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Quasipositivity Problem for 3-Braids

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Abstract

A braid is called *quasipositive* if it is a product of conjugates of standard generators of the braid group. We present an algorithm deciding if a given braid with three strings is quasipositive or not. The complexity (the time of work) of our algorithm is $O(n^{k+1})$ where n is the length of the word in standard generators representing the braid and k is the algebraic length of the braid. The algorithm is based on the Garside normal form.

The problem of quasipositivity in braid groups is motivated by the topology of plane real algebraic curves (16th Hilbert's problem). In particular, our result can be interpreted as a classification of trigonal real pseudoholomorphic curves on rational ruled surfaces.

Let G be a group and $\mathcal{X}$ a fixed set of its elements. An element $g \in G$ is called $\mathcal{X}$-*quasipositive* if $g = \prod_j a_j x_j a_j^{-1}$ where $a_j \in G$ and $x_j \in \mathcal{X}$. We shall give a solution for the quasipositivity problem (i.e. we shall present an algorithm deciding if a given element is quasipositive or not) for a free group with any number of generators (Section 1) and for the group of braids with three strings (the rest of the paper). In both cases $\mathcal{X}$ is the set of standard generators.

The complexity (the time of work) of our algorithm is $O(n^{k+1})$ where n is the length and k is the algebraic length (the exponent sum) of the word.

The result on the free group is not new (see Remark 1.1) but we present it here because our proof of this result serves as a model of the proof for 3-braids. The main ingredient of the proof is the Garside normal form of a braid.

The term "quasipositive braid" was introduced by Lee Rudolph [5]. For us, the main motivation of the quasipositivity problem comes from the topology of plane real algebraic curves (see details in [2, 3]). A necessary condition for existence of a real algebraic curve realizing a given isotopy type, is the quasipositivity of a certain braid. If one enlarge the class of real algebraic curves up to the class of real pseudoholomorphic curves, then this condition becomes necessary and sufficient. In particular, the result of this note can be interpreted as a classification of trigonal real pseudoholomorphic curves on rational ruled surfaces.

Key words and phrases. Braid group, quasipositive braids.

1. Quasipositivity problem in free group

Let $\mathcal{X}$ be any set and $\mathbf{F}_{\mathcal{X}}$ be the free group generated by $\mathcal{X}$. We shall call $\mathcal{X}$-quasipositive elements just *quasipositive*. In the set of words in alphabet "`[`", "`*`", "`]`", we define a subset of *regular bracket structures* (RBS) recursively: the empty word and the word `*` are RBS; if a and b are RBS then ab and `[`a`]` also are.

Let $\mathcal{X}^{-1} = \{x^{-1} \,|\, x \in \mathcal{X}\}$. A word $w = x_1 x_2 \dots x_n$ in alphabet $\mathcal{X} \cup \mathcal{X}^{-1}$ is called *quasipositive* if there exists an RBS $u_1 u_2 \dots u_n$ which *agrees with* w, i.e. such that

(1) if $u_j = *$ then $x_j \in \mathcal{X}$;
(2) if u_j is the bracket matching to u_i then $x_j = x_i^{-1}$.

Proposition 1.1. *Any word representing a quasipositive element of* $\mathbf{F}_{\mathcal{X}}$*, is quasipositive.*

Proof. By definition, any quasipositive element can be represented by a quasipositive word $\prod a_j x_j a_j^{-1}$. If a word w is obtained by inserting xx^{-1} or $x^{-1}x$ into a quasipositive word then w is also quasipositive (one should just insert "`[]`" into the corresponding place of the RBS). It remains to prove that if a word w' is obtained by removing xx^{-1} or $x^{-1}x$ from a quasipositive word w then w' is quasipositive.

Let $w = x_1 \dots x_n$. Let $u_1 \dots u_n$ be the corresponding RBS and let w' be obtained from w by removing $x_i x_{i+1}$ where either $x_i = x$ and $x_{i+1} = x^{-1}$, or $x_i = x^{-1}$ and $x_{i+1} = x$ for some $x \in \mathcal{X}$. Let us consider separately all the possibilities for the word $u_i u_{i+1}$.

Case 1. "`**`". Impossible because $x_i x_{i+1}$ contains x^{-1}.

Case 2. "`*]`" or "`*[`" (the case of "`[*`" and "`]*`" is analogous). We have $x_i = x \in \mathcal{X}$, $x_{i+1} = x^{-1}$. Let u_j be the bracket matching to u_{i+1}. Then we have $x_j = x$, and the word obtained from $u_1 \dots u_n$ by deleting $u_i u_{i+1}$ and replacing u_j with "`*`", is an RBS which agrees with w'.

Case 3. "`[]`" or "`][`". Deleting $u_i u_{i+1}$ yields an RBS which agrees with w'.

Case 4. "`[[`" (the case of "`]]`" is analogous). Let $u_j = u_k =$ `]`, $j < k$, be the brackets matching to u_i and u_{i+1}. Removing $u_i u_{i+1}$ and replacing u_j with "`[`", we obtain an RBS which agrees with w'.

To check that the obtained words are RBS, it is convenient to use the following criterion. A word in alphabet "`[`", "`*`", "`]`" is an RBS if and only if the number of "`[`" is equal to the number of "`]`", and for any initial subword, the number of "`[`" is not less than the number of "`]`". □

Corollary 1.2. *A word in alphabet* $\mathcal{X} \cup \mathcal{X}^{-1}$ *defines a quasipositive element if and only if after removing some positive generators one obtains a word representing the unit of the group* $\mathbf{F}_{\mathcal{X}}$*.*

Remark 1.1. According to a result of Blank [4], the question of extendibility of an immersion $S^1 \to \mathbf{R}^2$ to an immersion $D^2 \to \mathbf{R}^2$ (where D^2 is a disk and S^1 is its boundary) can be reduced to the quasipositivity problem in a free group. An algorithm of recognizing quasipositive words is given in [4], thus, our Proposition 1.1 is not new.

2. Garside normal form in the group of braids with three strings

Let $B_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be the group of braids with three strings. Let Δ be the Garside element: $\Delta = \sigma_1\sigma_2\sigma_1$. Let $\mathfrak{r} : B_3 \to B_3$ be the automorphism defined by $\sigma_1 \mapsto \sigma_2$, $\sigma_2 \mapsto \sigma_1$. It is easy to check that

$$\Delta\beta = \mathfrak{r}(\beta)\Delta, \qquad \beta \in B_3. \tag{1}$$

Let $\mathfrak{a}$ be the homomorphism of B_3 to the additive group of integers $\mathbf{Z}$, such that $\mathfrak{a}(\sigma_1) = \mathfrak{a}(\sigma_2) = 1$, i.e. $\mathfrak{a}(\beta)$ is the exponent sum (the algebraic length) of β.

Let B_3^+ be the submonoid of B_3 generated by σ_1, σ_2. Elements of B_3^+ are called *positive braids*. For $\alpha, \beta \in B_3^+$, let us say that β is *left* (resp., *right*) *divisible by* α if there exists $\gamma \in B_3^+$ such that $\beta = \alpha\gamma$ (resp., $\beta = \gamma\alpha$). A word in σ_1, σ_2, is called *positive*.

Lemma 2.1. (see [1], Theorem 4). *Two positive words are equal in B_3 if and only if they can be obtained from each other by applying successively the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.*

Corollary 2.2. *A positive word is left or right divisible by Δ if and only if it contains the subword $\sigma_1\sigma_2\sigma_1$ or $\sigma_2\sigma_1\sigma_2$.*

Proof. If w contains such a subword then $w = \alpha\Delta\beta = \Delta\,\mathfrak{r}(\alpha)\beta = \alpha\,\mathfrak{r}(\beta)\Delta$ by (1). Otherwise, by Lemma 2.1, the word w cannot be equal in B_3 to any other positive word. □

Corollary 2.3. *An element of B_3^+ not divisible by Δ can be represented by a positive word in a unique way.*

Definition. *A Garside decomposition* of a braid $\beta \in B_3$ is its presentation

$$\beta = \beta_+\Delta^m, \qquad \beta_+ \in B_3^+. \tag{2}$$

If β_+ is not divisible by Δ then the decomposition (2) is called the *Garside normal form* of β. In this case, m is called the *power* of β.

To find a Garside decomposition, we replace each occurrence of σ_1^{-1} with $\sigma_2\sigma_1\Delta^{-1}$, and σ_2^{-1} with $\sigma_1\sigma_2\Delta^{-1}$, and then push all Δ^{-1} to the right using (1). To find further the Garside normal form, one should successively replace all subwords $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ with Δ, and push them to the right using (1).

3. Quasipositivity problem in the group of braids B_3

A braid with three strings (an element of B_3) is called *quasipositive* if it is $\mathcal{X}$-quasipositive for $\mathcal{X} = \{\sigma_1, \sigma_2\}$. It is clear that if a braid admits a decomposition (2) with a non-negative m then it is quasipositive. A criterion of the quasipositivity in the case of a negative m is as follows.

Proposition 3.1. *Let $\beta = \beta_+\Delta^m$ where β_+ is a positive word (i.e. a word in σ_1, σ_2), and $m \le 0$. The braid β is quasipositive if and only if one can delete some letters from β_+ so that the obtained word is equal in B_3^+ to the word Δ^{-m}.*

This proposition follows immediately from Proposition 3.2 below. It provides the following algorithm to decide if a given word in $\sigma_1^{\pm 1}$, $\sigma_2^{\pm 1}$ represents a quasipositive braid. Let k be the algebraic length of the braid.

Algorithm.

1. Compute the Garside normal form $\beta_+\Delta^m$ of the given braid β.
2. Try to remove k letters from β_+ in all possible ways, each time computing the Garside normal form of the obtained braid. If we obtain at least once the trivial braid then the braid β is quasipositive. Otherwise it is not.

It is clear that the complexity (the time of work) of this algorithm is $O(n^{k+1})$ where n is the length of the initial word.

Now we proceed to a proof of Proposition 3.1. In the set of words composed of the characters "[", "|", "]", "*", let us define a subset of *regular bracket structures with delimiters* (RBSD) recursively as follows: the empty word and the word $*$ are RBSD; if a and b are RBSD then ab and $[a\,|\,b]$ also are. If a is an RBSD then its *wight* $\ell(a)$ is defined as the number of occurrences of the character "|". It is clear that any occurrence of one of the characters "[", "|", or "]" into an RBSD w, uniquely determines the occurrences of the other two characters such that $w = a\,[b\,|\,c]\,d$ where b, c, and ad are RBSD. Such a mutual occurrance of "[", "|", and "]" is called a *regular mutual occurrance.* Let us say that a positive word $x_1 \dots x_n$, $x_j \in \{\sigma_1, \sigma_2\}$, *agrees* with an RBSD $u_1 \dots u_n$, if for any regular mutual ocuurrance of $u_i = [$, $u_j = |$, $u_k =]$, into $u_1 \dots u_n = au_ibu_jcu_kd = a\,[b\,|\,c]\,d$, we have either

$$x_i = \sigma_1,\ \ x_j = \mathfrak{r}^{\ell(b)}(\sigma_2),\ \ x_k = \mathfrak{r}^{\ell(bc)}(\sigma_1), \quad \textit{or} \quad x_i = \sigma_2,\ \ x_j = \mathfrak{r}^{\ell(b)}(\sigma_1),\ \ x_k = \mathfrak{r}^{\ell(bc)}(\sigma_2).$$

Proposition 3.2. *Let $b = w\Delta^m$ where w is a positive word and $m \le 0$. The braid b is quasipositive if and only if there exists an RBSD of the weight $-m$ which agrees with w.*

This statement easily follows from the results of Section 2 and the following lemma.

Lemma 3.3. *Let $w = x_1 \dots x_n$ be a positive word which agrees with some RBSD of a non-zero weight. Suppose that w contains a subword $x_ix_{i+1}x_{i+2} = \sigma_1\sigma_2\sigma_1$. Then there exists an RBSD $u_1 \dots u_n$ of the same weight agreeing with w such that $u_iu_{i+1}u_{i+2} =$ [|] .*

Proof. In the RBSD agreeing with w, let us consider the subword v which corresponds to $x_ix_{i+1}x_{i+2}$. If $v = ***$, then we replace v with [|] and we replace an arbitrary regular mutual occurrance ... [... | ...] ... with $\cdots * \cdots * \cdots * \cdots$.

If v intersects only with one regular mutual occurrance of [|] , then we replace v with [|] , and we replace with $*$ each element of this regular mutual occurrance which does not belong to v. If v intersects with more than one regular mutual occurrance of [|] , then we shall consider separately all possibilities for v up to symmetry:

$$\begin{aligned}
&\underset{[}{1}\underset{[}{2}\underset{|}{1}\dots\underset{]}{2}\dots\underset{|}{1}\dots\underset{]}{2} \longrightarrow \underset{[}{1}\underset{|}{2}\underset{]}{1}\dots\underset{[}{2}\dots\underset{|}{1}\dots\underset{]}{2}\\
&\underset{[}{2}\dots\underset{|}{1}\underset{[}{2}\underset{|}{1}\dots\underset{]}{2}\dots\underset{]}{1} \longrightarrow \underset{[}{2}\dots\underset{[}{1}\underset{|}{2}\underset{]}{1}\dots\underset{|}{2}\dots\underset{]}{1}\\
&\underset{[}{2}\dots\underset{|}{1}\underset{]}{2}\underset{[}{1}\dots\underset{|}{2}\dots\underset{]}{1} \longrightarrow \underset{[}{2}\dots\underset{[}{1}\underset{|}{2}\underset{]}{1}\dots\underset{|}{2}\dots\underset{]}{1}
\end{aligned}$$

```
121...2...1...2...1 ⟶ 121...2...1...2...1
*[[   |   ]   |   ]     [|]   *   [   |   ]
1...121...2...1...2 ⟶ 1...121...2...1...2
[   *|[   |   ]   ]     *   [|]   [   |   ]
2...1...121...2...1 ⟶ 2...1...121...2...1
[   |   *][   |   ]     [   *   [|]   |   ]
1...2...1...121...2 ⟶ 1...2...1...121...2
[   [   |   *]|   ]     [   |   ]   [|]   *
2...1...2...1...121 ⟶ 2...1...2...1...121
[   |   [   |   *]]     [   |   ]   *   [|]
2...121...2...1...1 ⟶ 2...121...2...1...1
[   |*[   |   ]   ]     [   [|]   |   ]   *
1...2...121...2...1 ⟶ 1...2...121...2...1
[   |   ]*[   |   ]     [   |   [|]   ]   *

121...2...1...2...1...2...1 ⟶ 121...2...1...2...1...2...1
[[[   |   ]   |   ]   |   ]     [|]   [   |   ]   [   |   ]
2...121...2...1...2...1...2 ⟶ 2...121...2...1...2...1...2
[   |[[   |   ]   |   ]   ]     [   [|]   |   ]   [   |   ]
1...2...121...2...1...2...1 ⟶ 1...2...121...2...1...2...1
[   |   ][[   |   ]   |   ]     [   [   [|]   |   ]   |   ]
2...1...2...121...2...1...2 ⟶ 2...1...2...121...2...1...2
[   [   |   ]|[   |   ]   ]     [   |   ]   [|]   [   |   ]
```

To simplify notation, we omit $\mathfrak{r}^{\ell(\cdots)}$ in the above list of modifications and we write 1 and 2 instead of σ_1 and σ_2. For example, the first line on this page should be understood as

$$\underset{\texttt{*}\ \texttt{[}\ \texttt{[}}{\sigma_1\sigma_2\sigma_1}\ a\ \mathfrak{r}^{\ell(\tilde a)}\underset{\texttt{|}}{(\sigma_2)}\ b\ \mathfrak{r}^{\ell(\tilde a\tilde b)}\underset{\texttt{]}}{(\sigma_1)}\ c\ \mathfrak{r}^{\ell(\tilde a\tilde b\tilde c)}\underset{\texttt{|}}{(\sigma_2)}\ d\ \mathfrak{r}^{\ell(\tilde a\tilde b\tilde c\tilde d)}\underset{\texttt{]}}{(\sigma_1)}$$
$$\downarrow$$
$$\underset{\texttt{[}\ \texttt{|}\ \texttt{]}}{\sigma_1\sigma_2\sigma_1}\ a\ \mathfrak{r}^{\ell(\tilde a)}\underset{\texttt{*}}{(\sigma_2)}\ b\ \mathfrak{r}^{\ell(\tilde a\tilde b)}\underset{\texttt{[}}{(\sigma_1)}\ c\ \mathfrak{r}^{\ell(\tilde a\tilde b\tilde c)}\underset{\texttt{|}}{(\sigma_2)}\ d\ \mathfrak{r}^{\ell(\tilde a\tilde b\tilde c\tilde d)}\underset{\texttt{]}}{(\sigma_1)}\,,$$

where a, b, c, d are arbitrary words in σ_1, σ_2, and $\tilde a, \tilde b, \tilde c, \tilde d$ the corresponding RBSD ($\tilde a, \tilde b, \tilde c, \tilde d$ are not changed under the modification). □

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