Turk J Math
28 (2004) , 119 – 135.

The Theory of Jacobi Systems and Their Abelian Representations

M. Shahryari, Y. Zamani

Abstract

In this article we introduce a new generalization of the concept of Lie ring which we call *Jacobi system* and we investigate some elementary properties of these systems and their Abelian representations.

The aim of this article is to introduce a new generalization of the concept of Lie ring. The importance of Lie rings in the study of nilpotent groups as well as their role in the investigation of the Burnside problem is known. Researchers have been interested in those aspects of Lie rings which are concerned with the Burnside problem, nilpotent groups and regular automorphisms.

In [3], Zamani and Shahryari introduced an algebraic system, dropping the commutativity assumption in a Lie ring. These systems are called *Jacobi systems* and they are analogue to *near-rings*, about which hundreds of papers has been written, (See [2]).

The goodness of the theory of near-rings gives us the hope that we may bring the theory of Jacobi systems in the interest of doing further research in this area.

In this paper we give the generalities of this theory. Topics such as **J**-solvable and **J**-nilpotent Jacobi systems, Abelian representation and some other elementary topics are included in this paper. But we do not know how much interest could be gained from this subject.

We give our appreciation to the Office of Research at Tabriz University and also the Department of Research at Sahand University of Technology for their financial support during this research.

1. Introduction

Let J be a group and suppose that there is a bi-homomorphism

$$[\ ,\ ] : J \times J \to J$$

such that

i) $[\ x, x\ ] = 1$ for all $x \in J$,

ii) $[\ [\ x, y\ ], z\ ][\ [\ y, z\ ], x\ ][\ [\ z, x\ ], y\ ] = 1$ for all $x, y, z \in J$.

Then we say that J is a *Jacobi system*. Obviously any Lie ring is a Jacobi system in which the underling group is an Abelian group. Another example of a Jacobi system is a group J with $J' \leq Z(J)$ and the bi-homomorphism defined as ordinary commutator:

$$[\ x, y\ ] = xyx^{-1}y^{-1}.$$

Further examples of Jacobi systems will be presented later.

Now suppose J is a Jacobi system. We have

$$[\ xy, xy\ ] = 1$$

for all $x, y \in J$. This gives the identity

$$[\ x, y\ ] = [\ y, x\ ]^{-1}.$$

A subgroup $S \leq J$ is a *sub-system* if $[\ x, y\ ] \in S$ for all $x, y \in S$. Clearly every sub-system is a Jacobi system. An *ideal* of J is a normal subgroup I in J with the property $[\ x, y\ ] \in I$ for $x \in I, y \in J$.

For any $x \in J$ we define the **J**-*class* of x to be the subgroup $[\ x, J\ ]$. The **J**-*centralizer* of x is the normal subgroup

$$C^*_J(x) = \{y \in J \mid [x, y] = 1\}.$$

In fact $C^*_J(x)$ is a sub-system of J. It is easy to see that the map

$$\phi : \frac{J}{C^*_J(x)} \to [\ x, J\ ]$$

defined by $\phi(yC^*_J(x)) = [\ x, y\ ]$ is a well-defined isomorphism of groups.

We now define the *Jacobi center*, or **J**-*center* of J by

$$Z^*(J) = \bigcap_x C^*_J(x).$$

It is trivial that $Z^*(J)$ is an ideal of J. One can easily see that $J' \leq Z^*(J)$, where J' is the ordinary commutator subgroup of J. So, if J is a non-abelian group, then $Z^*(J) \neq 1$. Especially, J can not be **J**-simple in this case.

A Jacobi system J is said to be **J**-*abelian*, if its bracket is trivial, i.e.

$$[\ x, y\ ] = 1,$$

for all $x, y \in J$. If J has no ideals except itself and 1, and moreover, if $[\ J, J\ ] \neq 1$, we call J a **J**-simple Jacobi system. It is easy to see that the ordinary commutator subgroup J' is an ideal of J. Let $\pi : J \to J/J'$ be the canonical map. If we define

$$[\pi(x), \pi(y)] = \pi([x, y]),$$

then we obtain a Lie ring structure on the quotient group J/J'.

Let $X \subseteq J$. The ideal generated by X is the smallest ideal of J containing X. For example, let $X = \{[x, y] | x, y \in J\}$. Then we write $[J, J]$ for the ideal generated by X and we call it the **J**-*derived* ideal of J. Clearly $J/[J, J]$ is **J**-abelian and if J/I is **J**-abelian, then $[J, J] \subseteq I$.

Now we can define the derived series of J. Let $d^1(J) = [J, J]$ and define inductively $d^n(J) = [d^{n-1}(J), d^{n-1}(J)]$. So the series of ideals

$$J \geq d^1(J) \geq d^2(J) \geq \dots$$

is called the **J**-*derived* series of J.

A Jacobi system J is **J**-*solvable*, if $d^n(J) = 1$ for some $n \geq 1$. The smallest n with this property is called the **J**-*derived length* of J. Clearly if J simple, it is **J**-solvable only if the bracket is trivial. One can see that if J is **J**-solvable, then every sub-system and every quotient of J is **J**-solvable as well. Conversely, if I is a **J**-solvable ideal of J with **J**-solvable quotient J/I, then J is **J**-solvable.

A series of ideals

$$J = I_0 \geq I_1 \geq I_2 \geq \cdots \geq I_n = 1$$

is said to be **J**-*abelian series* if I_r/I_{r+1} is **J**-abelian for all r. So J is **J**-solvable if and only if it has a **J**-abelian series.

Similar to the concept of **J**-solvable Jacobi system, we can define the concept of **J**-*nilpotent* Jacobi system. Let $p^1(J) = J$ and define inductively $p^n(J) = [\ J, p^{n-1}(J)\ ]$. Then we get a series of ideals

$$J = p^1(J) \geq p^2(J) \geq \dots$$

and J is said to be **J**-*nilpotent* if $p^n(J) = 1$ for some $n \geq 1$. The smallest n with this property is called the **J**-*nilpotency class* of J.
Clearly, every **J**-nilpotent Jacobi system is also **J**-solvable. If J is a **J**-nilpotent Jacobi system, then so is every sub-system and every quotient of J. Other standard theorems of solvable and nilpotent groups can be proved for the **J**-solvable and **J**-nilpotent systems.

2. Examples

In this section we obtain some examples of Jacobi systems which are not Lie rings.

Example 2.1 *Let n be an even integer and $J = D_{2n}$ be the dihedral group generated by elements a and b subject to the relations*

$$a^n = b^2 = 1, bab = a^{-1}.$$

Every element of J can be expressed as $a^i b^j$ where $0 \leq i \leq n-1$ and $j = 0, 1$. We define

$$[\ a, b\ ] = ab$$

and we extend this map to whole of J as a bi-homomorphism, i.e.

$$\begin{aligned} [\ a^i b^j, a^r b^s\ ] &= [\ a, b\ ]^{si-rj} \\ &= (ab)^{si-rj}. \end{aligned}$$

To verify that this is a bi-homomorphism, let

$$A = a^i b^j, \quad B = a^\alpha b^\beta, \quad C = a^r b^s.$$

Then we have $AB = a^{i-\alpha} b^{\beta+j}$. So

$$\begin{aligned} [\ AB, C\ ] &= [\ a^{i-\alpha} b^{\beta+j}, a^r b^s\ ] \\ &= (ab)^{s(i-\alpha)-r(\beta+j)} \\ &= (ab)^{si-s\alpha-r\beta-rj}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [\ A, C\ ][\ B, C\ ] &= [\ a^i b^j, a^r b^s\ ][\ a^\alpha b^\beta, a^r b^s\ ] \\ &= (ab)^{si-rj}(ab)^{s\alpha-r\beta} \\ &= (ab)^{si-rj+s\alpha-r\beta}. \end{aligned}$$

But $o(ab) = 2$ *and we have*

$$si - s\alpha - r\beta - rj \equiv si - rj + s\alpha - r\beta \ (mod\ 2),$$

so $[\ AB, C\ ] = [\ A, C\ ][\ B, C\ ]$. *Now we show that the* Jacobi identity *holds, i.e.*

$$[\ [\ A, B\ ], C\ ][\ [\ B, C\ ], A\ ][\ [\ C, A\ ], B\ ] = 1.$$

We have

$$\begin{aligned} [[A,B],C][[B,C],A][[C,A],B] &= [(ab)^{\beta i-\alpha j}, C][(ab)^{s\alpha - r\beta}, A][(ab)^{jr-is}, B] \\ &= [\ ab, C\ ]^{\beta i-\alpha j}[\ ab, A\ ]^{s\alpha-r\beta}[\ ab, B\ ]^{jr-is} \\ &= (ab)^{(s-r)(\beta i-\alpha j)+(j-i)(s\alpha-r\beta)+(\beta-\alpha)(jr-is)} \end{aligned}$$

But the exponent of the last expression is even, so it equals to 1. *Other Jacobi structures can be defined over* $J = D_{2n}$ *by considering* $[\ a, b\ ]$ *to be another suitable element of* D_{2n}.

Example 2.2 *Let* G *be any non abelian group and let* A *be an Abelian subgroup of* G. *Let* ∞ *be a symbol with* $\infty \notin G$. *Suppose that* $\tilde{G} = G \cup \{\infty\}$. *Let* J *be the set of all functions*

$$f : \tilde{G} \to G$$

such that $f|_G \in Hom(G, A)$. *Now we define an operation on* J *as follows:*

$$(f.g)(x) = f(x)g(x)$$

for all $f, g \in J$. *It is easy to see that* J *is a non-abelian group together with this operation. We define another operation on* J *by*

$$(f * g)(x) = f(g(x)).$$

Having defined these two operations, J *becomes an algebraic system known as a* distributive near-ring, *(See [2]). Now we define a bracket on* J *as*

$$[\ f, g\ ] = (f * g).(g * f)^{-1}.$$

It is now easy to verify that J *is a Jacobi system.*

Example 2.3 *We can generalize the foregoing example by considering a distributive near-ring* J. *Then by definition*

$$[\ a, b\ ] = ab - ba,$$

and we obtain a Jacobi system.

3. J-solvable and J-nilpotent systems

In this section we give some sufficient conditions for a Jacobi system to be **J**-solvable or **J**-nilpotent. First we will prove an analogue version of Kreknin's theorem which asserts that every Lie ring with a regular automorphism of finite order is necessarily solvable, (See [1]). Recall that a regular automorphism is an automorphism which has no fixed element except 1.

Theorem 3.1 *Let J be a Jacobi system together with an automorphism $\alpha : J \to J$ with finite order and the property*

$$x^{-1}\alpha(x) \in J' \Rightarrow x \in J'.$$

Then J is solvable.

Proof. We define a map $\alpha^* : J/J' \to J/J'$ by

$$\alpha^*(xJ') = \alpha(x)J'.$$

This map is well defined because J' is a characteristic subgroup of J. Indeed it is a Lie ring automorphism, since

$$\begin{aligned}\alpha^*([\ xJ', yJ'\ ]) &= \alpha^*([\ x, y\ ]J')\\ &= \alpha([\ x, y\ ])J'\\ &= [\ \alpha(x), \alpha(y)\ ]J'\\ &= [\ \alpha^*(xJ'), \alpha^*(yJ')\ ].\end{aligned}$$

Now let $\alpha^*(xJ') = xJ$. Then

$$x^{-1}\alpha(x) \in J',$$

so $x \in J$ by the assumption. Hence α^* is a regular automorphism of finite order for the Lie ring J/J'. Hence J/J' is solvable as a Lie ring, so $d^n(J/J') = 0$ for some n. But

$$d^n(\frac{J}{J'}) = \frac{d^n(J)J'}{J'},$$

so $d^n(J) \leq J' \leq Z^*(J)$. Hence $d^{n+1}(J) = 1$.

We now prove some Engel type theorems for Jacobi systems. □

Theorem 3.2 *Let J be a Jacobi system such that J/J' is finitely generated and torsion free. Let for any $x \in J$ the homomorphism*

$$ad\ x : J \to J$$

$$ad\ x(y) = [\ x, y\ ]$$

be nilpotent. Then J is **J***-nilpotent.*

Proof. Let $L = J/J'$. Then L is a finitely generated torsion free Lie ring. It is evident that every element of L is ad-nilpotent. Let

$$L^* = L \bigotimes_Z Q.$$

We prove that every element of L^* is ad-nilpotent. Let $X \in L^*$. Then

$$X = \sum_i r_i(x_i \otimes \frac{m_i}{n_i})$$

for some $r_i, n_i, m_i \in Z$. Let $N = \prod_i n_i$. Then

$$NX = \sum_i r_i(x_i \otimes m_i N_i),$$

where $N_i = N/n_i$. So

$$\begin{aligned} NX &= \sum_i (r_i m_i N_i x_i) \otimes 1 \\ &= (\sum_i r_i m_i N_i x_i) \otimes 1. \end{aligned}$$

We now have $ad\ (NX) = ad\ (\sum_i r_i m_i N_i x_i) \otimes 1$. So $ad\ (NX)$ is nilpotent. Hence

$$N^l (ad\ X)^l = 0$$

for some l. But L^* is vector space over Q, so $(ad\ X)^l = 0$. This shows that every element of L^* is ad-nilpotent. Hence by the Engel's theorem L^* is a nilpotent Lie algebra, i.e.

$$(L^*)^n = 0$$

for some n. But $(L^*)^n = p^n(J/J') \bigotimes_Z Q$. So every element $p^n(J/J')$ has finite order. But this is impossible by the assumption, except the case $p^n(J/J') = 0$. This implies that

$$p^n(J) \leq J' \leq Z^*(J),$$

so $p^{n+1}(J) = 1$. □

Theorem 3.3 *Let J be a Jacobi system such that J/J' is finitely generated of exponent p, where p is a prime number. Suppose every element of J is ad-nilpotent. Then J is* **J***-nilpotent.*

Proof. We use a similar argument as in 3.2. Let $L = J/J'$ and suppose

$$L^* = L \bigotimes_Z Z_p.$$

Then L^* is a finitely generated Lie algebra over Z_p with ad-nilpotent elements. Hence L^* is a nilpotent Lie algebra, so

$$L^n \bigotimes_Z Z_p = 0.$$

But every element of L has order p, so $L^n = 0$. This shows that $p^{n+1}(J) = 1$. □

4. Abelian representations

In this section we introduce the concept of an Abelian representation of a Jacobi system. Let J be a Jacobi system. Every Jacobi homomorphism, $\phi : J \to gl(V)$ is called an *abelian representation*. In this article we will say *representation* instead of this longer expression.

Definition 4.1 *Let J be a Jacobi system and V be a vector space over a field K. We say that V is a* **J***-module if there exists a map $J \times V \to V$ (transforming (x, v) into an element $x.v \in V$) such that*

i) $x.(\lambda v) = \lambda(x.v)$,

ii) $x.(v + u) = x.v + x.u$,

iii) $(xy).v = x.v + y.v$,

iv) $[\ x, y\ ].v = x.(y.v) - y.(x.v)$,

where $x, y \in J, v, u \in V$ *and* $\lambda \in K$.

It is evident that if V is a **J**-module, then we can obtain a representation $\phi : J \to gl(V)$ by $\phi(x)v = x.v$. Conversely, if ϕ is a representation of J over V, then by the definition $x.v = \phi(x)v$ the vector space V becomes a **J**-module. The following proposition has a quite elementary proof.

Proposition 4.2 *Let J be a Jacobi system. Then there is a one-to-one correspondence between the set of representations of J and the set of representations of J/J'.*

The notions of **J**-*submodule*, *quotiont* **J**-module, *irreducible* **J**-module, **J**-module *homomorphism* and *isomorphism* of **J**-modules should be defined in a completely similar way to the corresponding notions for groups.

All isomorphism theorems are valid in the case of **J**-modules and we can construct new **J**-modules from old by using direct sum or tensor product. Also we have a Schur's Lemma about representations of **J**-modules.

Schur's Lemma 4.3 *Let $\phi : J \to gl(V)$ be an irreducible representation of a Jacobi system J over an algebraically closed field K. Let $T : V \to V$ be a linear map commuting with all $\phi(x), x \in J$. Then T is a scalar transformation.*

5. Invariant Form

Suppose V is a finite dimensional vector space over a field K and let $\phi : J \to gl(V)$ be a representation. We can define a form

$$< x, y >= \mathrm{Tr}(\phi(x)\phi(y)).$$

It is easy to see that

$$\begin{aligned} < x_1x_2, y > &= < x_1, y > + < x_2, y > \\ < x, y_1y_2 > &= < x, y_1 > + < x, y_2 > . \end{aligned}$$

It is also symmetric, i.e. $< x, y >=< y, x >$. We prove this form is invariant, i.e.

$$< [\ x, y\ ], z >=< x, [\ y, z\ ] > .$$

To do this, we write

$$\begin{aligned}
< [\, x, y\,], z > &= \mathrm{Tr}(\phi([\, x, y\,])\phi(z)) \\
&= \mathrm{Tr}((\phi(x)\phi(y) - \phi(y)\phi(x))\phi(z)) \\
&= \mathrm{Tr}(\phi(x)\phi(y)\phi(z)) - \mathrm{Tr}(\phi(y)\phi(x)\phi(z)) \\
&= \mathrm{Tr}(\phi(x)\phi(y)\phi(z)) - \mathrm{Tr}(\phi(x)\phi(z)\phi(y)) \\
&= \mathrm{Tr}(\phi(x)(\phi(y)\phi(z) - \phi(z)\phi(y))) \\
&= < x, [\, y, z\,] > .
\end{aligned}$$

Definition 5.1 *The radical of ϕ is the set*

$$Rad_\phi = \{x \in J| < x, J >= 0\}$$

Theorem 5.2 *Rad_ϕ is an ideal of J and we have*

$$J' \leq Ker\ \phi \leq Rad_\phi.$$

Proof. Let $x, y \in \mathrm{Rad}_\phi$ and $z \in J$. Then

$$\begin{aligned}
< xy, z > &= < x, z > + < y, z > \\
&= 0 + 0 \\
&= 0,
\end{aligned}$$

so $xy \in \mathrm{Rad}_\phi$. Also, $1 \in \mathrm{Rad}_\phi$, because

$$< 1, z >=< 1, z > + < 1, z > .$$

On the other hand,

$$< x^{-1}, z >= - < x, z >= 0$$

so $x^{-1} \in \mathrm{Rad}_\phi$. Hence we proved that Rad_ϕ is a subgroup of J. We now prove that it is a normal subgroup. Let $x \in \mathrm{Rad}_\phi$ and $g, z \in J$. Then

$$\begin{aligned}
< gxg^{-1}, z > &= < g, z > + < x, z > - < g, z > \\
&= 0.
\end{aligned}$$

So $gxg^{-1} \in \mathrm{Rad}_\phi$. Finally if $x \in \mathrm{Rad}_\phi$ and $y, z \in J$, then

$$\begin{aligned}
< [\, x, y\,], z > &= < x, [\, y, z\,] > \\
&= 0.
\end{aligned}$$

Hence $[\ x, y\ ] \in Rad_\phi$ and so Rad_ϕ is an ideal. □

Theorem 5.3 *Suppose* $exp(J) = \infty$ *or* $char\ K \not| exp(J)$ *if* $exp(J)$ *is finite. Let* ϕ *be a non-trivial representation of* J*. Then*

$$Ker\,\phi \neq Rad_\phi.$$

Proof. Let $Ker\,\phi = Rad_\phi$ and define $R_\phi = J/Rad_\phi$. Then R_ϕ is a Lie ring, because $J' \subseteq Rad_\phi$. We use the notation

$$\bar{x} = xRad_\phi.$$

Let $\phi^* : R_\phi \to gl(V)$ be defined by

$$\phi^*(\bar{x}) = \phi(x).$$

This map is well-defined. Also, we have

$$\begin{aligned} <\bar{x}, \bar{y}> &= Tr(\phi^*(\bar{x})\phi^*(\bar{y})) \\ &= Tr(\phi(x)\phi(y)) \\ &= <x, y>. \end{aligned}$$

We claim that $Rad_{\phi^*} = 0$. Since if $\bar{x} \in Rad_{\phi^*}$, then $<\bar{x}, R_\phi>= 0$, so $<x, J>= 0$. This shows that $x \in Rad_\phi$ and hence $\bar{x} = 0$. But

$$Ker\,\phi^* \leq Rad_{\phi^*},$$

so ϕ^* is faithful. Let $m = exp(J)$ and $x \in J$ such that $\phi(x) \neq 0$. Then $x \notin Rad_\phi$ so $\bar{x} \neq 0$. But $m\bar{x} = 0$, hence

$$\phi^*(m\bar{x}) = 0.$$

This shows that $m\phi^*(\bar{x}) = 0$. Since $\phi^*(\bar{x}) \neq 0$, we must have $charK|m$, a contradiction.

We can rewrite the above theorem with some weaker assumptions. □

Corollary 5.4 *Let* ϕ *be a non-trivial representation of* J *such that*

$$charK \not| \ exp(\frac{J}{Rad_\phi})$$

Then $Ker\,\phi \neq Rad_\phi$.

Definition 5.5 *We say that the corresponding form of* ϕ *is non-degenerated, if* $Rad_\phi = 1$.

Corollary 5.6 *If the corresponding form of a representation* $\phi : J \to gl(V)$ *is non-degenerated, then* $charK \mid exp(J)$.

6. n-dimensional J-modules

In this section we assume that K is a field and J is a Jacobi system. We investigate the structure of n-dimensional **J**-modules over K.
To do this we use $Hom(J, K^+)$, the K-space of all group homomorphism $\lambda : J \to K^+$, where K^+ is the additive group of K. If $\lambda, \mu \in Hom(J, K^+)$, then we define a map

$$[\ \lambda, \mu\ ] : J \times J \to K$$

by

$$[\ \lambda, \mu\ ](x, y) = \lambda(x)\mu(y) - \lambda(y)\mu(x).$$

Also we define $B_J : J \times J \to J$ by

$$B_J(x, y) = [\ x, y\ ].$$

Now let V be an n-dimensional **J**-module with a basis $v_1, \dots, v_n$. For any $1 \leq i \leq n$ and $x \in J$ we can write

$$x.v_i = \sum_{j=1}^{n} \lambda_{ij}(x)v_j,$$

where $\lambda_{ij}(x) \in K$. It is easy to see that $\lambda_{ij} \in Hom(J, K^+)$. Now suppose $x, y \in J$. Then we have

$$[\ x, y\ ].v_i = x.(y.v_i) - y.(x.v_i).$$

Hence,

$$\begin{aligned} \sum_{j=1}^{n} \lambda_{ij}([\ x, y\ ])v_j &= x.(\sum_{r=1}^{n} \lambda_{ir}(y)v_r) - y.(\sum_{r=1}^{n} \lambda_{ir}(x)v_r) \\ &= \sum_{j=1}^{n}(\sum_{r=1}^{n}(\lambda_{ir}(y)\lambda_{rj}(x) - \lambda_{ir}(x)\lambda_{rj}(y)))v_j. \end{aligned}$$

So we have

$$\lambda_{ij}([\ x, y\ ]) = \sum_{r=1}^{n} \lambda_{ir}(y)\lambda_{rj}(x) - \lambda_{ir}(x)\lambda_{rj}(y).$$

This is equivalent to

$$\lambda_{ij} o B_J = -\sum_{r=1}^{n} [\ \lambda_{ir}, \lambda_{rj}\ ];$$

thus we proved the following theorm.

Theorem 6.1 *Let V be an n-dimensional $\mathbf{J}$-module with a basis $v_1, v_2, \ldots, v_n$. Let*

$$x.v_i = \sum_{j=1}^{n} \lambda_{ij}(x) v_j.$$

Then $\lambda_{ij} \in Hom(J, K^+)$ and we have

$$\lambda_{ij} o B_J = -\sum_{r=1}^{n} [\ \lambda_{ir}, \lambda_{rj}\ ].$$

Conversely, let V be an n-dimensional vector space over K with a basis $v_1, v_2, \ldots, v_n$. Let the set

$$\lambda_{ij} \in Hom(J, K^+)$$

satisfy the equation

$$\lambda_{ij} o B_J = -\sum_{r=1}^{n} [\ \lambda_{ir}, \lambda_{rj}\ ].$$

Then V becomes a $\mathbf{J}$-module by definition

$$x.v_i = \sum_{j=1}^{n} \lambda_{ij}(x) v_j.$$

□

Suppose Λ be the $n \times n$ matrix with entries λ_{ij}. We will denote the corresponding $\mathbf{J}$-module by $V = V_\Lambda$. Note that the matrix Λ is not unique.

Proposition 6.2 *Let $V = V_\Lambda$ and $W = V_M$ be two n-dimensional $\mathbf{J}$-module with $\Lambda = [\lambda_{ij}]$ and $M = [\mu_{ij}]$. Then V is $\mathbf{J}$-isomorphic to W if and only if there exists $B \in GL_n(K)$ such that*

$$M = B^{-1} \Lambda B.$$

Proof. We know that V_Λ has a basis $v_1, \dots, v_n$ such that

$$x.v_i = \sum_{j=1}^{n} \lambda_{ij}(x) v_j$$

and that V_M has a basis $u_1, \dots, u_n$ such that

$$x.u_i = \sum_{j=1}^{n} \mu_{ij}(y) u_j.$$

Let $f : V \to W$ be a $\mathbf{J}$-isomorphism. Let $A = [a_{ij}]$ be the matrix representation of f with respect to the bases v_i and u_i. For any $x \in J$ we have

$$\begin{aligned} f(x.v_i) &= x.f(v_i) \\ &= x.(\sum_{r=1}^{n} a_{ri} u_r) \\ &= \sum_{r=1}^{n} a_{ri} x.u_r \\ &= \sum_{j=1}^{n} (\sum_{r=1}^{n} a_{ri} \mu_{rj}(x)) u_j. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(x.v_i) &= f(\sum_{j=1}^{n} \lambda_{ij}(x) v_j) \\ &= \sum_{j=1}^{n} \lambda_{ij}(x) f(v_j) \\ &= \sum_{j=1}^{n} (\sum_{r=1}^{n} a_{jr} \lambda_{ir}(x)) u_j. \end{aligned}$$

Comparing both sides of these two equations, we get

$$\sum_{r=1}^{n} a_{ri} \mu_{rj}(x) = \sum_{r=1}^{n} a_{jr} \lambda_{ir}(x).$$

So we have the equation

$$\sum_{r=1}^{n} a_{ri}\mu_{rj} = \sum_{r=1}^{n} a_{jr}\lambda_{ir}$$

in the **K**-space $\mathrm{Hom}(J, K^+)$. This is equivalent to the formal matrix equality

$$A^T M = \Lambda A^T.$$

Suppose $B = A^T$. Hence we have $M = B^{-1}\Lambda B$. □

Proposition 6.3 *Let V_Λ and V_M be* **J***-modules with dimensions n and m, respectively, where $\Lambda = [\lambda_{ij}]$ and $M = [\mu_{ij}]$. Then V_M can be embedded in V_Λ if and only if $A\Lambda = MA$ for some $m \times n$ matrix A.*

Proof. According to 6.2 it is enough to show that $V_M \leq_J V_\Lambda$ if and only if $A\Lambda = MA$ for some $m \times n$ matrix A. But this can be proved in a similar way as in the proof of 6.2. □

7. Results on irreducible representations

This final section deals with some properties of irreducible representations of a Jacobi system. We assume that the field K is algebraically closed and so we can apply Schur's lemma.

Proposition 7.1 *Let J be a Jacobi system with the property $[\ x, y\ ] = [\ y, x\ ]$ for all $x, y \in J$. Let V be an irreducible* **J***-module and also assume that $charK \neq 2$. Then $dimV = 1$.*

Proof. Let ϕ be the corresponding representation of V. For any $x, y \in J$ we have

$$\phi([\ x, y\ ]) = \phi([\ y, x\ ]),$$

so $2\phi(x)\phi(y) = 2\phi(y)\phi(x)$. But $\mathrm{char}K \neq 2$. Hence we have $\phi(x)\phi(y) = \phi(y)\phi(x)$. This shows that $\phi(x)$ commutes with all elements of $\phi(J)$. By Schur's lemma we must have $\phi(x) = \varepsilon I$ for some $\varepsilon \in K$, where I is the identity map.

Now let W be a subspace of V. Then for any $w \in W$ we have

$$x.w = \phi(x)w = \varepsilon w \in W$$

so W is a **J**-submodule. Since V is irreducible, we obtain $\dim V = 1$. □

Suppose that $\phi : J \to gl(V)$ is an irreducible representation and let $z \in Z^*(J)$. For any $x \in J$ we have $[\ x, z\ ] = 1$. So

$$\phi([\ x, z\ ]) = 0.$$

This shows that $\phi(z)$ commutes with all elements of $\phi(J)$. Hence by Schur's lemma we must have $\phi(z) = \varepsilon I$ for some $\varepsilon \in K$. Let $\lambda : Z^*(J) \to K$ be defined as $\lambda(z) = \varepsilon$. It is easily seen that λ is an irreducible representation of $Z^*(J)$. We have

$$\phi|_{Z^*(J)} = \lambda I.$$

Definition 7.2 *Let $\phi : J \to gl(V)$ be a representation. We define*

$$Z(\phi) = \{x \in J | \phi(x) = \varepsilon I,\ \textit{for some}\ \varepsilon \in K\}$$

Theorem 7.3 *The following statements hold:*
i) $Z(\phi)$ is an ideal of J,
ii) $Z(\phi)/Ker\,\phi \subseteq Z^*(J/Ker\,\phi)$.
iii) If ϕ is irreducible then

$$\frac{Z(\phi)}{Ker\,\phi} = Z^*(J/Ker\,\phi).$$

In particular, if ϕ is also faithful, then

$$Z(\phi) = Z^*(J).$$

Proof. Let $x, y \in Z(\phi)$. Then $\phi(x) = \varepsilon_1 I$ and $\phi(y) = \varepsilon_2 I$ for some $\varepsilon_1, \varepsilon_2 \in K$. But then $\phi(xy) = (\varepsilon_1 + \varepsilon_2)I$. So $xy \in Z(\phi)$. Since $\phi(1) = 0$ so $1 \in Z(\phi)$. Let $x \in Z(\phi)$. Then since $\phi(x^{-1}) = -\phi(x)$, so $x^{-1} \in Z(\phi)$, we have proved that $Z(\phi)$ is a subgroup.

Now suppose that $x \in Z(\phi)$ and $y \in J$. Then

$$\begin{aligned}\phi([\ x, y\ ]) &= \phi(x)\phi(y) - \phi(y)\phi(x)\\ &= 0.\end{aligned}$$

Hence $[\ x, y\ ] \in Z(\phi)$. Finally, we note that $Z(\phi)$ is a normal subgroup because

$$\phi(gxg^{-1}) = \phi(x),$$

so $Z(\phi)$ is an ideal.
To prove part (ii), let $\bar{x} \in Z(\phi)/Ker\,\phi$. Then $x \in Z(\phi)$ and so for any $y \in J$ we have

$$\phi([\ x, y\ ]) = 0,$$

thus $[\ x, y\ ] \in Ker\,\phi$. Hence we have $[\ \bar{x},\ \bar{y}] = 0$, so $\bar{x} \in Z^*(J/Ker\,\phi)$.

Finally, let ϕ be irreducible and $\bar{x} \in Z^*(J/Ker\,\phi)$. Then for any $y \in J$ we have $[\ \bar{x},\ \bar{y}] = 0$ so $[\ x, y\ ] \in Ker\,\phi$. Thus $\phi([\ x, y\ ]) = 0$. But this implies $\phi(x)\phi(y) = \phi(y)\phi(x)$. By Schur's lemma we conclude that $\phi(x) = \varepsilon I$ for some $\varepsilon \in K$. So $\bar{x} \in Z(\phi)/Ker\,\phi$, proving part (iii). □

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M. SHAHRYARI
Department of Pure Mathematics,
University of Tabriz, Tabriz, IRAN
Y. ZAMANI
Faculty of Sciences,
Sahand University of Technology,
Tabriz, IRAN

Received 31.10.2002