ of $\mathbb{C}^3/G$ from two perspectives: (1) the moduli space description of Craw and Ishii [CI], and (2) toric geometry. The interplay between these two views will allow us to gain more insight about the tautological bundles on $X$.

First a little bit of notation. Let $r := |G|$, the order of the group $G$. Let $R_0, \dots, R_{r-1}$ be the irreducible representations of $G$, and let $R$ be its regular representation $R = \oplus_{i=0}^{r-1} R_i$. We denote by $R(G)$ the representation ring of $G$.

### 2.1. Moduli description of crepant resolutions

Let $Q = \mathbb{C}^3$ be the natural 3-dimensional representation of $G$ induced by its embedding into $SL(3, \mathbb{C})$. For $B \in \mathrm{Hom}(R, Q \otimes R)$ we have $B \wedge B \in \mathrm{Hom}(R, \Lambda^2 Q \otimes R)$. We consider the algebraic variety

$$\mathcal{N} = \{B \in Hom^G(R, Q \otimes R) | B \wedge B = 0\}.$$

The group $GL^G(R)$ acts on $\mathcal{N}$ by conjugation. Its quotient by the scalar matrices

$$PGL^G(R) := GL(R_0) \times \dots GL(R_{r-1})/\mathbb{C}^*.$$

acts faithfully. Each element $\theta \in \mathrm{Hom}^{\mathbb{Z}}(R(G), \mathbb{Z})$ satisfying $\theta(R) = 0$ induces a character $\chi_\theta$ of $PGL^G(R)$ by: $\chi_\theta([g_0, \dots, g_{r-1}]) = \Pi_{i=0}^{r-1} \det{(g_i)}^{\theta_i}$. We take the GIT quotient of $\mathcal{N}$ by $PGL^G(R)$ with respect $\chi_\theta$

$$\mathcal{M}_\theta := \mathcal{N} /\!\!/_{\chi_\theta} PGL^G(R).$$

Recall that the character $\chi_\theta$ defines a $PGL^G(R)$-equivariant structure on the line bundle $L = \mathbb{C} \times \mathcal{N} \to \mathcal{N}$ – call this $L_\theta$. A point $p \in \mathcal{N}$ is said to be $\chi_\theta$*-semistable* if there exists as $PGL^G(R)$-invariant section of $L_\theta^k$ – for some positive integer $k$ – which does not vanish at $p$; the set of $\chi$-semistable points of $\mathcal{N}$ is denoted by $\mathcal{N}^{\mathrm{ss}}(L_\theta)$. Among the points of $\mathcal{N}^{\mathrm{ss}}(L_\theta)$ we have an *extended G-equivalence* induced by the closure of $PGL^G(R)$-orbits: $p \sim q$ if and only if the closure of the $PGL^G(R)$-orbits of $p$ and $q$ intersect. The GIT quotient of $\mathcal{N}$ by $PGL^G(R)$ with respect $\chi_\theta$ is $\mathcal{N} /\!\!/_{\chi_\theta} PGL^G(R) := \mathcal{N}^{\mathrm{ss}}(L_\chi)/_\sim$. Moreover since $\mathcal{M}_\theta \cong \mathcal{M}_{k\theta}$ for any positive integer $k$, this quotient is well-defined for parameters $\theta$ in

$$\Theta := \{\theta \in \mathrm{Hom}^{\mathbb{Z}}(R(G), \mathbb{Q}) \,|\, \theta(R) = 0\}\,. \tag{6}$$

The space $\mathcal{M}_\theta$ comes together with a projective morphism

$$\rho_\theta : \mathcal{M}_\theta \to \mathcal{M}_0.$$

**Theorem 2** (Sardo-Infirri [SIa])**.** *Let $G \subset SL(3, \mathbb{C})$ be a finite subgroup of $SL(3, \mathbb{C})$ which acts with an isolated singularity on $\mathbb{C}^3$. Then*

(i) $\mathcal{M}_0 \cong \mathbb{C}^3/G$*;*
(ii) *The morphism $\rho_\theta : \mathcal{M}_\theta \to \mathbb{C}^3/G$ is a (partial) resolution of singularities.*

The following result gives a characterization of all the crepant resolutions of $\mathbb{C}^3/G$ as a GIT quotient and also extends the result of Bridgeland, King and Reid [BKR].

**Theorem 3** (Craw and Ishii [CI]). *Let $G \subset SL(3,\mathbb{C})$ be a finite abelian subgroup. Then any projective crepant resolution $X$ of $\mathbb{C}^3/G$ is isomorphic to $\mathcal{M}_\theta$ for some $\theta \in \Theta$. Moreover the projective morphism $\rho_\theta : \mathcal{M}_\theta \to \mathbb{C}^3/G$ induces – via the universal bundle of $\mathcal{M}_\theta \times \mathbb{C}^3$ – an equivalence*

$$\Phi_\theta : \mathcal{D}(\mathcal{M}_\theta) \to \mathcal{D}^G(\mathbb{C}^3)$$

*between the bounded derived categories of coherent sheaves on $\mathcal{M}_\theta$ and the derived category of $G$-equivariant coherent sheaves on $\mathbb{C}^3$.*

**Corollary 1.** *For each irreducible representation $R_j$ of $G$, the associated sheaf*

$$\mathcal{R}_{\theta,j} := \Phi_\theta^{-1}(R_j \otimes \mathcal{O}_{\mathbb{C}^3})^\vee \tag{7}$$

*is locally free. We call $\mathcal{R}_j$ the* tautological $R_j$-bundle *on $\mathcal{M}_\theta$; the set of all these bundles form a $\mathbb{Z}$-basis in $K(\mathcal{M}_\theta)$.*

*Remark.* Ito and Nakajima showed that for $\theta \in \Theta$ so that $\theta(R_0) < 0$ and $\theta(R_i) > 0$ for $i \neq 0$ the GIT quotient $\mathcal{M}_\theta$ is isomorphic to Nakamura's $G$-Hilbert scheme $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$, and that the description (2) of the tautological bundles coincides to (7), [IN].

*Remark.* The moduli spaces $\mathcal{M}_\theta$ where studied by Sardo-Infirri as moduli of representations of the McKay quiver of $G$ [SIb]. In the work of Craw and Ishii the spaces $\mathcal{M}_\theta$ arrive as moduli spaces of $G$-constellations. The equivalence between the two perspectives is supplied by a result of King [Ki].

The stability condition induces a chamber structure on $\Theta$, [CI]. The moduli space $\mathcal{M}_\theta$ and its tautological bundles are independent of the chamber $C$ in which $\theta$ lies. For this reason we will refer to $\mathcal{M}_C$ and the corresponding tautological bundles $\mathcal{R}_{C,0}, \ldots, \mathcal{R}_{C,r-1}$ from now on.

Following Craw and Ishii [CI], a *wall* $W_Q$ of a chamber $C$ corresponds to a subrepresentation $Q \subset R$ so that $\theta(Q) = 0$ – and also $\theta(R/Q) = 0$ – for a generic $\theta \in W_Q$; each such wall has an unstable locus. The change in the geometry of the moduli space $\mathcal{M}_C$ while crossing the wall determines the *type of the wall.* In this note we will be concerned with walls of type 0 and of type 1. A *wall of type* 0 has an unstable locus $E$ given by a compact, reduced and connected divisor of $\mathcal{M}_C$; there is no change in the geometry of $\mathcal{M}_C$ while crossing it, but the tautological line bundles change according to the following rule

$$\begin{aligned} &\text{if } R_0 \subset Q, \text{ then } \mathcal{R}'_j \cong \mathcal{R}_j \text{ for } \rho_j \subset Q \text{ and } \mathcal{R}'_j \cong \mathcal{R}_j(-E) \text{ for } \rho_j \subset R/Q \\ &\text{if } R_0 \subset R/Q, \text{ then } \mathcal{R}'_j \cong \mathcal{R}_j(E) \text{ for } \rho_j \subset Q \text{ and } \mathcal{R}'_j \cong \mathcal{R}_j \text{ for } \rho_j \subset R/Q \end{aligned} \tag{8}$$

The divisor $E$ can be written as $E = \sum_{age(g)=1} a(g) D_g$, with $a(g) = 0$ or 1 and also so that $E$ is connected.

The *walls of type* 1 correspond to flops and they have the unstable locus given by the $(-1,-1)$-curve $l_0$ which is flopped. The tautological line bundles on the flop are the proper transforms of the initial tautological bundles.

There are also some other type of walls, which are not significant for our note (the wall of type 0 and 1 are the only ones coming into the game when we perform flops). For the general discussion we refer to the original article of Craw and Ishii.

### 2.2. Toric description of a crepant resolution

Again, assume that $G \subset SL(3,\mathbb{C})$ is a finite subgroup acting with an isolated singularity on $\mathbb{C}^3$. Such a group $G$ must be cyclic of order $r = |G|$ and so that the $\frac{1}{r}$-integers $a(g)$, $b(g)$ and $c(g)$ of Definition 1 do not vanish for any nontrivial $g \in G$. Since $G$ is abelian the orbifold singularity $\mathbb{C}^3/G$ is toric and therefore we can study its toric crepant resolutions.

Let $\overline{L} = \mathbb{Z}^3$ be the standard lattice and consider the supralattice of index $r$

$$L := \overline{L} + \sum_{g\in G} \mathbb{Z}\cdot(a(g), b(g), c(g)). \tag{9}$$

We consider the *junior triangle* in $L$

$$\Delta := \{\, (m,n,p) \in L \,|\, m+n+p = 1\}. \tag{10}$$

This triangle has vertices at

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1).$$

Its interior lattice points are in one-to-one correspondence with the elements of $g \in G$ with $age(g) = 1$. We denote these points by $v_g := (a(g), b(g), c(g))$. Since $G$ acts with an isolated singularity on $\mathbb{C}^3$ there are $\frac{r-1}{2}$ such points.

The cone in $L$ supported on the junior triangle $\Delta$ represents the orbifold $\mathbb{C}^3/G$. In order to get a (partial) resolution of singularities for $\mathbb{C}^3/G$ we need a subdivision of this cone. We look at subdivisions arising from triangulations of the junior triangle. Such a triangulation is called *basic* if it cannot be further triangulated, i.e. each triangle in the triangulation contains exactly three lattice points: its vertices. Since $L$ is a supralattice of index $r$ of $\overline{L}$ the condition that a triangle is basic is equivalent to the condition that the determinant of the $3\times 3$-matrix formed with the coordinates of its vertices is $\pm\frac{1}{r}$. The following theorem is now folklore for the people working in the field [CR, and the references therein].

**Theorem 4.** *Let $G \in SL(3,\mathbb{C})$ be a finite abelian subgroup. Then the crepant resolutions of $\mathbb{C}^3/G$ are in one-to-one correspondence to the basic triangulations of the junior triangle.*

In order to use this theorem we need to go to the dual picture. Let $\overline{\Lambda}$ be the dual lattice to the standard lattice $\overline{L}$; we think of it as the lattice of Laurent monomials in three variables. Let $\Lambda$ be the dual lattice of $L$: it is a sublattice of order $r$ in $\overline{\Lambda}$, and we identify it with the lattice of $G$-invariant Laurent monomials. The positive cone in $\Lambda$ is the ring of $G$-invariant monomials

$$\Lambda_+ \equiv \mathbb{C}^G[X, Y, Z],$$

and corresponds to the coordinate ring of the algebraic variety $\mathbb{C}^3/G$.

This duality allows us to translate the purely combinatorial picture which takes place in $L$ into a commutative algebra picture. Let $\mathcal{T}$ be a basic triangulation of the junior triangle $\Delta$. Each triangle $\tau \in \mathcal{T}$ determines a cone in $L$. We denote by $\tau^\vee$ its dual cone in $\Lambda$. Each such $\tau^\vee$ determines the ring $\Lambda_\tau$ – generated by the three primitive vectors along its edges – to which we associate the affine open set

$$U_\tau := \operatorname{Spec} \Lambda_\tau.$$

Given two basic triangles which share a common edge, we glue together the corresponding affine open sets according to the identification given by it. In this way we obtain a smooth algebraic variety $X_{\mathcal{T}}$ associated to $\mathcal{T}$ together with a morphism $\pi_{\mathcal{T}} : X_{\mathcal{T}} \to \mathbb{C}^3/G$ – it is one of the crepant resolutions given by Theorem 4.

Let $\operatorname{Star}(v_g)$ denote the star of an interior lattice point $v_g$ in the junior triangle. It consists of the basic triangles with this point as a vertex. Gluing together the affine open sets corresponding to these cones we obtain a torus-invariant surface $D_g$. These surfaces are exactly the compact prime divisors induced by the resolution $(X_{\mathcal{T}}, \pi_{\mathcal{T}})$. In this way we see the motivation for Theorem 1.

### 2.2.1. Toric description of the tautological bundles

The orbifold $\mathbb{C}^3/G$ inherits $r$ distinct coherent sheaves $\mathcal{O}_0, \ldots, \mathcal{O}_{r-1}$ corresponding to the irreducible representations of $G$ by defining their module of holomorphic sections to be

$$M_j := \operatorname{Hom}^G(R_j, \mathbb{C}[X, Y, Z]). \tag{11}$$

Note that $M_0 \equiv \Lambda_+$ and all the other $M_j$'s are finitely generated $\Lambda_+$-modules. Equivalently, each irreducible representation $R_j$ gives rise to a $G$-equivariant sheaf on $\mathbb{C}^3$: $R_j \otimes \mathcal{O}_{\mathbb{C}^3}$. The pushforward of this sheaf to $\mathbb{C}^3/G$ via the projection map is exactly $\mathcal{O}_j$. The sheaves $\mathcal{O}_0, \ldots, \mathcal{O}_{r-1}$ are locally free on the smooth locus $\mathbb{C}^3/G \setminus \{[O]\}$ – their associated line bundles represent the $r$ elements in

$$H^2(S^5/G, \mathbb{Z}) \cong G.$$

But, for any $j \neq 0$ they do not extend in a locally free way over the singular point $[O]$. However, upon the identification of $X_{\mathcal{T}} = \mathcal{M}_C$ for some chamber $C \subset \Theta$, there is a unique way to extend them to locally free sheaves over any crepant resolution $X_{\mathcal{T}}$, giving exactly the tautological bundles $\mathcal{R}_j$ of Craw and Ishii (see Corollary 1).

For each irreducible representation $R_j$ of $G$ and for each lattice point $w$ in the junior triangle (it could be a vertex or an interior lattice point) we define

$$\frac{1}{r}\rho_j(w) := \min\{\langle \mathrm{m}, \mathrm{w}\rangle \mid \mathrm{m} \in \mathrm{M_j}\}.$$

Note that $\langle m, w\rangle$ is a $\frac{1}{r}$-integer so this definition gives that $\rho_j(w)$ is a well-defined integer.

**Lemma 1.** *Let $\mathcal{T}$ be a basic triangulation of the junior triangle, and let $\tau$ be one of its basic triangles. There exists a unique element $m_j(\tau) \in \Lambda M_j$ so that*

$$\langle m_j(\tau), w\rangle = \rho_j(w), \quad \textit{for all vertices of } \tau. \tag{12}$$

*Proof.* Let $w_1, w_2, w_3$ be the vertices of $\tau$. The statement of the Lemma follows easily from Cramer's rule applied to the system: the determinant of the matrix of coefficients is $\det[w_1, w_2, w_3]^{\mathrm{tr}} = \pm\frac{1}{r}$ and all its $2\times 2$-minors are $\frac{1}{r}$-integers. □

**Proposition 2.** *Let $\mathcal{T}$ be a basic triangulation of the junior triangle $\Delta$ and let $X_{\mathcal{T}}$ be the associated crepant resolution of singularities. Then for each irreducible representation $R_j$ of $G$ there exists a unique holomorphic line bundle $\mathcal{S}_j$ on $X_{\mathcal{T}}$ so that for each $\tau$ a basic triangle, the $\Lambda_\tau$-module of holomorphic sections of $\mathcal{S}_j|_{U_\tau}$ is generated by $m_j(\tau)$.*

*Moreover, in the case when $\mathcal{T}$ gives Nakamura's $G$-Hilbert scheme $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$, these are exactly its tautological line bundles.*

We postpone the proof of this proposition to the next subsection.

The problem with the GIT construction of the crepant resolutions of singularities is that we do not know that the bundles introduced by the above Proposition are a basis in $K$-theory for an arbitrary crepant resolution of singularities. Rather, we know that starting from a crepant resolution $\mathcal{M}_C$, there is a sequence of crossing walls of type 0 and type 1 to arrive to any other crepant resolution. If we did not have any walls of type 0 then the above Proposition gave exactly the tautological line bundles; otherwise, the bundles $\mathcal{S}_0, \ldots, \mathcal{S}_{r-1}$ give the tautological line bundles after some twisting.

**Proposition 3.** *Let $\mathcal{T}$ be a basic triangulation of the junior triangle $\Delta$ and let $X_{\mathcal{T}}$ be the associated crepant resolution of singularities. Then for each chamber $C \subset \Theta$ so that $\mathcal{M}_C = X_{\mathcal{T}}$, and for each irreducible representation $\mathcal{R}_j$ there exists a compact divisor $E_j \subset X_{\mathcal{T}}$ so that the tautological line bundle $\mathcal{R}_{C,j}$ satisfies*

$$\mathcal{R}_{C,j} = \mathcal{S}_j \otimes \mathcal{O}_{X_{\mathcal{T}}}(-E_j). \tag{13}$$

The divisors $E_j = \sum_{age(g)=1} a_j(g) D_g$ with $a_j(g)$ integers are determined by the walls of type 0 which we need to cross in order to arrive from the chamber which gives $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ to the chamber $C$. The result follows from Proposition 2 and (8).

Now we have the following characterization of the tautological line bundles in terms of the crepant divisors $D_g$.

**Corollary 2.** *For each irreducible representation $R_j$ of $G$, and for each basic triangulation $\mathcal{T}$ of the junior triangle $\Delta$, the corresponding tautological bundle $\mathcal{R}_j$ corresponding to the chamber $C$ satisfies*

$$\mathcal{R}^r_{C,\,j} \;=\; \mathcal{O}_X(-\sum_{\substack{g\in G \\ age(g)=1}} (\rho_j(g) + r\,a_j(g))\; D_g). \tag{14}$$

*Proof.* Let $\mathcal{T}$ be a basic triangulation of $\Delta$ and denote by $X$ the corresponding crepant resolution. Each of the tautological bundles $\mathcal{R}_j$ give, after tensoring by $\mathcal{O}_{X_{\mathcal{T}}}(E_j)$, the Cartier divisor $\{(U_\tau, -r\,m_j(\tau))\}$ on $X$ (corresponding to $\mathcal{S}^r_j$). The corresponding principal divisor on $U_\tau$ is

$$\mathrm{div}(-\mathrm{r}\,\mathrm{m_j}(\tau)) \;=\; \sum_{\mathrm{v_g}\in\Delta} \langle\, -\mathrm{r}\,\mathrm{m_j}(\tau), \mathrm{v_g}\,\rangle \mathrm{D_g}.$$

From the definition of $m_j(\tau)$ the conclusion of the corollary follows. ☐

*Remark*. The search for such a result was inspired by a similar result of Mrowka, Ozsváth and Yu [MOY, Section 11] for the case of tautological line bundles on minimal resolutions of surface singularities. Recently Logvinenko [Lo] proved this result in much greater generality.

### 2.2.2. Proof of Proposition 2

We will first show that the Proposition is true for the triangulation which gives $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ with the tautological sheaves $\mathcal{R}_0, \ldots, \mathcal{R}_{r-1}$. Then we show that the result holds for any flop of $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ (i.e. when we cross a wall of type 1) with the proper transforms of the tautological sheaves $\mathcal{R}_j$.

Nakamura proved that there exists a basic triangulation which gives $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ [Nak]. Craw and Reid gave an algorithm to construct this triangulation [CR]. Using a combinatorial procedure involving continued fractions they showed that there exists a coarser triangulation called the *regular triangulation*. The triangles of the regular triangulation are equilateral triangles of size $s$ – call them *regular triangles*. Any basic triangulation is obtained by futher subdividing every regular triangle of size $s \geq 2$ into $s^2$ basic triangles. The basic triangulation which gives $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ is the most symmetric one: each regular triangle is subdivided using $s-1$ parallel lines to its sides. Craw and Reid call this process *regular tesselation*. Each new edge $l_0$ arising through the regular tesselation gives a $(-1,-1)$-curve $C_0$ on $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$.

**Lemma 2** (Nakamura [Nak])**.** *Let $\mathcal{T}$ be the triangulation which gives* $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$. *Then for any basic triangle $\tau$ we have $m_j(\tau) \in \Lambda_+ M_j$. The tautological sheaves satisfy $\mathcal{R}_j = \pi^*\mathcal{O}_j$ and are locally free. Moreover, the natural map*

$$\mathcal{O}_j \to \pi_*\pi^*\mathcal{O}_j$$

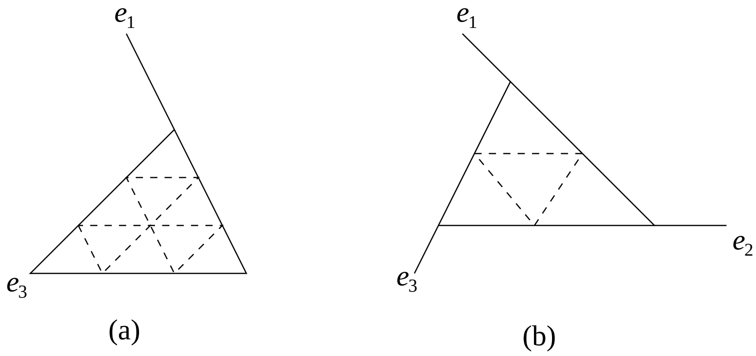


**Figure 1**. (a) corner triangle of size 3 and its regular tesselation; (b) meeting of champions of size 2 and its regular tesselation

*is an isomorphism.*

*Proof.* This Lemma was also proved by Craw and Reid [CR]. Our proof is similar to theirs.

We show that for each basic triangle $\tau = \tau(w_1, w_2, w_3)$ of the triangulation $\mathcal{T}$ there are exactly $r$ lattice points in $\Lambda^+$ (thus monomials in $\mathbb{C}[X, Y, Z]$) which satisfy Lemma 1. Moreover, each such point belongs to a different representation space $M_j$ which gives the conclusion to the first part of the lemma.

In order to show this claim, we need an analysis on different type of basic triangles which might show up in the basic triangulation $\mathcal{T}$. We base our analysis on the discussion in [CI, Subsection 9.1]. We start with the easier case of triangles of size 1 to set up the notations and the line of reasoning. However, this situation is also handled by the general case.

*Case* 1*a:* Corner triangle of size 1. This means that the triangle is basic with vertices $w_1, w_2, w_3$ and the dual cone spanned (up to a permutation of the coordinates) by

$$w_1^\perp := (a, -b, 0), \quad w_2^\perp := (-a+1, b+c+1, 0), \quad w_3^\perp := (0, -c, 1),$$

The functional $\langle m, w_1 \rangle$ reaches its minimum at lattice points in $\overline{\Lambda}_+$ with the property that $m - w_1^\perp$ is not in $\overline{\Lambda}_+$ – meaning that $m - w_1^\perp$ does not represent a honest monomial – giving that $0 \leq m_1 < a$. In the same way the condition for minimum for the other two functionals $\langle m, w_2 \rangle$ and $\langle m, w_3 \rangle$ gives the bounds $0 \leq m_2 < b + c + 1$ and $0 \leq m_3 < 1$. Moreover the point $w_1^\perp + w_2^\perp = (1, c+1, 0)$ corresponds to an invariant monomial and therefore the candidates for the minimum belong to

$$[0, a-1] \times [0, c] \times \{0\} \cup \{0\} \times [c+1, b+c] \times \{0\}.$$

(see Figure 2 a). The number of lattice points in this set is $a(c+1) + b = r$ since $\det[w_1^\perp, w_2^\perp, w_3^\perp] = r$ (being the dual of the triangle $\tau$ of volume $\frac{1}{r}$).

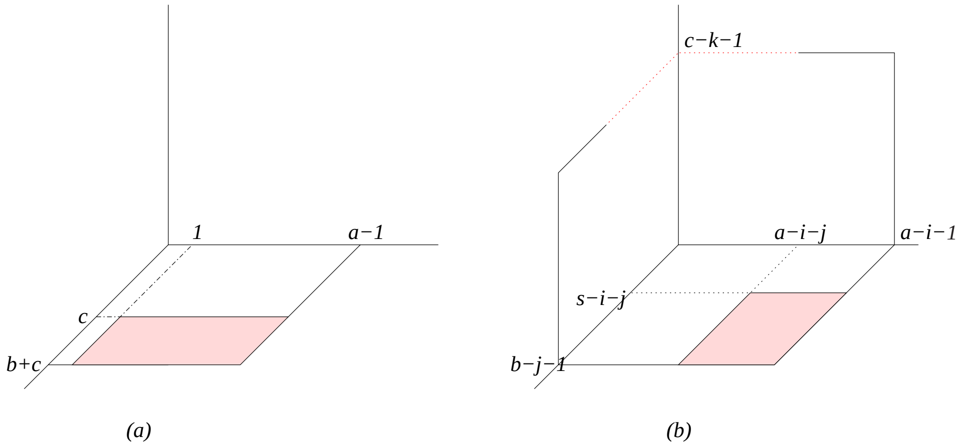


**Figure 2**. Possible $m_j(\tau)$ for: (a) Corner triangle of size 1; (b) Up triangle in a meeting of champions of size $s$.

*Case* 1*b:* Meeting of champions of size 1. This is the easiest of the cases. The dual cone is spanned (up to a permutation of the coordinates) by

$$w_1^{\perp} = (a, -b+1, 0), \quad w_2^{\perp} = (0, b, -c+1), \quad w_3^{\perp} = (-a+1, 0, c). \tag{15}$$

The points where the functionals $\langle\, m, w_i \,\rangle$ attain their minimum in $\overline{\Lambda}_+$ satisfy respectively

$$0 \le m_1 < a, \quad 0 \le m_2 < b, \quad 0 \le m_3 < c.$$

Since $(1,1,1)$ corresponds to an invariant monomial, the points of minimum can only lie on the coordinate planes. Because of this and that fact that the triangle is basic (i.e. $\det [w_1^{\perp}, w_2^{\perp}, w_3^{\perp}] = r$), the total number of lattice points satisfying the above condition is $r$.

*Case sa:* Corner triangle of size $s$. This case and the next one are similar in their treatment.

*Case sb:* meeting of champions of size $s$. The regular tessellation of this triangle gives $s^2$ basic triangles. They correspond to $\alpha, \beta, \gamma = 0, \ldots, s-1$ with $\alpha+\beta+\gamma = s-1$ (and we call this an *up* triangle) or $\alpha+\beta+\gamma = s+1$ (and we call this a *down* triangle).

*Subcase sb-up:* We have $\tau = \tau(w_1, w_2, w_3)$ an up triangle with the dual cone spanned by

$$w_1^{\perp} = (a-\alpha, -b+s-\alpha, -\alpha), \quad w_2^{\perp} = (-\beta, b-\beta, -c+s-\beta), \quad w_3^{\perp} = (-a+s-\gamma, -\gamma, c-\gamma),$$

and $\alpha+\beta+\gamma = s-1$. From the existence of this triangle we must also have $\alpha+\beta < a$ and the corresponding cyclic permutations. The points where the functionals attain their minimum must satisfy

$$0 \le m_1 < a-\alpha, \quad 0 \le m_2 < b-\beta, \quad 0 \le m_3 < c-\gamma$$

due to the fact that $m-w_i^\perp$ does not belong to $\overline{\Lambda}_+$. Moreover $w_1^\perp+w_2^\perp=(a-\alpha-\beta, s-\alpha-\beta, -c+s-\alpha-\beta)$ giving that the monomial corresponding to $(a-\alpha-\beta, s-\alpha-\beta, 0)$ is obtained from the monomial associated to $(0,0,c-\gamma-1)$ by multiplication with an element in the ring $\Lambda_\tau$. Therefore the red rectangle in Figure 2 (b) does not contain points of minima for all three functionals. We deal in the same way with the conditions coming out of $w_2^\perp+w_3^\perp$ and $w_1^\perp+w_3^\perp$, obtaining at the end that the number of possible lattice points is exactly $\det[w_1^\perp, w_2^\perp, w_3^\perp]=r$.

*Subcase sb-down:* In this case the dual cone is spanned by

$$w_1^\perp=(-a+\alpha, b-s+\alpha, \alpha), \quad w_2^\perp=(\beta, -b+\beta, c-s+\beta), \quad w_3^\perp=(a-s+\gamma, \gamma, -c+\gamma), \tag{16}$$

with $\alpha+\beta+\gamma=s+1$ and $\alpha+\beta+1<a$ and the corresponding cyclic permutations. From the condition that $m-w_i^\perp$ does not belong to $\overline{\Lambda}_+$ we obtain

$$(m_2-b+s-\alpha)(m_3-\alpha)<0, \quad (m_3-c+s-\beta)(m_1-\beta)<0, \quad (m_1-a+s-\gamma)(m_2-\gamma)<0.$$

Also the fact that $w_1^\perp+w_2^\perp$ and its cyclic permutations are in the invariant lattice give new relations for the possible minima. Counting the lattice points which satisfy these relations we see we get exactly $\det[w_1^\perp, w_2^\perp, w_3^\perp]=r$.

Now, we need to prove that for each $\tau$ a basic triangle of $\mathcal{T}$, the $\Lambda_\tau$-module $\Lambda_\tau M_j$ is free and generated by the element $m_j(\tau)$. For this, we must verify that the map

$$\Lambda_\tau \to \Lambda_\tau M_j$$

which takes

$$m \to m+m_j(\tau)$$

is surjective (since it is clearly injective). Let $m$ be in $M_j$. By the definition of $\rho_j(w_i)$ we must have $\langle m, w_i \rangle \geq \frac{1}{r}\rho_j(w_i)$. We need to show that $m-m_j(\tau)$ is in $\Lambda_\tau$. By construction $m-m_j(\tau)\in\Lambda$ and $m-m_j(\tau)=\alpha_1 w_1^\perp+\alpha_2 w_2^\perp+\alpha_3 w_3^\perp$, for some integers $\alpha_1,\alpha_2,\alpha_3$. But now $\alpha_1=\langle m-m_j(\tau), w_1^\perp \rangle \geq 0$ etc, and it gives that $m\in\Lambda_\tau$.

Moreover, given any $u\in\bigcap \Lambda_\tau M_j$ it is actually in $M_j$. This is true since for any $\tau$ basic triangle in $\mathcal{T}$ the element $m-m_j(\tau)$ is in all $\Lambda_\tau$. Since $\bigcap\Lambda_\tau=\Lambda_+$, it follows that $u\in M_j$. In this way we proved that $\mathcal{R}_j=\pi^*(\mathcal{O}_j)$ are locally free sheaves of rank 1. ☐

**Lemma 3.** *Let $l_0$ be an edge in the triangulation $\mathcal{T}$ of $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$ which determines a $(-1,-1)$ curve $C_0$. Then $\deg\mathcal{R}_j|_{C_0}$ is $0$ or $1$.*

*Proof.* This is a result already noticed by Craw and Ishii [CI, Proposition 9.7.]. We reprove it here using our notation.

Let $\tau_1$ and $\tau_2$ be two basic triangles arriving via the regular tesselation which have a common edge $l_0$. Without loss of generality we can assume that $\tau_1$ is an up triangle – corresponding to $\alpha+\beta+\gamma=s-1$ – and $\tau_2$ is a down triangle – corresponding to

$(\alpha+1)+(\beta+1)+\gamma = s+1$. We denote by $C_0$ the $(-1,-1)$-curve corresponding to $l_0$ (see left side of Figure 3).

For $R_j$ an irreducible representation of $G$, we have $m_j(\tau_1)$ and $m_j(\tau_2)$ given by Lemma 2 which correspond respectively to the generators of $\mathcal{R}_j|_{U_{\tau_1}}$ and $\mathcal{R}_j|_{U_{\tau_1}}$.

If $m_j(\tau_1) = m_j(\tau_2)$, then $\deg \mathcal{R}_j|_{C_0} = 0$. On the other hand, if $m_j(\tau_1) \neq m_j(\tau_2)$ it means that $m_j(\tau_1)$ is not a minimum for the point $D$. Therefore we get the corresponding minimum $m_j(\tau_2) = m_j(\tau_1) + k l_0^{\perp}$, where $k$ is a strictly positive integer. We claim that $k = 1$ which gives that $\deg(\mathcal{R}_j|_{C_0}) = 1$.

Assume that $k \geq 2$. Then from (15) we get $-l_0^{\perp} = (-a+s-\gamma, -\gamma, c-\gamma)$. Also the discussion in Lemma 2 for the case of an up triangle gives that $0 \leq (m_j(\tau_2))_3 < c - \gamma$. We have a contradiction for $k \geq 2$. □

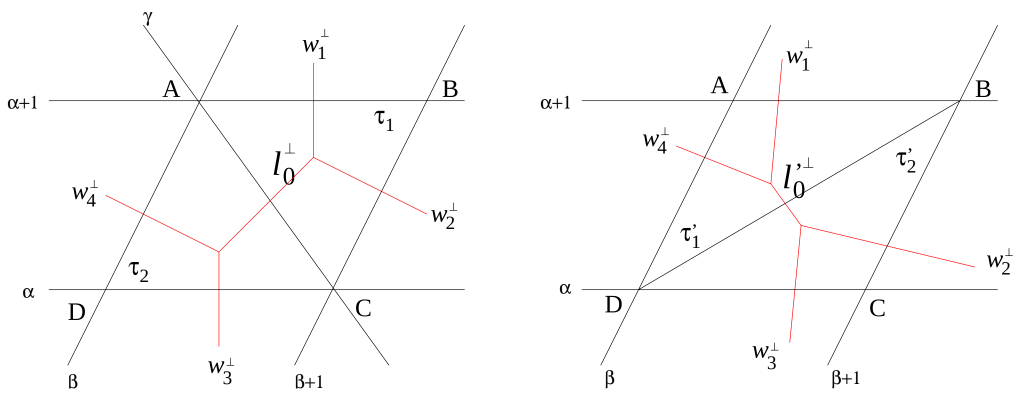


**Figure 3**. The flop of the curve corresponding to $l_0$.

**Lemma 4.** *Let $\mathcal{T}'$ be the triangulation obtained by flopping the edge $l_0$. Then, the proper transforms $\mathcal{R}'_j$ on the crepant resolution $(X_{\mathcal{T}'}, \pi_{\mathcal{T}'})$ have the property that $\mathcal{R}'_j|_{U_\tau}$ is generated by $m_j(\tau)$, for each $\tau$ a basic triangle in $\mathcal{T}'$.*

*Proof.* We continue in the same set-up as in the proof of Lemma 3.

In the new triangulation $\mathcal{T}'$ we denote by $l'_0$ the transformation of the edge $l_0$ (see Figure 3), and by $\tau'_1$ and $\tau'_2$ the new basic triangles.

If $m_j(\tau_1) = m_j(\tau_2)$ then they are also equal to $m_j(\tau'_1) = m_j(\tau'_2)$, and the conclusion follows.

Now assume that $m_j(\tau_1) \neq m_j(\tau_2)$. The proof of the previous Lemma gives $m_j(\tau_2) = m_j(\tau_1) + l_0^{\perp}$. Also, since we are dealling with basic triangles we have

$$l_0^{\perp} \;=\; (1,1,1) - w_1^{\perp} - w_2^{\perp} \;=\; -(1,1,1) + w_3^{\perp} + w_4^{\perp}.$$

The monomials corresponding to $m_j(\tau_1)$ and $m_j(\tau_2)$ are a set of generators for the $\Lambda_+$-module $\Lambda_+ M_j$. However the module of holomorphic sections of $\mathcal{R}'_j|_{\mathcal{U}_{\tau'_1}}$ is $\Lambda_{\tau'_1} M_j$ (recall that $\Lambda_{\tau'_1} = \mathbb{C}[w_1^{\perp}, w_4^{\perp}, l_0'^{\perp}]$). Take $m_j(\tau_1) - w_1^{\perp}$. Of course this belongs to $\Lambda_{\tau'_1} M_j$, and it can be seen that it generate it as a $\Lambda_{\tau'_1}$-module. This is because

$$\begin{aligned}
\langle\, m_j(\tau_1) - w_1^{\perp}, A\,\rangle &= \langle\, m_j(\tau_1), A\,\rangle = \frac{1}{r}\rho_j(A),\\
\langle\, m_j(\tau_1) - w_1^{\perp}, B\,\rangle &= \langle\, m_j(\tau_1), B\,\rangle = \frac{1}{r}\rho_j(B),\\
\langle\, m_j(\tau_1) - w_1^{\perp}, D\,\rangle &= \langle\, m_j(\tau_2), D\,\rangle - \langle\,(1,1,1), D\,\rangle + \langle\, w_2^{\perp}, D\,\rangle\\
&= \langle\, m_j(\tau_2), D\,\rangle = \frac{1}{r}\rho_j(D).
\end{aligned}$$

For the last equation we used the fact that $D$ has the sum of its components adding up to 1. It follows that $m_j(\tau_1) - w_1^{\perp} = m_j(\tau'_1)$. □

*Remark.* In the same way, we have $m_j(\tau'_2) = m_j(\tau_2) - w_3^{\perp}$ proving that $m_j(\tau'_2) = m_j(\tau'_1) + l_0^{\perp}$, i.e. that $\deg(\mathcal{R}'_j|_{l'_0}) = 1$. This is actually true in general, under a flop if $\deg(\mathcal{L}|_{l_0}) = a$ then the proper transform satisfies $\deg(\mathcal{L}'|_{l'_0}) = a$.

To finish the proof, we take the bundle $\mathcal{S}_j$ to be $\mathcal{R}_j$ in the case of $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3, \mathrm{G})$ and we take them to be the proper transforms of $\mathcal{R}_j$ under a sequence of flops which makes us arrive at the given crepant resolution.

**Corollary 3.** *With the notations from the previous Lemma, it also follows that*

$$\rho_j(A) + \rho_j(C) - \rho_j(B) - \rho_j(D) = -k,$$

*where* $k = \deg(\mathcal{S}_j|_{l_0})$. *(Note that in fact* $k$ *it is* 0 *or* 1.*)*

*Proof.* This follows since if $m_j(\tau_2) = m_j(\tau_1) + k l_0^{\perp}$ then

$$\begin{aligned}
\rho_j(A)+\rho_j(C)-\rho_j(B)-\rho_j(D) &= \langle\, m_j(\tau_1), A\,\rangle + \langle\, m_j(\tau_2), C\,\rangle - \langle\, m_j(\tau_2), B\,\rangle - \langle\, m_j(\tau_1), D\,\rangle\\
&= \langle\, m_j(\tau_1), A - D\,\rangle + \langle\, m_j(\tau_2), C - B\,\rangle\\
&= k\langle\, l_0^{\perp}, C - B\,\rangle\\
&= -k.
\end{aligned}$$

□

## 3. Triple Products

We base our analysis here on previous results from [De2]. Let $G \subset SL(3,\mathbb{C})$ be a finite subgroup which acts with an isolated singularity on $\mathbb{C}^3$. Let $X$ be a crepant resolution of $\mathbb{C}^3/G$. It is endowed with a Ricci-flat ALE metric – meaning that at infinity it is close to the orbifold metric $h_0$ on $\mathbb{C}^3/G$ – of the form $g = h_0 + \mathcal{O}(r^{-6})$ and with appropriate decays on the derivative of the metric. Also, the tautological line bundles $\mathcal{R}_j$ come with canonical $L^2$-connections. In this way we think of the first Chern class $c_1(\mathcal{R}_j)$ as being a class in the $L^2$-cohomology of $X$. It is known that the $L^2$-cohomology in degree $k$ of an ALE manifold $X$ is the image of $H^k_c(X,\mathbb{Q})$ into $H^k(X,\mathbb{Q})$ [APS].

**Theorem 5.** *Let $G$ be a finite subgroup of $SL(3,\mathbb{C})$ which acts with an isolated singularity at the origin. Let $X$ be a crepant resolution of $\mathbb{C}^3/G$. For each irreducible representation $R_j$ of $G$ we consider the tautological line bundle $\mathcal{R}_j$ – upon choosing a chamber $C \subset \mathcal{M}_C$ so that $X = \mathcal{M}_C$. Then*

$$\int_X c_1(\mathcal{R}_j)^3 = \frac{1}{|G|}\sum_{\substack{g\in G\\ age(g)=1}} \rho_j(v_g) + ra_j(g))\,(n_g-6) + \frac{3}{|G|}\sum_{\substack{g\in G\\ g\neq 1}} \frac{\mathrm{Trace}\,(g,R_j)}{-\mathrm{Trace}\,(g,Q)+\mathrm{Trace}\,(g,\Lambda^2 Q)}. \tag{17}$$

*Here $n_g$ denotes the number of edges emanating out of the lattice point $v_g$ in the regular triangulation $\mathcal{T}$ of the junior triangle $\Delta$ determined by $X$. The integers $a_j(g)$ come from $E_j = \sum_{age(g)=1} a_j(g)D_g$ which is the twisting divisor for each irreducible representation $R_j$ introduced by the choice of the chamber $C$.*

*Proof.* We consider the Dirac operator on $X$ twisted by the line bundle $\mathcal{R}_j$. We complete the space of smooth twisted spinors in $\alpha$-weighted Sobolev norm – a function which is bounded in this norm behaves like $r^{-\alpha}$ at infinity. For all, but a discrete number of weights the completion of this operator is Fredholm, and moreover for the weight in the interval $(0,5)$ the index is given

$$\mathrm{index} D^+_{\mathcal{R}_j} = \int_X \mathrm{ch}(\mathcal{R}_\mathrm{j})\hat{\mathrm{A}}(\mathrm{X}) - \frac{\eta_{\mathrm{R_j}}}{2}. \tag{18}$$

Here $R_j$ is the fiber at infinity of the line bundle $\mathcal{R}_j$, and $\eta_{R_j}$ is the $\eta$-invariant, a spectral invariant of $S^5/G$ the boundary at infinity. For our choice of the weight $\alpha \in (0,5)$

$$\eta_{R_j} = -\frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq 1}} \frac{\mathrm{Trace}\,(g,\rho_j)}{-\mathrm{Trace}\,(g,\mathbb{C}^3)+\mathrm{Trace}\,(g,\Lambda^2\mathbb{C}^3)}.$$

Also, the main result in [De2], a vanishing theorem for the index of the Dirac operator, gives that

$$\mathrm{index} D^+_{\mathcal{R}_j} = 0, \tag{19}$$

so that the formula (18) becomes

$$\int_X \mathrm{ch}(\mathcal{R}_\mathrm{j})\hat{\mathrm{A}}(\mathrm{X}) = \frac{\eta_{\mathcal{R}_\mathrm{j}}}{2}. \tag{20}$$

The left-hand-side of the above formula can be written explicitly as

$$\frac{1}{6}\int_X c_1(\mathcal{R}_j)^3 \; - \; \frac{1}{12}\int_X c_1(\mathcal{R}_j)\, c_2(X).$$

From formula (14) it follows that the second term is

$$-\frac{1}{12\,|G|} \sum_{\substack{g\in G \\ \mathrm{age\,(g)=1}}} (\rho_j(v_g) + ra_j(g))\; c_2(X)|_{D_g}.$$

Recall that $D_g$ is the crepant divisor corresponding to $\mathrm{Star}(v_g)$. We have

$$\begin{aligned} c_2(X)|_{D_g} &= c_2(D_g) + c_1(D_g)\, c_1(\mathcal{N}_{D_g/X}) \\ &= c_2(D_g) - c_1(D_g)^2, \end{aligned}$$

where the last equality comes from $c_1(D_g) = -c_1(\mathcal{N}_{D_g/X})$ due to the fact that $X$ is Calabi-Yau.

Since the crepant divisors are toric, it follows that they are rational surfaces. Therefore they are obtained either from the projective plane $\mathbb{C}P^2$ or the Hirzebruch surfaces $\mathbb{F}_n$ through a finite number of blow-ups [GH].

In the situation when the crepant divisor $D_g$ is obtained from $\mathbb{C}P^2$ through a series of $k$ blow-ups

$$c_2(D_g) = 3 + k,$$

being the Euler number of $D_g$. Also

$$c_1(D_g)^2 = c_1(\mathbb{C}P^2)^2 - k = 9 - k.$$

On the other hand, in the toric picture there will be $k+3$ edges emanating from $v_g$ the junior lattice point corresponding to $g$. This means

$$c_2(X)|_{D_g} = 2(k+3) - 12. \tag{21}$$

In the same way, in the situation when the crepant divisor $D_g$ is obtained from the Hirzebruch surface $\mathbb{F}_n$ through a series of $k$ blow-ups, it follows that

$$c_2(X)|_{D_g} = 2(k+4) - 12, \tag{22}$$

with $k+4$ being the number of edges emanating from $v_g$.

The above two formulas can be put together into

$$c_2(X)|_{D_g} = 2(n_g - 6), \tag{23}$$

with $n_g$ the number of edges out of $v_g$ in the triangulation $\mathcal{T}$. □

*Remark.* When the crepant resolution is $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^3)$, Craw and Reid's algorithm gives that every interior lattice point $v_g$ has valency 3, 4, 5 or 6, thus giving crepant divisors which are respectively the projective plane $\mathbb{C}P^2$, a Hirzebruch surface $\mathbb{F}_n$, a Hirzebruch surface blown up at one or two points. The last one includes $dP_6$, the del Pezzo surface of degree 6, which corresponds to the lattice points which are in the interior of the regular triangles. In this way the crepant divisors corresponding to $dP_6$ do not contribute to the formula (17).

**Corollary 4.** *Let $G$ be a finite subgroup of $SL(3,\mathbb{C})$ which acts with an isolated singularity at the origin. Let $X$ be a crepant resolution of $\mathbb{C}^3/G$ and let $C \subset \Theta$ be a chamber so that $X = \mathcal{M}_C$ and so that the tautological line bundles on $X$ are $\mathcal{R}_0, \dots, \mathcal{R}_{r-1}$. Let $X'$ be the crepant resolution obtained by crossing a wall of type* 1 *of $C$ – the unstable locus is a $(-1,-1)$-curve $l_0$ – and let $\mathcal{R}'_0, \dots, \mathcal{R}'_{r-1}$ denote the corresponding tautological line bundles. Then*

$$\begin{aligned} \int_X c_1(\mathcal{R}'_j)^3 &= \int_X c_1(\mathcal{R}'_j)^3 \qquad && \textit{if } \deg(\mathcal{S}_j|_{l_0}) = 0 \\ \int_X c_1(\mathcal{R}'_j)^3 &= \int_X c_1(\mathcal{R}'_j)^3 - 1 \qquad && \textit{if } \deg(\mathcal{S}_j|_{l_0}) = 1. \end{aligned} \tag{24}$$

*Here $\mathcal{S}_0, \dots, \mathcal{S}_{r-1}$ are the holomorphic line bundles on $X$ uniquely associated to each irreducible representation by Proposition 2.*

*Proof.* By Corollary 2, for each irreducible representation $R_j$ there exists a divisor $E_j = \sum_{age(g)=1} a_j(g) D_g$ so that $\mathcal{R}_j = \mathcal{S}_j \otimes \mathcal{O}_X(-E_j)$. The proper transform of this bundle is $\mathcal{R}'_j = \mathcal{S}'_j \otimes \mathcal{O}_X(-E'_j)$ where $E'_j$ is the proper transform of $E_j$, i.e. $E'_j = \sum_{age(g)=1} a_j(g) D'_g$ with $D'_g$ the crepant divisors on $X'$. Theorem 5 gives

$$\begin{aligned} \int_X c_1(\mathcal{R}'_j)^3 - \int_X c_1(\mathcal{R}_j)^3 &= \rho_j(A) + \rho_j(C) - \rho_j(B) - \rho_j(D) + r(a_j(A) + a_j(C) - a_j(B) - a_j(D)) \\ &= -\deg(\mathcal{S}_j|_{l_0}) + r\left(a_j(A) + a_j(C) - a_j(B) - a_j(D)\right). \end{aligned}$$

Here the curve which is flopped is $l_0$ corresponding to a line $AC$ in the junior triangle $\Delta$ and the transformed curve $l'_0$ corresponds to the line $BD$ (see Figure 3). We applied Corollary 3 to deduce the last equality. Now the last parenthesis is nothing else but $r(\deg(\mathcal{O}_X(E_j)|_{l_0}) - \deg(\mathcal{O}_{X'}(E'_j)|_{l'_0}))$ which is zero. The conclusion follows. $\square$

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