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Moments Equalities for Nonnegative Integer-Valued Random Variables

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Abstract

We present and prove two theorems about equalities for the nth moment of nonnegative integer-valued random variables. These equalities generalize the well known equality for the first moment of a nonnegative integer-valued random variable X in terms of its cumulative distribution function, or in terms of its tail distribution.

Key words and phrases: Expectation, Moments, Equalities.

1. Introduction

There is a well-known equality for the nth moment of a nonnegative random variable Y as an integral of a function of its tail distribution. A similar equality for the first moment of a nonnegative integer-valued random variable as a sum over x of a function of its tail distribution is also well known and used a lot in the literature (See [1, p. 43], for example). What we prove in this paper is a generalization of this sum equality when the random variable is integer-valued.

In the next section we will prove a generalization of the well-known equality in the discrete case. Our equality gives a neat formula of the nth moment, when it exists, for nonnegative integer-valued variables.

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2. Main Theorems

In this section we prove two identities that each will be used to prove one of our main theorems. The first identity is used to express a product of terms of the form $(X-i)$ as a finite sum of products of similar terms when the sum ranges from 1 to a nonnegative integer x. The second identity is used to express x^n as a finite sum that ranges from 1 to a nonnegative integer x.

Before we proceed to the main theorems, we need the following lemma.

Lemma 2.1 *Let x and n be nonnegative integers such that $x > n$. Then*

$$\binom{x}{n+1} = \sum_{j=1}^{x-n} \binom{x-j}{n}.$$

Proof. Apply Pascal's identity, namely

$$\binom{x}{n+1} = \binom{x-1}{n} + \binom{x-1}{n+1},$$

to each last term on the right-hand side of the resulting equation. Continue this procedure $x-n-1$ times to get

$$\begin{aligned}\binom{x}{n+1} &= \sum_{j=1}^{x-n-1} \binom{x-j}{n} + \binom{n+1}{n+1} \\ &= \sum_{j=1}^{x-n} \binom{x-j}{n}.\end{aligned}$$

□

Lemma 2.2 *Let $g_n(x) := \sum_{i=1}^{x} \prod_{j=1}^{n}(i-j)$ and $f_n(x) := \prod_{j=0}^{n}(x-j)$, where x and n are nonnegative integers such that $x > n$. Then $(n+1)g_n(x) = f_n(x) = (x)_{n+1}$, where $(z)_m = z(z-1)\dots(z-m+1)$.*

Proof. Notice that

$$f_n(x) = x(x-1)\cdots(x-n) = (x)_{n+1} = (n+1)!\binom{x}{n+1}.$$

We also notice that $g_n(x)$ is a finite sum of terms of the form $(x-j)(x-j-1)(x-j-2)\cdots$ of length n. Such a general term can be expressed as $(x-j)_n$ for $j = 1, 2, \ldots, n$. The number of these terms is $x - n$. Therefore,

$$g_n(x) = \sum_{j=1}^{x-n} (x-j)_n. \tag{2.1}$$

Since $(x-j)_n = n!\binom{x-j}{n}$, (2.1) can be written as

$$g_n(x) = n! \sum_{j=1}^{x-n} \binom{x-j}{n}.$$

By Lemma 2.1, we have

$$g_n(x) = n! \binom{x}{n+1}.$$

This implies that $(n+1)g_n(x) = f_n(x)$. □

Remark 2.1 *Lemma 2.2 simply says that, for $x > n \geq 2$,*

$$\prod_{i=0}^{n-1} (x-i) = n \sum_{y=1}^{x} \prod_{i=1}^{n-1} (y-i). \tag{2.2}$$

Theorem 2.1 *Let X be a nonnegative integer-valued random variable and $n \geq 2$. Then*

$$E\left(\prod_{i=0}^{n-1} (X-i)\right) = \begin{cases} n \sum_{x=n-1}^{\infty} \left(\prod_{i=0}^{n-2} (x-i) P(X > x)\right) & \text{if } X > n \\ 0 & \text{if } X \leq n, \end{cases} \tag{2.3}$$

provided that the sum on the right-hand side of (2.3) *exists.*

Proof. Note that when $X < n$, the left-hand side of (2.3) is 0. The proof of this theorem when $X > n$ mainly depends on (2.2) which is proved in Lemma 2.2.

$$\begin{aligned}
\mathrm{E}\left(\prod_{i=0}^{n-1}(X-i)\right) &= \sum_{x\geq 0}\prod_{i=0}^{n-1}(x-i)P(X=x) \\
&= n\sum_{x\geq 0}P(X=x)\sum_{y=1}^{x}\prod_{i=1}^{n-1}(y-i) \qquad \text{(by (2.2))} \\
&= n\sum_{y=1}^{\infty}\sum_{x=y}^{\infty}\prod_{i=1}^{n-1}(y-i)P(X=x) \quad \text{(Fubini's Theorem)} \\
&= n\sum_{y=1}^{\infty}\prod_{i=1}^{n-1}(y-i)P(X\geq y) \\
&= n\sum_{x=n}^{\infty}\prod_{i=1}^{n-1}(x-i)P(X\geq x) \\
&= n\sum_{x=n-1}^{\infty}\prod_{i=0}^{n-2}(x-i)P(X>x),
\end{aligned} \tag{2.4}$$

where we have changed the dummy variable y to x. □

Remark 2.2 *We may start the sum on the right-hand side of* (2.3) *at* $x = 0$*, since the first* $n-1$ *terms of the product* $n\prod_{i=0}^{n-2}(x-i)$ *are equal to 0.*

Remark 2.3 *The case when* $n = 1$ *is well known and can be stated separately for notational convenience, where the proof can be found in [1], for example, and will be part of our next main theorem.*

$$E(X) = \sum_{x=0}^{\infty} P(X > x).$$

Our second main theorem provides an explicit formula for the nth moment of a nonnegative integer-valued random variable X. We need the following lemma before we state and prove the second main theorem.

Lemma 2.3 *Let $A_n(x)$ be defined for $n = 1, 2, \ldots$ as follows:*

$$A_n(x) := \sum_{y=1}^{n} (-1)^{y+1} \binom{n}{y} x^{n-y}, \quad x = 1, 2, \ldots$$

Then we have the following:

(1) $A_n(x+1) = \sum_{y=1}^{n} \binom{n}{y} x^{n-y}$

(2) $\sum_{i=1}^{x} A_n(i) = x^n$

(3) $\sum_{i=1}^{x} A_n(i+1) = (1+x)^n - 1.$

Proof. To prove (1), note first that by adding and subtracting the term of the sum when $y = 0$ we get

$$\begin{aligned} A_n(x) &= \sum_{y=0}^{n} (-1)^{y+1} \binom{n}{y} x^{n-y} + x^n \\ &= -\sum_{y=0}^{n} (-1)^{y} \binom{n}{y} x^{n-y} + x^n \\ &= x^n - (x-1)^n \qquad \text{(by the binomial theorem).} \end{aligned}$$

Therefore, we have

$$\begin{aligned} A_n(x+1) &= (1+x)^n - x^n \\ &= \sum_{y=0}^{n} \binom{n}{y} x^{n-y} - x^n \\ &= \sum_{y=1}^{n} \binom{n}{y} x^{n-y}. \end{aligned}$$

To prove (2) we see that

$$\begin{aligned} \sum_{i=1}^{x} A_n(i) &= \sum_{i=1}^{x} i^n - \sum_{i=1}^{x} (i-1)^n \\ &= \sum_{i=1}^{x} i^n - \sum_{j=0}^{x-1} j^n \qquad \text{(by setting } j = i-1\text{)} \\ &= \sum_{j=0}^{x} j^n - \sum_{j=0}^{x-1} j^n = x^n. \end{aligned}$$

To prove (3) we see that

$$\begin{aligned}\sum_{i=1}^{x} A_n(i+1) &= \sum_{i=1}^{x}(1+i)^n - \sum_{i=1}^{x} i^n \\ &= \sum_{j=2}^{x+1} j^n - \sum_{i=1}^{x} i^n \qquad \text{(by setting } j = i+1\text{)} \\ &= \sum_{j=1}^{x} j^n - \sum_{j=1}^{x} j^n - 1 + (1+x)^n \\ &= (1+x)^n - 1.\end{aligned}$$

□

Theorem 2.2 *Let X be a nonnegative integer-valued random variable and $n \geq 1$. If the sum $\sum_{x=0}^{\infty} A_n(x+1)P(X > x)$ exists, then*

$$\begin{aligned}E(X^n) &= \sum_{x=0}^{\infty} A_n(x+1)P(X > x) \\ &= \sum_{x=0}^{\infty}[(1+x)^n - x^n]P(X > x),\end{aligned} \tag{2.5}$$

where $A_n(x+1) := \sum_{y=1}^{x} \binom{n}{y} x^{n-y}$.

Proof. Note that $X^n = \sum_{i=1}^{X} A_n(i)$. Take the expectation of both sides to get

$$\begin{aligned}\mathrm{E}(X^n) &= \mathrm{E}(\sum_{i=1}^{X} A_n(i)) \\ &= \sum_{x=0}^{\infty}\sum_{i=1}^{x} A_n(i)P(X = x) \\ &= \sum_{i=1}^{\infty}\sum_{x=i}^{\infty} A_n(i)P(X = x) \qquad \text{(by Fubini's Theorem)}\end{aligned}$$

$$= \sum_{i=1}^{\infty} A_n(i) P(X \geq i)$$

$$= \sum_{i=0}^{\infty} A_n(i+1) P(X > i)$$

$$= \sum_{x=0}^{\infty} A_n(x+1) P(X > x),$$

by changing the dummy variable i to x.

Now the last part of (2.5) follows from the fact that

$$A_n(x+1) = \sum_{y=1}^{n} \binom{n}{y} x^{n-y} = (1+x)^n - x^n.$$

□

References

[1] Kai Lai Chung, *A Course in Probability Theory*, Second Edition, Academic Press, New York, 1974.

[2] George Casella, Roger L. Berger, *Statistical Inference*, Duxbury Press, California, 1990.

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